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WEIBULL BURR X DISTRIBUTION PROPERTIES AND APPLICATION

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ABSTRACT

In this paper, we developed a new continuous distribution called the Weibull-Burr type X distribution which extends the Burr type X distribution. We obtained expressions for the density and the cumulative function. We also derived various structural properties of the new distribution are included Quantile function, the rth moment, moment generating function, Renyi entropy and Order statistics. We estimate parameters by using Least Square, Weighted Least Square and Maximum Likelihood methods. additionally, the asymptotic confidence intervals for the parameters are derived from the Fisher information matrix. Finally, the obtained results are validated using a real data set and is shown that the new family provides a better fit than some other known distributions. This new distribution will serve as an alternative model to other models available in the literature for modelling positive real data in many areas.

KEYWORDS

Quantile function, Moment, Order Statistics, Estimation.

1. INTRODUCTION

Recently, attempts have been made to define new models that extend well known distributions and provide a greater flexibility in modelling real data and to improve the goodness-of-fit the generated family. Eugene et al. (2002) proposed a general class of distributions based on the logit of a beta random variable called the beta-G family distribution by adding two shape parameters. Cordeiro and de Castro (2011) using Kumaraswamy distribution on the unit interval (0,1) to generate a family of distribution named Kumaraswamy-G by adding two shape parameters. There are many families appeared after (2008) for generating the baseline distribution like Gamma-G family (type 1) Zagrafos and Balakrishnan (2009), Gamma - G family (type 2) Ristic and Balakrishnan (2012), Gamma-X family Alzaatreh et al. (2014), Gamma-G family (type 3) by Torabi and Hedash (2012), the generalized transmuted-G by Nofel et al. (2016), the transmuted geometric-G by Afify et al. (2016a), the Kumaraswamy transmuted-G by Afify et al. (2016b), the exponentiated transmuted-G by Merovci et al. (2016), the Burr X-G by Yousof et al. (2016), Odd-Burr generalized family Alizadeh et al. (2016a), the beta transmuted-H by Afify et al. (2017) and the exponentiated generalized-G Poisson family by Aryal and Yousof (2017) and many others. Alzaatrah et al. (2013b) using a new technique and he proposed a general form to generate a new family named transformed-transformer (T-X) family. Recently, Bourguignon et al. (2014) proposed and studied in the generality family of a univariate distribution with two additional parameters.

`The Burr type X distribution is one from twelve distributions was explored by using the method of differential equation Burr (1942). This distribution has found many applications in many areas such as health, agricultural, biological, reliability study, the lifetime of random phenomenon and engineering. Many authors studied Burr type X with one parameter distribution like Sartawi and Abo-Salih (1991), Ahmed et al. (1997), Raqab (1998), Jaheen (1996), Mousa (2001), Surles and Padgett (1998) and Khaleel et al. (2017a). In (2001), Surles and Padgett (2001) proposed the generalized Burr type X with one parameter and add one scale parameter and called Burr type X (BX) distribution. They found that the BX distribution can be used a quite effectively in modelling strength data as well as general lifetime data. Raqab and Kundu (2005) developed a two-parameter Burr type X distribution that has a closed form of the generalized Rayleigh distribution. This distribution was further compared with the Weibull exponentiated exponential, gamma, and generalized exponential distributions. The cumulative distribution function (CDF) of the Burr type X (BX) can be written as

$$G(x,\lambda,\theta)=\left(1-e^{-(\lambda x)^2}\right)^{\theta}\lambda>0,\theta>0 \tag{1}$$

and the probability density function (PDF) of (BX) distribution corresponding to CDF is

$$g(x,\lambda,\theta)=2\,\lambda^2\theta x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\theta-1}x>0 \tag{2}$$

where λ and θ are the scale and shape parameters, respectively. The survival $S(x,\lambda,\theta)$and hazard rate functions $h(x,\lambda,\theta)$ of the Burr type X distribution are

$$S(x,\lambda,\theta)=1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}$$

and

$$h(x,\lambda,\theta)=2\,\lambda^2\theta x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\theta-1}\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{-1}$$

respectively. The rth moment for Burr type X distribution is given as:

$$\boldsymbol{\mu'_r=\frac{\theta}{\lambda^r}\Gamma\left(\frac{r}{2}+1\right)\sum_{j=0}^{\theta-1}\binom{\theta-1}{j}\frac{(-1)^j}{(j+1)^{\frac{r}{2}+1}}} \tag{3}$$

The BX is a special case of the exponentiated Weibull distribution. Raqab and Kundu (2005) observe that the probability density function PDF of BX distribution is monotonic decreasing and unbounded in the left tail when $\boldsymbol{\theta\le\frac{1}{2}}$ and monotonic decreasing with left tail ordinate equal to 1 when $\boldsymbol{\theta}=\frac{1}{2}$. It is a right skewed unimodal function with a short-left tail when $\boldsymbol{\theta}>\frac{1}{2}$. Furthermore, they also observed that the hazard function of BX distribution can be either an increasing function when $\boldsymbol{\theta}>\frac{1}{2}$ and bathtub function when $\boldsymbol{\theta}<\frac{1}{2}$ or $\boldsymbol{\theta}=\frac{1}{2}$. Many authors' study BX distribution, such as Afify et al. (2017), Ahmed et al. (2009), Merovic et al. (2016), Khaleel et al. (2016), Aludaat et al. (2008), Lio et al. (2014), Khaleel et al. (2017b) and Abd et al. (2014).

