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THE MODIFIED EXPONENTIAL DISTRIBUTION WITH APPLICATIONS

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ABSTRACT

In this article, we generalize the exponential distribution by compounding the extended exponential distribution (Gomez et al., 2014) and generalized exponential distribution (Gupta and Kundu, 2001) and call it modified exponential (ME) distribution. It includes as special submodels,Kumaraswamy exponential, generalized exponential and exponential distributions. We providea comprehensive description of the mathematical properties of the proposed distribution. Theestimation of the model parameters is performed by the maximum likelihood method. A simulation study is performed to assess the performance of the maximum likelihood estimators.The usefulness of the modified exponential distribution for modeling data is illustrated usingreal data set by comparison with some generalizations of the exponential distribution.

KEYWORDS

Exponential distribution; Hazard rate function; Reliability function; Maximum likelihood estimation.

1. INTRODUCTION

In many fields of sciences such as medicine, engineering and finance, amongst others, modeling lifetime data is very important. Several lifetime distributions introduced to model such types of data. The quality of the methods used in statistical analysis is highly dependent on the underlying statistical distributions. In the last two decades, several ways of generating new continuous distributions from classical ones were developed and studied. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

In this article, we present a new generalization of the exponential distribution via a new method and call it the modified exponential (ME) distribution. Other generalization of the exponential distribution are generalized exponential (Gupta and Kundu, 2001), beta exponential (Nadarajah and Kotz, 2006), beta generalized exponential (Barreto-Souza et al., 2010), Kumaraswamy exponential (Cordeiro and de Castro, 2011), gamma exponentiated exponential (Ristic and Balakrishnan, 2012), Transmuted exponentiated exponential distribution (Merovci, 2013), exponentiated exponential geometric (Louzada et al., 2014) and Kumaraswamy transmuted exponential (Afify et al., 2016) distributions.

For a baseline random variable having pdf $g(x)$ and cdf $G(x)$, Gupta and Kundu (2001) defined the one-parameter ($\gamma > 0$) exponentiated-G cdf by

$$F(x) = G^{\gamma}(x), x \in \mathbb{R}. \tag{1}$$

The pdf corresponding to (1) becomes

$$f(x) = \gamma g(x) G^{\gamma-1}(x).$$

On the other hand, Gomez et al. (2014) introduced a weighted exponential distribution ($EE(\alpha, \beta)$) with the pdf and cdf

$$g(x; \alpha, \beta) = \frac{\alpha^2(1+\beta x)e^{-\alpha x}}{\alpha+\beta}, x, \alpha > 0\ , \beta \geq 0$$

and

$$G(x) = 1 - (1 + \frac{\alpha\beta}{\alpha+\beta} x)e^{-\alpha x},$$

respectively. If $\beta = 0$, $EE(\alpha, \beta)$ is reduced to the exponential distribution and If $\beta = 1$ the distribution reduces to simple Lindly distribution. The pdf of $EE(\alpha, \beta)$ is a mixture of an exponential density, $E(\alpha)$, and a Gamma density, $G(2, \alpha)$, as shown below

$$g(x; \alpha, \beta) = \frac{\alpha}{\alpha+\beta} \alpha e^{-\alpha x} + \frac{\beta}{\alpha+\beta} \alpha^2 x e^{-\alpha x}.$$

Now, we propose a new extended family of distributions with name Modified-G (M-G) by

$$F(x) = 1 - [1 - G^{\gamma}(x)]^{\alpha} \left\{1 - \tfrac{\alpha\beta}{\alpha+\beta} \log[1 - G^{\gamma}(x)]\right\}, \tag{2}$$

where $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ are three additional shape parameters for the baseline cdf G.

Equation (2) provides a more flexible family of continuous distributions in term of pdf and hrf functions. It includes the Kumaraswamy-G family of distributions (Cordeiro and de Castro, 2011) ($\beta = 0$), the exponentiated family of distributions ($\beta = 0, \alpha = 1$) and the base distribution ($\beta = 0, \alpha = 1, \gamma = 1$).

The density function corresponding to (2) is given by

$$f(x) = \frac{\alpha^2\gamma}{\alpha+\beta} g(x)G^{\gamma-1}(x)\{1 - \beta\log[1 - G^{\gamma}(x)]\}[1 - G^{\gamma}(x)]^{\alpha-1}. \tag{3}$$

Equation (3) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

1.1 Quantile Function

Let $G^{-1}(.) = Q_G(.)$ quantile function of G, then for $\beta = 0$, if $u \sim U(0,1)$, then

$$x_u = G^{-1}\left[1 - (1 - u)^{1/\beta}\right]^{1/\alpha}.$$

For $\beta \neq 0$, we give two algorithms for simulation. The first algorithm is based on generating random data from the Lindley distribution using the exponential-gamma mixture.

Algorithm 1 (Mixture Form of the new exponential distribution)

1. *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n;$
2. *Generate* $V_i \sim Exponential(\alpha), i = 1, \cdots, n;$
3. *Generate* $W_i \sim Gamma(2, \alpha), i = 1, \cdots, n;$
4. *If* $U_i \leq \frac{\alpha}{\alpha+\beta}$ *set* $X_i = G^{-1}\left\{[1 - e^{-V_i}]^{\frac{1}{\gamma}}\right\}$, otherwise, set $G^{-1}\left\{[1 - e^{-W_i}]^{\frac{1}{\gamma}}\right\}, i = 1, \cdots, n.$

The second algorithm is based on generating random data from the inverse cdf in (2) distribution.

Algorithm 2 (Inverse cdf)

1. *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n;$
2. *Set*

$$X_i = G^{-1}\left\{\left\{1 - e^{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\alpha}W\left[(U_i-1)(1+\frac{\alpha}{\beta})e^{-1-\frac{\alpha}{\beta}}\right]}\right\}^{\frac{1}{\gamma}}\right\}, i = 1, \cdots, n,$$

where $W(.)$ is the Lambert function. The Lambert W function (Corless et al., 1996; Jodra, 2010) has been applied to solve several problems in mathematics, physics and engineering. It is implicitly defined as the branches of the inverse relation of the function$\tau(z) = ze^z$,$z \in \mathbb{C}$, that is

$$z = \tau^{-1}(ze^z) = W(ze^z), z \in \mathbb{C}.$$

The Lambert function cannot be expressed in terms of elementary functions. However, a feature that makes the Lambert function attractive is that it is analytically differentiable and integrable.

2. THE ME DISTRIBUTION

Taking in $G(x) = 1 - e^{-\lambda x}, x > 0$, the cdf of the ME distribution is given (for $x \geq 0$) by

$$F(x) = 1 - \left\{1 - \frac{\alpha\beta}{\alpha+\beta}\log\left[1 - \left(1 - e^{-\lambda x}\right)^{\gamma}\right]\right\}\left[1 - \left(1 - e^{-\lambda x}\right)^{\gamma}\right]^{\alpha}, \quad (4)$$

with corresponding pdf

$$f(x) = \frac{\alpha^2\gamma\lambda}{\alpha+\beta}e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\gamma-1}\left\{1 - \beta\log\left[1 - \left(1 - e^{-\lambda x}\right)^{\gamma}\right]\right\}\left[1 - \left(1 - e^{-\lambda x}\right)^{\gamma}\right]^{\alpha-1}. \quad (5)$$

Note that the ME distribution is an extended model to analyze more complex data. Clearly, for $\beta = 0$, we obtain the Kumaraswamy exponential distribution, for $\beta = 0$ and $\alpha = 1$, we have the generalized exponential distribution and for $\beta = 0$ and $\alpha = \gamma = 1$, the ME distribution reduces to the exponential distribution. Figure 1 illustrates some of the possible shapes of the pdf of a ME distribution for selected values of the parameters β, α, γ and λ.

3. USEFUL EXPANSIONS

By using binomial expansion, we can demonstrate that the cdf (4) admits the expansion

$$F(x) = 1 - \sum_{i=0}^{\infty}\binom{\alpha}{i}(-1)^i(1 - e^{-\lambda x})^{\gamma i}\left\{1 - \frac{\alpha\beta}{\alpha+\beta}\log\left[1 - (1 - e^{-\lambda x})^{\gamma}\right]\right\}. \quad (6)$$

Then with Taylor expansion of logarithm function, equation (6) can be expanded as

$$F(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\binom{\alpha}{i}(-1)^{i+j}\frac{\alpha\beta}{\alpha+\beta}(1 - e^{-\lambda x})^{\gamma(i+j)} - \sum_{i=1}^{\infty}\binom{\alpha}{i}(-1)^i(1 - e^{-\lambda x})^{\gamma i}.$$

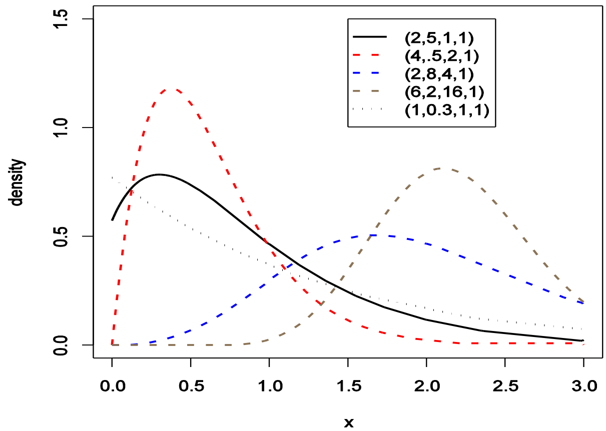

Figure 1: The pdfs of Various ME Distributions $ME(\alpha, \beta, \gamma, \lambda)$.