Let $\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\phi})$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\phi})$ be a cumulative and density functions of the baseline model with parameter vector $\boldsymbol{\phi}$ and the Weibull CDF is $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta})=\mathbf{1}-\boldsymbol{e}^{-(\boldsymbol{\alpha x})^{\boldsymbol{\beta}}}$ for $\boldsymbol{x}>\mathbf{0}$ with positive parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Based on this density, by replacing $\boldsymbol{x}$ with $\frac{G(x,\phi)}{[1-G(x,\phi)]}$. The CDF of the Weibull - G distribution with twoextra parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is defined as [8].

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\phi})=\int_{0}^{\frac{G(x,\phi)}{[1-G(x,\phi)]}}\alpha\beta x^{\beta-1}e^{-(\alpha x)^{\beta}}=1-e^{-\alpha\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}} \tag{4}$$

where $\boldsymbol{G}(\boldsymbol{x},\boldsymbol{\phi})$is a baseline CDF, which depends on the parameter vector $\boldsymbol{\phi}$. The PDF corresponding to (3) can be written as

$$f(x,\alpha,\beta,\phi)=\alpha\beta g(x,\phi)\left[\frac{G(x,\phi)}{[1-G(x,\phi)]}\right]^{\beta-1}e^{-\alpha\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}},\alpha>0,\beta>0. \tag{5}$$

The CDF is a special case of the T-X family for Alzaatreh et al. (2013). In this context, we propose and study the Weibull - Burr type X distribution (for short WBX) based on equations (4) and (5). The aims of this paper are to explore and study the mathematical properties of the new model. The rest of this paper is organized as follows: In Section 2, we define the WBX distribution. We expansion the cumulative and density functions in Section 3. In Section 4, we study and discuss some mathematical properties of the new model including, quantile function, moments, moment generating function, Renyi entropy and order statistics. The least square estimation, weighted least square estimation and maximum likelihood estimation are proposed to estimate the parameter in Section 5. In Section 6, two real data sets are used to illustrate the usefulness of the new model. Finally, concluding remarks are presented in Section 7.

2. WEIBULL BURR TYPE X

In this section, we study the four parameter Weibull Burr type X (WBX) distribution. By inserting (1) in (3) yields the four parameter WBX which the CDF is given as

$$F(x,\alpha,\beta,\lambda,\theta)=1-e^{-\alpha\left(\frac{\left(1-e^{-(\lambda x)^{2}}\right)^{\theta}}{\left[1-\left(1-e^{-(\lambda x)^{2}}\right)^{\theta}\right]}\right)^{\beta}}=1-e^{-\alpha\frac{\left(1-e^{-(\lambda x)^{2}}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x)^{2}}\right)^{\theta}\right]^{\beta}}}, \tag{6}$$

the corresponding PDF of the WBX is given by

$$f(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\theta}) = \mathbf{2}\ \boldsymbol{\alpha\beta\lambda^2\theta} e^{-(\lambda x)^2} \frac{\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta+1}} e^{-\alpha \frac{\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta}}} \tag{7}$$

where $\boldsymbol{\alpha} > 0$, $\boldsymbol{\beta} > 0$, $\boldsymbol{\lambda} > 0$ and $\boldsymbol{\theta} > 0$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two additional shape parameter, we denote by X~WBX $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\theta})$ a random variable having the PDF (6). The survival function $\boldsymbol{S}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\theta})$ and the hazard function $\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\theta})$ of X are given by

$$\boldsymbol{S}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\theta}) = e^{-\alpha \frac{\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta}}}$$

and

$$\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda},\boldsymbol{\theta}) = \mathbf{2}\ \boldsymbol{\alpha\beta\lambda^2\theta} e^{-(\lambda x)^2} \frac{\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta+1}}$$

respectively.

Figures 1 display some plots of WBX density for selected values of the parameter $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda}$ and $\boldsymbol{\theta}$. The plot of the hazard function for some parameters value given in Figure 2. The hazard function of WBX distribution can increasing, decreasing and bathtub depending on the parameter values.

The WBX distribution is a very flexible model that approaches to different distributions when its parameters are changed. It contains the following **new special models**:

- The case $\boldsymbol{\theta} = 1$ refers to the Weibull Rayleigh (WRa) distribution.
- For $\boldsymbol{\lambda} = 1$ the WBX model reduce to the Weibull Burr type X distribution with one parameter (WBX1) distribution.
- For $\boldsymbol{\alpha} = 1$ and $\boldsymbol{\beta} = 1$ we obtain the BX distribution.
- For $\boldsymbol{\alpha} = 1$, $\boldsymbol{\beta} = 1$ and $\boldsymbol{\theta}= 1$ it follows the Rayleigh (Ra) distribution.
- For $\boldsymbol{\alpha} = 1$, $\boldsymbol{\beta} = 1$ and $\boldsymbol{\lambda} = 1$ it follows the Burr type X distribution with one parameter (BX1) distribution.