On the other hand, an expression for $(1-e^{-\lambda x})^b$ ($b>0$ real non-integer) is

$$(1-e^{-\lambda x})^b = \sum_{k=0}^{\infty} S_k(b)\,(1-e^{-\lambda x})^k, \tag{7}$$

where

$$S_k(b) = \sum_{p=k}^{\infty} (-1)^{r+j}\binom{b}{j}\binom{j}{p}. \tag{8}$$

Thus by using (7) and (8) we obtain

$$F(x) = \sum_{k=0}^{\infty} w_k^*(1-e^{-\lambda x})^k,$$

where

$$w_k^* = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\binom{\alpha}{i}(-1)^{i+j}\frac{\alpha\beta}{\alpha+\beta}S_k(\gamma(i+j)) - \sum_{i=1}^{\infty}\binom{\alpha}{i}(-1)^i S_k(\gamma i) \tag{9}$$

and $(1-e^{-\lambda x})^k$ denotes the generalized exponential cdf with power parameter $k>0$. The last results hold for real non-integer k. For integer k, it is clear that the indices should stop at integers and we can easily update the formula.

The density function of ME distribution can be expressed as an infinite linear combination of generalized exponential densities as following

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} w_{k+1}^*(k+1)\lambda e^{-\lambda x}(1-e^{-\lambda x})^k \\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{\mathrm{k}}(-1)^r\binom{k}{r} w_{k+1}^*(k+1)\lambda e^{-\lambda x(1+r)}, \end{aligned} \tag{10}$$

where $(k+1)\lambda e^{-\lambda x}(1-e^{-\lambda x})^k$ is the generalized exponential density with power parameter $k+1$. Thus, some mathematical properties of the new distribution can be derived from those of the generalized exponential distribution based on (10) such as the ordinary and incomplete moments and generating function.

4. MOMENTS

Let X has a ME distribution.Then, it can be shown that the$n$th moment of X can be written as

$$\begin{aligned} E(X^n) &= \sum_{k=0}^{\infty}\sum_{r=0}^{\mathrm{k}} -1^r\binom{k}{r} w_{k+1}^*(k+1)\lambda\int_0^{\infty} x^n e^{-\lambda x(1+r)}dx \\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{\mathrm{k}}(-1)^r\binom{k}{r} w_{k+1}^*(k+1)\lambda\frac{\Gamma(n+1)}{[\lambda(1+r)]^{n+1}}. \end{aligned} \tag{11}$$

The mean and variance can be simply calculated from (11). It is immediate from (11) that the moment generating function (mgf) of X, say $M_X(t)$, is

$$M_X(t) = \sum_{n=0}^{\infty}\frac{\mu_n^{'}}{n!}t^n = \sum_{n,k=0}^{\infty}\sum_{r=0}^{k}\frac{(-1)^r\binom{k}{r}w_{k+1}^*(k+1)\lambda\Gamma(n+1)t^n}{n![\lambda(1+r)]^{n+1}}. \tag{12}$$

The reliability function $R(x)$, which is the probability of an item not failing prior to some time x, is defined by $R(x) = 1 - F(x)$. The reliability function of the ME distribution is given by

$$R(x) = \left\{1 - \frac{\alpha\beta}{\alpha+\beta}\log\left[1 - \left(1 - e^{-\lambda x}\right)^{\gamma}\right]\right\}\left[1 - \left(1 - e^{-\lambda x}\right)^{\gamma}\right]^{\alpha}, x > 0.$$

The other characteristic property of interest of a random variable is the hazard rate function defined by

$$h(x) = \frac{f(x)}{1 - F(x)},$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to time t. The hazard rate function (hrf) for a ME random variable is given by

$$h_X(x) = \frac{\frac{\alpha^2\gamma}{\alpha+\beta}\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\gamma-1}\left\{1 - \beta\log\left[1 - (1 - e^{-\lambda x})^{\gamma}\right]\right\}}{\left\{1 - \frac{\alpha\beta}{\alpha+\beta}\log[1 - (1 - e^{-\lambda x})^{\gamma}]\right\}[1 - (1 - e^{-\lambda x})^{\gamma}]}. \quad (13)$$

The shape of hrf (13) is illustrated in Figure 2. This hazard rate shapes includes increasing and S shape.