3. LINEAR REPRESENTATION

In this section, we derive expansions for the CDF and PDF of the WBX distribution that are useful to study its statistical properties. From 7 we have

$$\boldsymbol{A} = e^{-\alpha \frac{\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta}}}$$

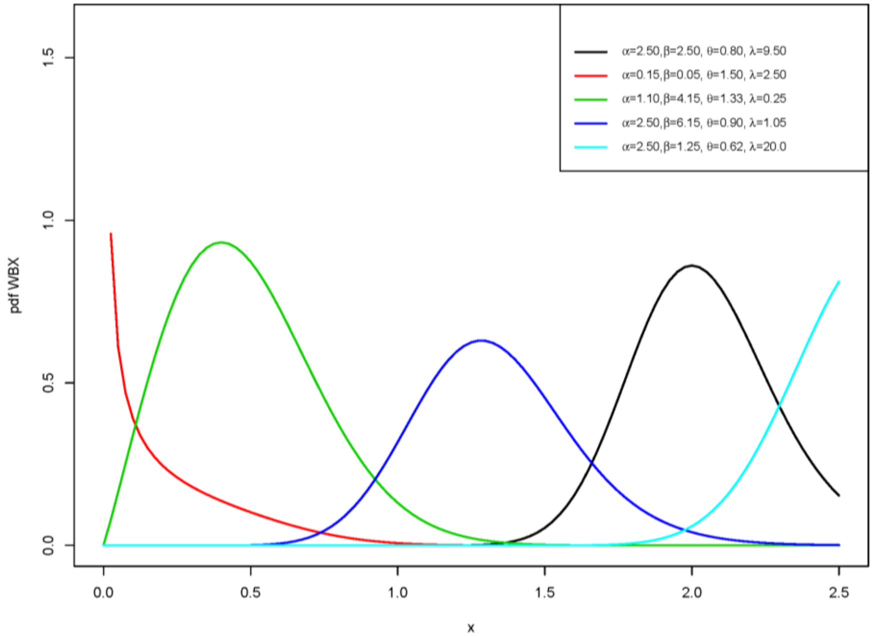

Figure 1: Plot of the WBX Density Function for Some Parameter Values

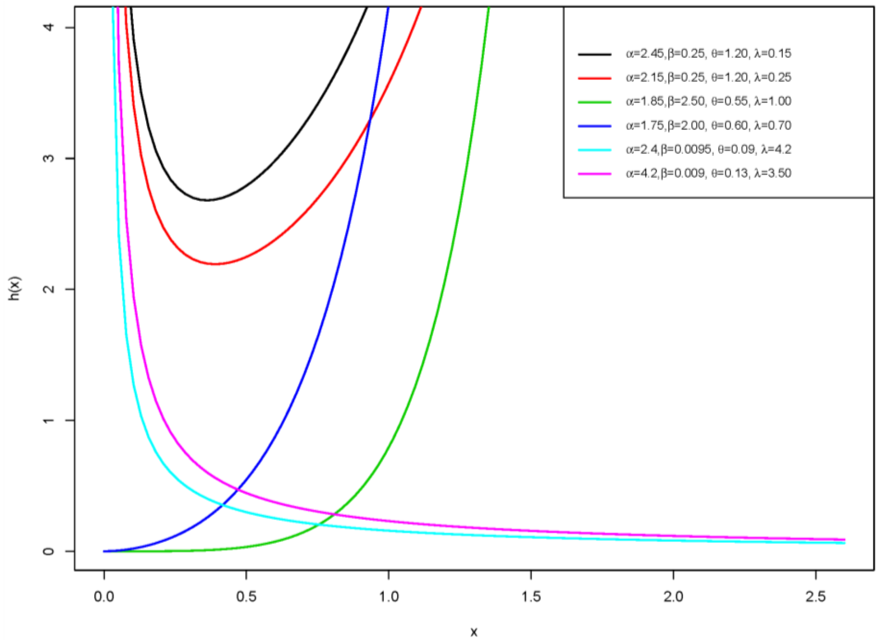

Figure 2: Plot of the WBX Hazard Function for Some Parameter Values

By expanding the exponential function in A, we obtain

$$A = \sum_{j=0}^{\infty} \frac{(-1)^j (\alpha)^j \left(1 - e^{-(\lambda x)^2}\right)^{\theta\beta j}}{j! \left[1 - \left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta j}}$$

Inserting the expansion in (7) and after some algebra, we obtain

$$\mathrm{f}(x, \alpha, \beta, \lambda, \theta) = 2\,\alpha\,\beta\lambda^2\theta e^{-(\lambda x)^2} \sum_{j=0}^{\infty} \frac{(-1)^j (\alpha)^j \left(1 - e^{-(\lambda x)^2}\right)^{\theta\beta j}}{j!} * \left[1 - \left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right]^{-[\beta(j+1)+1]}. \tag{8}$$

By expanding binomial terms in (8), we have

$$\left[1 - \left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right]^{-[\beta(j+1)+1]} = \sum_{k=0}^{\infty} (-1)^k \binom{-[\beta(j+1)+1]}{k} \left(1 - e^{-(\lambda x)^2}\right)^{\theta k}.$$