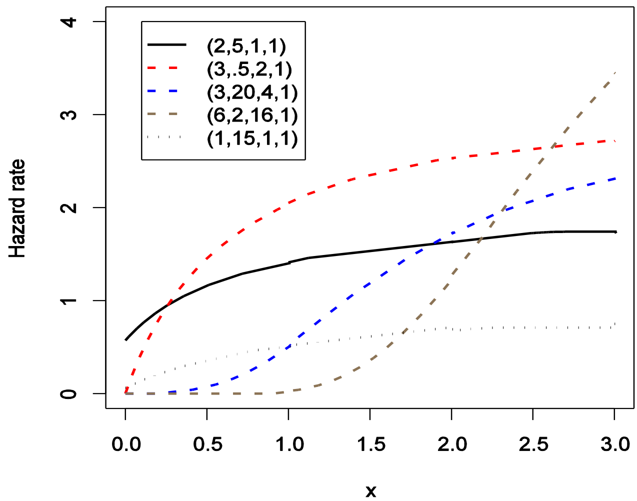

Figure 2: The hrf of Various ME Distributions $ME(\alpha, \beta, \gamma, \lambda)$.

5. ORDER STATISTICS

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from any G distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = cf(x)F^{i-1}(x)\{1 - F(x)\}^{n-i}$$
$$= c\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1},$$

where

$$c = \frac{1}{B(i, n-i+1)}$$

We use the result 0.314 of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i} u^i,$$

where the coefficients $c_{n,i}$ (for $i = 1,2, \ldots$) are determined from the recurrence equation (with $c_{n,0} = a_0^n$)

$$c_{n,i} = (ia_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m c_{n,i-m}.$$

We can demonstrate that the density function of the ith order statistic of ME distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k} h_{r+k+1}(x), \tag{14}$$

where $h_{r+k+1}(x)$ denotes the generalized exponential density function with power parameter $r + k + 1$,

$$m_{r,k} = \frac{n!\ (r+1)\ (i-1)!\ w_{r+1}^*}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!\ j!},$$

w_r^* is given by (9) and the quantities $f_{j+i-1,k}$ can be determined given that $f_{j+i-1,0} = {w_0^*}^{j+i-1}$ and recursively for $k \geq 1$

$$f_{j+i-1,k} = (kw_0^*)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\, w_m^* f_{j+i-1,k-m}.$$

We can obtain the ordinary and incomplete moments, generating function and mean deviations of the ME order statistics from equation (14).

6. CHARACTERIZATIONS

Characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This section deals with various characterizations of distribution (2). The characterizations of the special case of (2) (namely (4)) are given in the Appendix B. These characterizations are based on a simple relationship between two truncated moments. It should be mentioned that these characterization can be applied when the cdf does not have a closed form. The first characterization result employs a theorem due to Glanzel (1987) see Theorem 1 in Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glanzel (1990) this characterization is stable in the sense of weak convergence.

Proposition 1:

Let $X: \Omega \to \mathbb{R}$ be a continuous random variable and let, $q_1(x) = \left[1 - \beta log\left(1 - G^{\gamma}(x)\right)\right]^{-1}$ and $q_2(x) = q_1(x)\left(1 - G^{\gamma}(x)\right)$ for $x \in \mathbb{R}$. The random variable X has pdf (3) if and only if the function ξ defined in Theorem 1 has the form

$$\xi(x) = \frac{\alpha}{\alpha + 1}\left(1 - G^{\gamma}(x)\right), x \in \mathbb{R}.$$

Proof:

Let X be a random variable with pdf(3), then

$$\left(1 - F(x)\right)E[q_1(x)|X \geq x] = \frac{\alpha}{\alpha + \beta}\left(1 - G^{\gamma}(x)\right)^{\alpha}, x \in \mathbb{R},$$

and

$$\left(1 - F(x)\right)E[q_2(x)|X \geq x] = \frac{\alpha^2}{(\alpha + 1)(\alpha + \beta)}\left(1 - G^{\gamma}(x)\right)^{\alpha+1}, x \in \mathbb{R},$$

and finally

$$\xi(x)q_1(x) - q_2(x) = -\frac{1}{\alpha + 1}q_1(x)\left(1 - G^{\gamma}(x)\right) < 0, for x \in \mathbb{R}.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x)q_1(x)}{\xi(x)q_1(x) - q_2(x)} = \frac{\alpha\gamma g(x)G^{\gamma-1}(x)}{1 - G^{\gamma}(x)} x \in \mathbb{R},$$

and hence

$$s(x) = -\alpha \log\left(1 - G^{\gamma}(x)\right), x \in \mathbb{R}.$$

Now, in view of Theorem 1, X has density (3).