Equation (8) can be expressed as

$$f(x, \alpha, \beta, \lambda, \theta) = 2\,\alpha\,\beta\lambda^2\theta e^{-(\lambda x)^2} \sum_{j,k=0}^{\infty} \omega_{j,k} \left(1 - e^{-(\lambda x)^2}\right)^{\theta[\beta(j+1)+k]-1} \tag{9}$$

where

$$\omega_{j,k} = \frac{(-1)^j (\alpha)^j}{j!} \binom{-[\beta(j+1)+1]}{k}.$$

Equation (9) can reduce to

$$f(x, \alpha, \beta, \lambda, \theta) = \alpha\,\beta \sum_{j,k=0}^{\infty} \omega_{j,k}\, g(x; \lambda, \theta[\beta(j+1)+k]), \tag{10}$$

where $g(x; \lambda, \theta[\beta(j+1)+k])$ is the BX density with shape parameter $\theta[\beta(j+1)+k]$ and scale parameter λ. Thus, the WBX density can be expressed as a double mixture representation of BX densities. Many of its structural properties can be derived from equation (9) and those properties of BX distribution.

By integrating (9), the CDF of X can be given in the mixture form

$$F(x, \alpha, \beta, \lambda, \theta) = \alpha\,\beta \sum_{j,k=0}^{\infty} \omega_{j,k}\, G(x; \lambda, \theta[\beta(j+1)+k]),$$

where $G(x; \lambda, \theta[\beta(j+1)+k])$ is the CDF of BX distribution.

4. MATHEMATICAL PROPERTIES

In this section we explore and study some important mathematical properties of the WBX distribution specially Quantile function, moment, moment generating function, Renyi entropy, and Order statistics.

4.1 Quantile Function

Quantile functions are in widespread use in general statistics. The quantile function for Weibull Burr type X distribution can be found by inverting of (6) as

$$\boldsymbol{Q(u)=\frac{1}{\lambda}\left[-ln\left(1-\left\{\frac{\left[\frac{-ln(1-u)}{\alpha}\right]^{\frac{1}{\beta}}}{1+\left[\frac{-ln(1-u)}{\alpha}\right]^{\frac{1}{\beta}}}\right\}^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}}. \tag{11}$$

The equation (11) very important to find some essential measure such as Bowley's skewness and Moor's kurtosis this two measure are used to find the heave tail distributions and we can use these measure in distributions that have not third and fourth moments, also these measure less sensitive to outliers from the original measures. Simulating the WBX random variable is straight forward, if U be a continuous uniform variable on the unit interval (0, 1). Using the inverse transformation method, the random variable X is given by

$$\boldsymbol{X=\frac{1}{\lambda}\left[-ln\left(1-\left\{\frac{\left[\frac{-ln(1-U)}{\alpha}\right]^{\frac{1}{\beta}}}{1+\left[\frac{-ln(1-U)}{\alpha}\right]^{\frac{1}{\beta}}}\right\}^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}} \tag{12}$$

has the WBX distribution. Equation (12) used to generate random numbers from the WBX distribution when the parameters are known.

4.2 Moments

The *r*th moment of the WBX distribution can be defined as

$$\boldsymbol{\mu_r'=\int_0^\infty x^r f(x,\alpha,\beta,\lambda,\theta)dx}, \tag{13}$$

using the PDF of the WBX distribution in equation (10), we obtain

$$\boldsymbol{\mu_r'=\int_0^\infty x^r g(x;\lambda,\theta[\beta(j+1)+k])dx}.$$

Using the rth moment of BX equation (3), we obtain

$$\boldsymbol{\mu_r'=\frac{\alpha\beta}{\lambda^r}\sum_{j,k,l=0}^{\infty}\omega_{j,k}\binom{\theta[\beta(j+1)+k]}{l}\frac{\Gamma\left(\frac{r}{2}+1\right)}{(j+1)^{\frac{r}{2}+1}}}. \tag{13}$$

Equation (14) very important to find many measures such as mean, Coefficient of variation central moments and cumulants and also skewness and kurtosis or anthers properties by sitting r = 1 in equation (14) we obtain the mean of X. Furthermore, the central moment ($\boldsymbol{\mu}_n'$) and the cumulants ($\boldsymbol{\kappa}_n$) of X are obtained from (14) as

$$\boldsymbol{\mu}_n' = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, \mu_n'^k \mu_{n-k}'$$

and

$$\boldsymbol{\kappa}_n = \mu_n' - \sum_{k=1}^{n-1} \binom{n-1}{k-1} (-1)^k \, \kappa_k \mu_{n-k}'$$

respectively. Where $\boldsymbol{\kappa}_1 = \mu_1'$, $\boldsymbol{\kappa}_2 = \mu_2' - \mu_1'^2$, $\boldsymbol{\kappa}_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$ and $\boldsymbol{\kappa}_4 = \mu_4' - 4\mu_3'\mu_1' + 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$. The Coefficient of variation (CV), Skewness (Sk) and Kurtosis (Ku) can be determined from the rth moment. Moreover, we can use the equation (14) to find the Moment generating function (mgf).