Corollary 1:

Let $X: \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The pdf of X is (3) if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x) - q_2(x)} = \frac{\alpha\gamma g(x)G^{\gamma-1}(x)}{1 - G^{\gamma}(x)} x \in \mathbb{R}.$$

The general solution of the differential equation in Corollary 1 is

$$\xi(x) = \left(1 - G^{\gamma}(x)\right)^{-\alpha}\left[-\int \alpha\gamma g(x) G^{\gamma-1}(x)\left(1 - G^{\gamma}(x)\right)^{\alpha-1}(q_1(x))^{-1} q_2(x) + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. However, it should be also noted that there are other triplets (q_1, q_2, ξ) satisfying the conditions of Theorem 1.

7. PARAMETER ESTIMATION

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the ME distribution from complete samples only. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from $ME(\alpha, \beta, \gamma, \lambda)$, $\theta = (\alpha, \beta, \gamma, \lambda)$. The likelihood function for the vector of parameters

$$\begin{aligned} L(\theta; x_1, \dots, x_n) &= \left(\frac{\alpha^2\gamma\lambda}{\alpha+\beta}\right)^n e^{-\lambda\sum_{i=1}^n x_i} \prod_{i=1}^n \left(1 - e^{-\lambda x_i}\right)^{\gamma-1}\left\{1 - \beta\log\left[1 - \left(1 - e^{-\lambda x_i}\right)^{\gamma}\right]\right\} \\ &\quad \times \prod_{i=1}^n \left[1 - \left(1 - e^{-\lambda x_i}\right)^{\gamma}\right]^{\alpha-1}. \end{aligned}$$

Taking the log-likelihood function for the vector of parameters $\theta = (\alpha, \beta, \gamma, \lambda)$ we get

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (15). We can find the estimates of the unknown parameters by maximum likelihood method by setting the derivates equal to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

Note that the ME log-likelihood has second derivatives with respect to the parameters, so that Fisher information matrix (FIM), $I_{ij}(\theta)$ can be expressed as

$$I_{ij}(\theta) = E\left(\frac{\partial^2 \ell(\theta; X_1, \dots, X_n)}{\partial\theta_i \, \partial\theta_j}\right), i, j = 1,2,3,4.$$

Elements of the FIM can be numerically obtained by MAPLE software. The total FIM $I_n(\theta)$ can be approximated by

$$J_n(\hat{\theta}) = \left[\frac{\partial^2 \ell(\theta; X_1, \dots, X_n)}{\partial\theta_i \, \partial\theta_j}\Big|_{\theta=\hat{\theta}}\right]_{4\times 4}, i, j = 1,2,3,4. \tag{16}$$

For real data, the matrix given in equation (16) is obtained after the convergence of the Newton-Raphson procedure in R software. Let $\hat{\theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda}\right)$ be the maximum likelihood estimate of $\theta = (\alpha, \beta, \gamma, \lambda)$: Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} N_4(0, I^{-1}(\theta))$, where $I(\theta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\theta)$ is replaced by the observed information matrix evaluated at $\hat{\theta}$, that is $J(\hat{\theta})$. The multivariate normal distribution with mean vector

$0 = (0,0,0,0)^T$ and covariance matrix $I^{-1}(\theta)$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)$ percent two-sided confidence intervals for α, β, γ and λ are given, respectively, by

$$\hat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\theta})}, \hat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\theta})}\,, \hat{\gamma} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\gamma\gamma}^{-1}(\hat{\theta})} \text{ and } \hat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\lambda\lambda}^{-1}(\hat{\theta})},$$

where $I_{\alpha\alpha}^{-1}(\hat{\theta})$, $I_{\beta\beta}^{-1}(\hat{\theta})$, $I_{\gamma\gamma}^{-1}(\hat{\theta})$ and $I_{\lambda\lambda}^{-1}(\hat{\theta})$ are diagonal elements of $I_n^{-1}(\hat{\theta}) = \left(nI_n(\hat{\theta})\right)^{-1}$ and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}$th percentile of a standard normal distribution.

8. SIMULATION STUDY

In this section, the performance of the maximum likelihood method forestimating the ME parameters are discussed by means of a Monte Carlo simulation study. The coverage probabilities (CP), mean square errors (MSEs), bias of the parameter estimates and estimated average lengths (ALs) are calculated using the R software. We generate $N = 10{,}000$ samples of sizes $n = 50, 55, 60, \ldots, 1{,}000$ from the ME distribution with $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 2, \lambda = 2$. Let $\left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda}\right)$ be the MLEs of the new model parameters and $\left(s_{\hat{\alpha}}, s_{\hat{\beta}}, s_{\hat{\gamma}}, s_{\hat{\lambda}}\right)$ be thestandard errors of the MLEs. The estimated biasesand MSEs are given by

$$\widehat{Bias}_{\epsilon}(n) = \frac{1}{\mathrm{N}}\sum_{i=1}^{\mathrm{N}} \left(\widehat{\epsilon}_i - \epsilon\right)$$