The moment generating function for WBX distribution is given by

$$M_X(\mathrm{t}) = \alpha\beta \sum_{j,k,l,r=0}^{\infty} \frac{t^r}{r!\,\lambda^r} \omega_{j,k} \binom{\theta[\beta(j+1)+k]}{l} \frac{\Gamma\left(\frac{r}{2}+1\right)}{(j+1)^{\frac{r}{2}+1}} \tag{14}$$

4.3 Renyi Entropy

Renyi entropy is a measure of variation or uncertainty of random variable. It is very popular entropy measure in many fields of science such as (engineering, theory of communication, and probability). The Renyi entropy for a random variable with any pdf of distribution can find from the definition:

$$I_R(\zeta) = \frac{1}{1-\zeta} log \int_{-\infty}^{\infty} [f^{\zeta}(x)] dx \zeta > 0, \zeta \neq 1 \,. \tag{15}$$

We will find Renyi entropy for the WBX random variable. By raising equation (5) to the power $\boldsymbol{\zeta}$.

$$f^{\zeta}(x) = \frac{(\alpha\beta)^{\zeta}[g(x,\phi)]^{\zeta}[G(x,\phi)]^{\zeta(\beta-1)}}{[1-G(x,\phi)]^{\zeta(\beta+1)}} e^{\left\{-\zeta\alpha\left[\frac{G(x,\phi)}{1-G(x,\phi)}\right]^{\beta}\right\}} \tag{17}$$

By using the power series for the exponential function, we obtain

$$e^{\left\{-\zeta\alpha\left[\frac{G(x,\phi)}{1-G(x,\phi)}\right]^{\beta}\right\}} = \sum_{i=0}^{\infty} \frac{(-1)^i(\zeta\alpha)^i}{i!} \left\{\frac{[G(x,\phi)]^{\beta}}{[1-G(x,\phi)]^{\beta}}\right\}^i,$$

Inserting this expansion in equation (17) with some algebra, we have

$$f^{\zeta}(x) = (\alpha\beta)^{\zeta}[g(x,\phi)]^{\zeta} \sum_{i=0}^{\infty} \frac{(-1)^i(\zeta\alpha)^i}{i!} \frac{[G(x,\phi)]^{\beta(\zeta-1)-\zeta}}{[1-G(x,\phi)]^{\beta(\zeta+i)+\zeta}} \,. \tag{18}$$

By using the power series expansion

$$[1-G(x,\phi)]^{\beta(\zeta+i)+\zeta}=\sum_{j=0}^{\infty}\frac{\Gamma(\beta(\zeta+i)+\zeta+j)}{j!\ \Gamma(\beta(\zeta+i)+\zeta)}G(x,\phi)^{j}\,. \tag{19}$$

After power series expansion and combining the last two result, by inserting (1) and (2) in (18), we obtain

$$f^{\zeta}(x)=\ (2\alpha\beta\lambda^{2}\theta)^{\zeta}x^{\zeta}e^{-\zeta(\lambda x)^{2}}\sum_{i,j=0}^{\infty}v_{i,j}\left(1-e^{-(\lambda x)^{2}}\right)^{\beta(\zeta+i)+j-\zeta}, \tag{20}$$

where

$$v_{i,j}=\frac{(-1)^{i}(\zeta\alpha)^{i}\Gamma(\beta(\zeta+i)+\zeta+j)}{i!\,j!\ \Gamma(\beta(\zeta+i)+\zeta)}$$

By using the generalized binomial theorem, we have

$$\left(1-e^{-(\lambda x)^{2}}\right)^{\beta(\zeta+i)+j-\zeta}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\beta(\zeta+i)+j-\zeta)}{k!\ \Gamma(\beta(\zeta+i)+j-\zeta-k)}\left(e^{-(\lambda x)^{2}}\right)^{k}$$

Inserting the last expansion in (20), we obtain

$$f^{\zeta}(x)=\ (2\alpha\beta\lambda^{2}\theta)^{\zeta}\sum_{i,j=0}^{\infty}w_{i,j,k}\,x^{\zeta}e^{-(\zeta+k)(\lambda x)^{2}}\,, \tag{21}$$

where

$$w_{i,j,k}=\frac{(-1)^{i+k}(\zeta\alpha)^{i}\Gamma(\beta(\zeta+i)+\zeta+j)\Gamma(\beta(\zeta+i)+j-\zeta)}{i!\,j!\,k!\,\Gamma(\beta(\zeta+i)+\zeta)\ \Gamma(\beta(\zeta+i)+j-\zeta-k)}\,.$$

By calculated the integral for equation (21), we obtain

$$\int_{0}^{\infty}f^{\zeta}(x)=(2\alpha\beta\lambda^{2}\theta)^{\zeta}\sum_{i,j=0}^{\infty}w_{i,j,k}\int_{0}^{\infty}x^{\zeta}e^{-(\zeta+k)(\lambda x)^{2}}\,,$$

Then

$$\int_{0}^{\infty}x^{\zeta}e^{-(\zeta+k)(\lambda x)^{2}}=\frac{\Gamma\left(\frac{\zeta+1}{2}\right)}{2\lambda^{\zeta+1}[\zeta+k]^{\left(\frac{\zeta+1}{2}\right)}}\,.$$

The Renyi entropy is

$$I_{R}(\zeta)=\frac{1}{1-\zeta}log\left\{2^{\zeta-1}\lambda(\alpha\beta\theta)^{\zeta}\sum_{i,j,k=0}^{\infty}w_{i,j,k}\frac{\Gamma\left(\frac{\zeta+1}{2}\right)}{[\zeta+k]^{\left(\frac{\zeta+1}{2}\right)}}\right\}. \tag{22}$$

Equation (22) the main result in this section it is very important to find some entropy such as q-entropy and Shannon entropy.