and

$$\widehat{MSE}_{\epsilon}(n) = \frac{1}{\mathrm{N}}\sum_{i=1}^{\mathrm{N}} \left(\widehat{\epsilon}_i - \epsilon\right)^2$$

for $\epsilon = \alpha, \beta, \gamma, \lambda$. The CPs and ALs are given, respectively, by

$$CP_{\epsilon}(n) = \frac{1}{\mathrm{N}}\sum_{i=1}^{\mathrm{N}} I(\widehat{\epsilon}_i - 1.95996 s_{\widehat{\epsilon}_i}, \widehat{\epsilon}_i + 1.95996 s_{\widehat{\epsilon}_i})$$

and

$$AL(n) = \frac{3.919928}{\mathrm{N}}\sum_{i=1}^{\mathrm{N}} s_{\widehat{\epsilon}_i}.$$

The initial value for optimization problem is real value of parameters. The numerical results for the above measures are displayed in the Figure 3. Based on the Figure 3, the following results are concluded:

- ✓ Biases for all parameters are positive,
- ✓ Estimated biases decrease when the sample size n increases,
- ✓ Estimated MSEs decay toward zero when the sample size n increases,
- ✓ The CPs are near to 0.95 and approach the nominal value when the sample size n increases,
- ✓ The ALs decrease when the sample size n increases.

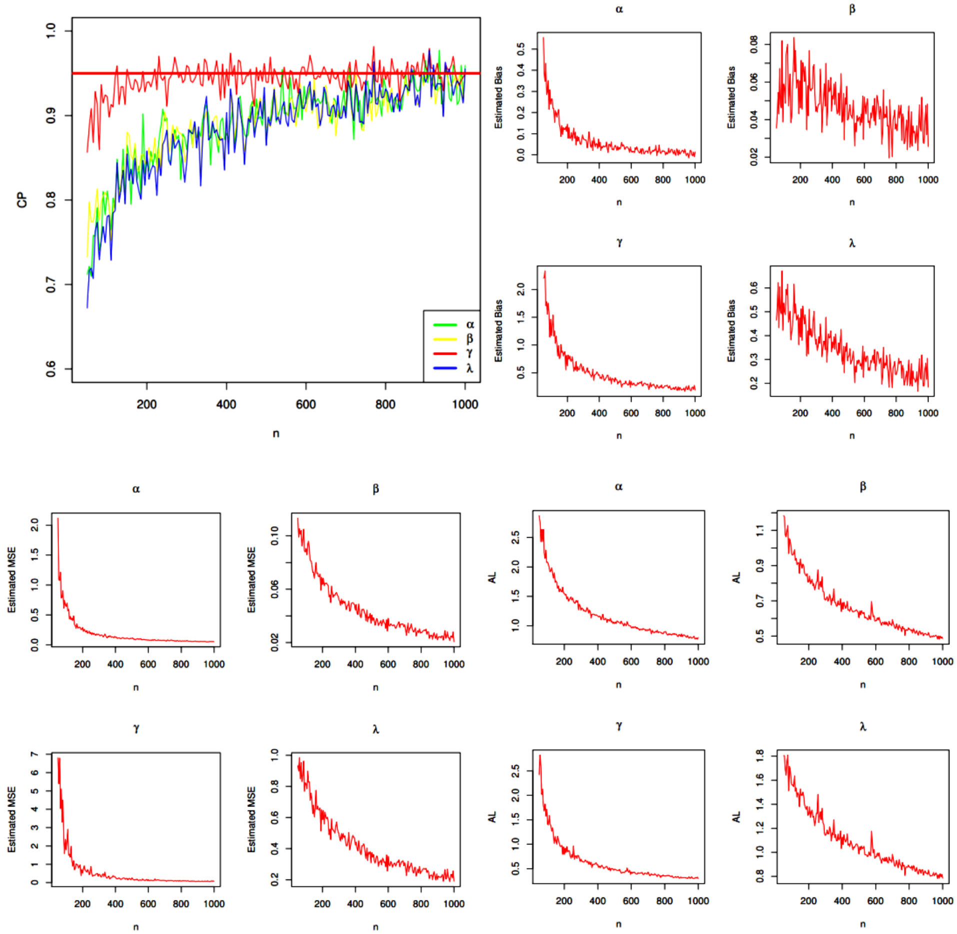

Figure 3: Estimated CPs, Biases, MSEs and ALs for the Selected Parameter Values

9. APPLICATION

Now we use a real data set to show that the ME distribution can be a better model than the Kumaraswamy exponential distribution (KE), generalized exponential (GE), exponential, beta exponential (BE), exponentiated exponential geometric (EEG), gamma exponentiated exponential (GEE) and transmuted exponentiated exponential distributions.

The data set represent the total milk production in the first birth of 107 cows from SINDI race. These cows are property of the Carnaúba farm which belongs to the Agropecuária Manoel Dantas Ltda (AMDA), located in Taperoá City, Paraíba (Brazil). This data is presented by Cordeiro and Brito (2012). These data are 0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747.

In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $G(r/n) = \left(\left(\sum_{i=1}^{r} y_{(i)}\right) + (n-r)y_{(r)}\right) / \sum_{i=1}^{n} y_{(i)}$ where $r = 1,\dots,n$ and $y_{(i)} (i = 1,\dots,n)$ are the order statistics of the sample, against r/n. If the shape is a straight diagonal the hazard is constant. It is convex shape for decreasing hazards and concave shape for increasing hazards. The TTT plot for datasetis presented in Figure 4. This figures indicates that dataset has increasing failure rate function.

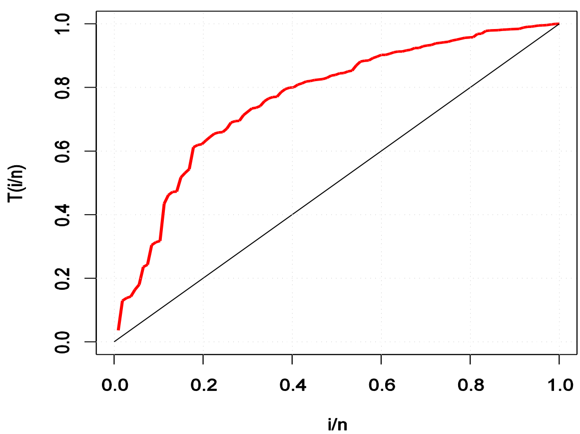

Figure 4: TTT-Plot for Cows' Milks Dataset

These numerical values with MLEs and their corresponding standard errors (in parentheses) of the model parameters are listed in Tables 1. Table 2 includes likelihood ratio test results for comparing ME distribution with submodels. Table 3 present goodness of fit statistics (Akaike information criterion(AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC)) for comparing ME with some other exponential extension models. These reports indicate ME model is superior. The plots of the fitted distributions to real datset are shown in Figures 6 and 7.

Table 1
MLEs of the Parameters (Standard Errors in Parentheses) and Goodness-of-Fit Statistics for the Cows Milks Dataset

Model	**Estimates**				$-2\ell(\hat{\theta})$
$ME(\alpha,\beta,\gamma,\lambda)$	3066.46 (21.74)	5244.90 (20.17)	2.28 (0.02)	0.007 ($1* 10^{-3}$)	**-46.784**
$KE(a,b,\lambda)$(sub-model)	101.99 (9.86)	2.73 (0.05)	0.39 (0.01)		-38.947
$GE(\alpha,\lambda)$(sub-model)	3.71 (0.35)	4.20 (0.23)			-10.06
$E(\lambda)$(sub-model)	2.13 (25.95)				51.901
$EEG(\alpha,\theta,\lambda)$	3.71 (0.35)	0.99 (0.14)	4.19 (0.23)		-10.06
$GEE(\lambda,\alpha,\delta)$	1.75 (0.09)	4.54 (0.24)	3.25 (0.16)		-15.272
$BE(a,b,\lambda)$	3.69 (0.17)	60.26 (3.09)	0.12 ($6* 10^{-3}$)		-18.84
$TEE(\alpha,\beta,\lambda)$	3.43 (0.24)	2.80 (0.14)	0.99 (0.29)		-18.94

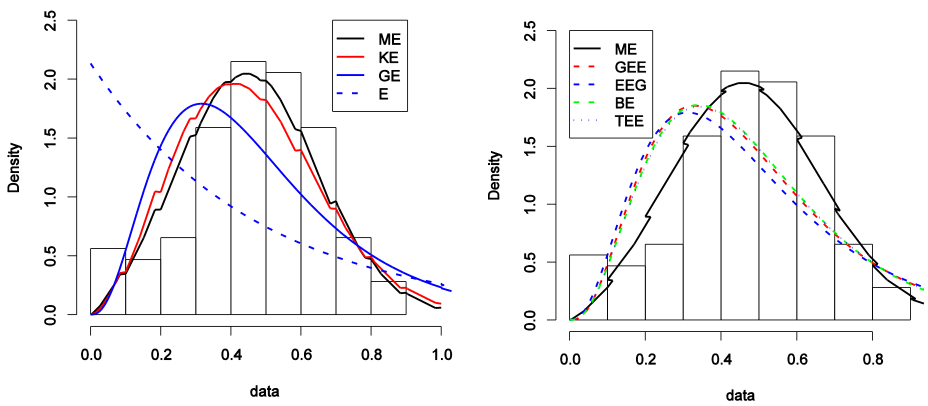

Figure 5: Fitted Densities: (Left) ME with Submodel (Right) ME and Some other Generalized Exponential Distributions