4.4 Order Statistics

Many areas of practice and statistical theory used order statistics. Let $\boldsymbol{X_1, X_2, \ldots X_n}$ be a random sample size n from the WBX distribution. The pdf of ith order statistics $\boldsymbol{X_{i;n}}$ is given by

$$f_{i,n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-1} (-1)^i \binom{n-i}{j} [F(x)]^{i+j-1}. \tag{23}$$

Based on CDF of Weibull - G distribution equation (4), we can write

$$[F(x, \alpha, \beta, \phi)]^{i+j-1} = \left[1 - e^{-\alpha\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}}\right]^{i+j-1}.$$

By expanding the exponential function in power series, we have

$$\left[1 - e^{-\alpha\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}}\right]^{i+j-1} = \sum_{k=0}^{\infty} (-1)^k \binom{i+j-1}{k} e^{-\alpha\, k\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}}.$$

The $\boldsymbol{f_{i,n}(x)}$ can be written as

$$f_{i,n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} (-1)^{i+k} \binom{n-i}{j} \binom{i+j-1}{k} e^{-\alpha\, k\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}}.$$

Based on the PDF of Weibull -G family distribution, we have

$$f_{i,n}(x) = \frac{f(x)[G(x,\phi)]^{\beta-1}}{\mathrm{B}(i, n-i+1)[1-G(x,\phi)]^{\beta+1}}$$
$$\times \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} (-1)^{i+k} \binom{n-i}{j} \binom{i+j-1}{k} e^{-\alpha\, (k+1)\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}}.$$

By using the power series for the exponential function, we obtain

$$e^{-\alpha\, (k+1)\left(\frac{G(x,\phi)}{[1-G(x,\phi)]}\right)^{\beta}} = \sum_{l=0}^{\infty} \frac{(-1)^l [\alpha(k+1)]^l}{l!} \frac{[G(x,\phi)]^{\beta l}}{[1-G(x,\phi)]^{\beta l}}.$$

and then

$$f_{i,n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-1} \sum_{k,l=0}^{\infty} (-1)^{i+k+l} \frac{[\alpha(k+1)]^l}{l!}$$
$$\times \binom{n-i}{j} \binom{i+j-1}{k} \frac{[G(x,\phi)]^{\beta(l+1)-1}}{[1-G(x,\phi)]^{\beta(l+1)+1}}.$$

By using the binomial expansion again, we obtain

$$[1-G(x,\phi)]^{-[\beta(l+1)+1]}=\sum_{m=0}^{\infty}\frac{\Gamma(\beta(l+1)+m+1)}{m!\,\Gamma(\beta(l+1)+1)}[G(x,\phi)]^{m}.$$

The $f_{i,n}(x)$ reduce to

$$f_{i,n}(x)=\alpha\beta g(x)\sum_{l,m=0}^{\infty}[\beta(l+1)+m]v_{l,m}[G(x,\phi)]^{\beta(l+1)+m-1}$$

where

$$v_{l,m}=\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\frac{(-1)^{i+k+l}[\alpha(k+1)]^{l}\Gamma(\beta(l+1)+m+1)}{\mathrm{m}!\,\mathrm{l}!\,\mathrm{B}(i,n-i+1)\Gamma(\beta(l+1)+1)}\binom{n-i}{j}\binom{i+j-1}{k}.$$

Using the equation (1) and (2), we have

$$f_{i,n}(x)=2\,\alpha\beta\lambda^{2}\theta x e^{-(\lambda x)^{2}}\sum_{l,m=0}^{\infty}[\beta(l+1)+m]v_{l,m}\left[1-e^{-(\lambda x)^{2}}\right]^{\theta[\beta(l+1)+m]-1}. \tag{24}$$

Finally, the ith order statistics can be written as a mixture of WBX densities with a new parameter.

$$f_{i,n}(x)=\sum_{l,m=0}^{\infty}v_{l,m}g(x;\alpha,\beta,\lambda,\theta[\beta(l+1)+m]) \tag{25}$$

Equation (25) is the main result in this subsection. Based on equation (25), we can obtain many structural properties of $X_{i;n}$. For example, the rth moment of $X_{i;n}$ follows from (14) and (25) as

$$E\left(X_{i,n}^{r}\right)=\frac{\alpha\beta}{\lambda^{2}}\sum_{l,m=0}^{\infty}v_{l,m}\theta[\beta(l+1)+m]\frac{\Gamma\left(\frac{r}{2}+1\right)}{(i+1)^{\frac{r}{2}+1}} \tag{26}$$

5. PARAMETER ESTIMATION

In this section, we estimate the unknown parameters of the WBX distribution by using the method of least square estimation and maximum likelihood function.

5.1 Least Squares and Weighted Least Squares Estimators

In this subsection, we provide the regression based methods to estimate the parameters of WBX distribution. The method was originally used by Swain et al. (1988) for estimating the parameter of the beta distribution. Let $x_1, x_2 \ldots, x_n$ be a random sample of size n from the Weibull Burr type X distribution and support that $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of the observed sample. It is well known

$$E\left(G(X_i)\right)=\frac{i}{1+n} \tag{26}$$

and

$$V\left(G(X_i)\right) = \frac{i(n+i+1)}{(1+n)^2(n+2)}$$

expectations and the variances of $G(X_i)$, two variances of the least square methods can be obtained.

5.1.1 Least Squares Estimation Method

The least squares estimate $\hat{\alpha}_{LS}, \hat{\beta}_{LS}, \hat{\lambda}_{LS}$ and $\hat{\theta}_{LS}$ of α, β, λ and θ respectively, are obtained by minimizing the

$$Q(\alpha,\beta,\lambda,\theta) = \sum_{i=1}^{n}\left(1 - e^{-\alpha\frac{\left[1-e^{-(\lambda x)^2}\right]^{\theta\beta}}{\left[1-(1-e^{-(\lambda x)^2})^{\theta}\right]^{\beta}}} - \frac{i}{1+n}\right)^2 . \tag{27}$$

The minimize equation (28) with respect to α, β, λ and θ we differentiate with respect these parameters. By equation to zero and using Newton's method or fixed point iteration techniques to solve the nonlinear equations.

5.1.2 Weighted Least Squares Estimators

The weighted least squares estimators can be obtained by minimizing

$$\sum_{i=1}^{n} w_i\left(G(X_i) - \frac{i}{1+n}\right)^2 . \tag{28}$$

with respect to the unknown parameters, where

$$w_i = \frac{1}{V\left(G(X_i)\right)} = \frac{(1+n)^2(n+2)}{i(n+i+1)}.$$

Therefore, in case of WBX distribution the weighted least square estimation of α, β, λ and θ say $\hat{\alpha}_{WLS}, \hat{\beta}_{WLS}, \hat{\lambda}_{WLS}$ and $\hat{\theta}_{WLS}$ respectively, can be obtained by minimizing

$$Q(\alpha,\beta,\lambda,\theta) = \sum_{i=1}^{n} w_i\left(1 - e^{-\alpha\frac{\left[1-e^{-(\lambda x)^2}\right]^{\theta\beta}}{\left[1-(1-e^{-(\lambda x)^2})^{\theta}\right]^{\beta}}} - \frac{i}{1+n}\right)^2 .$$

with respect to the unknown parameters only

5.2 Maximum Likelihood Function

Let $x_1, x_2 \dots, x_n$ be a random sample of size n from the WBX distribution with parameter α, β, λ and θ. The logarithm of likelihood function for the vector of parameter $\phi = (\alpha, \beta, \lambda, \theta)^T$ is given by

$$\begin{aligned}\log(L) &= n\log(2\alpha\beta\lambda^2\theta) - \sum_{i=1}^{n}(\lambda x_i)^2 + \sum_{i=1}^{n}\log(x_i) \\ &+ (\theta\beta - 1)\sum_{i=1}^{n}\log\left(\left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right) \\ &- (\beta - 1)\sum_{i=1}^{n}\log\left(1 - \left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right) \\ &- \alpha\log\left(\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\beta}}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta}}\right)\end{aligned} \tag{9}$$

The first partial derivative of the log likelihood function with respect the vectors of parameters and by equating the derivative to zero we obtain

$$\frac{\partial\log(L)}{\partial\alpha} = \frac{n}{\alpha} - \log\left(\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\beta}}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta}}\right) = 0\,. \tag{10}$$

$$\begin{aligned}\frac{\partial\log(L)}{\partial\beta} &= \frac{n}{\beta} + \theta\sum_{i=1}^{n}\log\left(\left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right) - \sum_{i=1}^{n}\log\left(1 - \left(1 - e^{-(\lambda x)^2}\right)^{\theta}\right) \\ &- \alpha\theta\log\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\beta}\log(1 - e^{-(\lambda x)^2})}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta}} \\ &- \alpha\log\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\beta}\log\left(1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right)}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta}} \\ &= 0\,.\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial\log(L)}{\partial\lambda} &= \frac{2n}{\lambda} + 2\sum_{i=1}^{n}\lambda {x_i}^2 + (\theta\beta - 1)\sum_{i=1}^{n}\frac{2e^{-(\lambda x)^2}\lambda {x_i}^2}{1 - e^{-(\lambda x)^2}} \\ &+ 2\theta(\beta - 1)\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta-1}e^{-(\lambda x)^2}\lambda {x_i}^2}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]} \\ &- 2\alpha\beta\lambda\theta\log\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\beta-1}e^{-(\lambda x)^2}\lambda {x_i}^2}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta}} \\ &- 2\alpha\beta\lambda\theta\log\sum_{i=1}^{n}\frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta(\beta+1)-1}e^{-(\lambda x)^2}\lambda {x_i}^2}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta+1}} = 0\,.\end{aligned} \tag{12}$$

$$\frac{\partial \log(L)}{\partial \theta} = \frac{n}{\theta} + \beta \sum_{i=1}^{n} \log\left(\left(1 - e^{-(\lambda x)^2}\right)\right)$$
$$- (\beta - 1) \sum_{i=1}^{n} \frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta} \log\left(1 - e^{-(\lambda x)^2}\right)}{1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}}$$
$$- \alpha\beta \log \sum_{i=1}^{n} \frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\beta} \log\left(1 - e^{-(\lambda x)^2}\right)}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta}}$$
$$- \alpha\beta \log \sum_{i=1}^{n} \frac{\left[1 - e^{-(\lambda x)^2}\right]^{\theta\,(\beta+1)} \log\left(1 - e^{-(\lambda x)^2}\right)}{\left[1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right]^{\beta+1}} = 0 . \quad (13)$$

Equations (31) - (34) so difficult to solve analytically. There is many statistical software can be used to maximize the likelihood function by using the package in R like (*Adequacy Model*) or any software such as SAS, OX program, and Mathematica, also we can solve numerically using iterative methods such as Newton- Raphson.