Table 2
Likelihood Ratio Test Statistics with P-Values for the Cows Milk Data

Model	LR Statistic	P-value
$ME(\alpha,\beta,\gamma,\lambda)$vs $KE(a,b,\lambda)$	**7.836**	**0.005**
$ME(\alpha,\beta,\gamma,\lambda)$vs $GE(\alpha,\lambda)$	36.706	$(1* 10^{-8})$
$ME(\alpha,\beta,\gamma,\lambda)$vs $E(\lambda)$	98.685	0

Table 3
Goodness-of-Fit Statistics for the Cows Milk Data

Model	AIC	BIC	CAIC	HQIC
$ME(\alpha,\beta,\gamma,\lambda)$	**-38.784**	**-28.092**	**-38.391**	**-34.449**
$EEG(\alpha,\theta,\lambda)$	-4.061	3.957	-3.828	-0.810
$GEE(\lambda,\alpha,\delta)$	-9.272	-1.254	-9.039	-6.021
$BE(a,b,\lambda)$	-12.840	-4.821	-12.607	-9.589
$TEE(\alpha,\beta,\lambda)$	-10.940	-0.248	-10.548	-6.606

10. CONCLUSION

We have proposed the new modified exponential (ME) distribution generated by a new class of generated distributions. We have derived important properties of the ME distribution like hazard rate function, moments, asymptotic distribution, characterizations and maximum likelihood estimation of parameters. We have illustrated the application of ME distribution to two real data sets used by researchers earlier. By comparing ME distribution with other popular generalization of exponential models we conclude that ME distribution performs better.

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APPENDIX A

Theorem 1:

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ *be a given probability space and let* $H = [d, e]$ *be an interval for some* $d < e$ ($d = -\infty, e = \infty$ *might as well be allowed*). *Let* $X: \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* q_1 *and* q_2 *be two real functions defined on* H *such that*

$$\mathbf{E}[q_2(X)|X \geq x] = \mathbf{E}[q_1(X)|X \geq x]\xi(x), x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ , particularly

$$F(x) = \int_a^x C \left| \frac{\xi^{'}(u)}{\xi(u)q_1(u) - q_2(u)} \right| \exp\big(-s(u)\big)du,$$

where the function s is a solution of the differential equation $s^{'} = \frac{\xi^{'} q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

APPENDIX B

Characterization of distribution (4) based on two truncated moments

Proposition 2:

Let $X:\Omega\to(0,\infty)$ *be a continuous random variable and let ,* $q_1(x)=\left\{1-\beta log\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\}^{-1}$ *and* $q_2(x)=q_1(x)\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}$ *for* $x\in(0,\infty)$. *The random variable* X *has pdf (6) if and only if the function* ξ *defined in Theorem 1 has the form*

$$\xi(x)=\frac{\alpha}{\alpha+1}\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\},x>0.$$

Proof:

Let X be a random variable with pdf(5), then

$$\left(1-F(x)\right)E[q_1(x)|X\geq x]=\frac{\alpha}{\alpha+\beta}\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\}^{\alpha},x>0,$$

and

$$\left(1-F(x)\right)E[q_2(x)|X\geq x]=\frac{\alpha^2}{(\alpha+1)(\alpha+\beta)}\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\}^{\alpha+1},x>0,$$

and finally

$$\xi(x)q_1(x)-q_2(x)=-\frac{1}{\alpha+1}q_1(x)\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\}<0, for x>0.$$

Conversely, if ξ is given as above, then

$$s^{'}(x)=\frac{\xi^{'}(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\frac{\alpha\gamma\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\gamma-1}}{1-[1-(1-e^{-\lambda x})]^{\gamma}}x>0,$$

and hence

$$s(x)=-\alpha\log\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\},x>0.$$

Now, in view of Theorem 1, X has density (5).

Corollary 2:

Let $X:\Omega\to\mathbb{R}$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition 2. The pdf of* X *is (5) if and only if there exist functions* q_2 *and* ξ *defined in Theorem 1 satisfying the differential equation*

$$\frac{\xi^{'}(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\frac{\alpha\gamma\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\gamma-1}}{1-[1-(1-e^{-\lambda x})]^{\gamma}},x>0.$$

The general solution of the differential equation in Corollary 2 is

$$\xi(x)=\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\}^{-\alpha}$$
$$\times\left[-\int\alpha\gamma\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\gamma-1}\left\{1-\left[1-\left(1-e^{-\lambda x}\right)\right]^{\gamma}\right\}^{\alpha-1}(q_1(x))^{-1}q_2(x)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 2 with $D=0$. However, it should be also noted that there are other triplets (q_1,q_2,ξ) satisfying the conditions of Theorem 1.