6. SIMULATION STUDY

In this section, the algorithm that can be used to generate random sample of size (n) from WBX by using the quantile function (12).We consider three different sample size n = 50, 150 and 300. Also, we examine three different sets for the parameters ($\alpha, \beta, \lambda, \theta$) and the values are, Set 1 = (3, 3, 3, 3), Set 2 = (2.5, 4, 3, 2.5) and Set 3 = (1.2, 3, 2, 0.5). The process is repeated 1000 times. The AvE, bias and RMSE are presented in Table 1.
Table 1 presents the AvE, bias and RMSE values of parameters α, β, λ and θ for different sample sizes n. From the results in Table 1 it can be seen that when the sample size n is increased the AvEs are close to the real values. Also, the RMSEs decrease toward zero as the sample size n increases. Based on the simulation study we can conclude that the maximum likelihood estimators are appropriate for estimating the WBX parameters.

Table 1
Average of MLEs (AvE), Bias and Root Mean Square Errors (RMSE) for Different Parameter Values

Set 1	n	α=3			β=3		
		AvE	Bias	RMSE	AvE	Bias	RMSE
	50	3.1898	0.1898	0.2433	3.0642	0.0642	0.3759
	150	3.1555	0.1555	0.1942	3.0096	0.0096	0.2065
	300	3.1058	0.1058	0.1731	2.9969	-0.0031	0.1404
	n	λ=3			θ=3		
		AvE	Bias	RMSE	AvE	Bias	RMSE
	50	3.0134	0.0134	0.0401	3.0482	0.0482	0.1301
	150	3.0054	0.0054	0.0216	3.0395	0.0395	0.0718
	300	3.0022	0.0022	0.0148	3.0318	0.0318	0.0571

Table 1 (Contd….)

Set	n	AvE	Bias	RMSE	AvE	Bias	RMSE
Set 2	**n**	α=2.5			β=4		
		AvE	**Bias**	**RMSE**	**AvE**	**Bias**	**RMSE**
	50	2.7091	0.2091	0.3801	4.0882	0.0882	0.4810
	150	2.6310	0.1310	0.2051	4.0169	0.0169	0.2618
	300	2.6200	0.1200	0.1805	3.9954	-0.0046	0.1795
	n	λ=3			θ=2.5		
		AvE	**Bias**	**RMSE**	**AvE**	**Bias**	**RMSE**
	50	3.0101	0.0101	0.0374	2.5352	0.0352	0.0884
	150	3.0035	0.0035	0.0222	2.5302	0.0302	0.0579
	300	3.0028	0.0028	0.0170	2.5283	0.0283	0.0511
Set 3	**n**	α=1.2			β=3		
		AvE	**Bias**	**RMSE**	**AvE**	**Bias**	**RMSE**
	50	1.3127	0.1127	0.2354	3.0461	0.0461	0.3261
	150	1.2933	0.0933	0.1676	3.0074	0.0074	0.1902
	300	1.2862	0.0862	0.1495	2.9954	-0.0046	0.1311
	n	λ=2			θ=0.5		
		AvE	**Bias**	**RMSE**	**AvE**	**Bias**	**RMSE**
	50	2.0148	0.0148	0.0689	0.5127	0.0127	0.0237
	150	2.0016	0.0016	0.0380	0.5088	0.0088	0.0153
	300	1.9989	-0.0011	0.0276	0.5075	0.0075	0.0125

7. APPLICATIONS

In this section, we illustrate the usefulness of the WBX distribution by using two real data sets. We compare the fits of the WBX distribution with some of its special sub - models such as Burr type X two parameter (BX), Burr type X one parameter ($BX1$) and non - nested Beta Burr type X (BBX) and Beta Burr type X with one parameter ($BBX1$) distributions. There density functions (for $x > 0$) are given by:

$$BX: g(x,\lambda,\theta) = 2\,\lambda^2 \theta x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{\theta-1}.$$

$$BX1: g(x,\theta) = 2\,\theta x e^{-(x)^2}\left(1 - e^{-(x)^2}\right)^{\theta-1}.$$

$$BBX: g(x,\alpha,\beta,\lambda,\theta) = \frac{2\,\lambda^2 \theta e^{-(\lambda x)^2 \left[1-e^{-(\lambda x)^2}\right]^{\theta-1}}}{\mathrm{B}(\alpha,\beta)} \left\{\left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right\}^{\alpha-1} \left\{1 - \left[1 - e^{-(\lambda x)^2}\right]^{\theta}\right\}^{\beta-1}.$$

$$BBX1: g(x,\alpha,\beta,\theta) = \frac{2\,\theta e^{-(x)^2 \left[1-e^{-(x)^2}\right]^{\theta-1}}}{\mathrm{B}(\alpha,\beta)} \left\{\left[1 - e^{-(x)^2}\right]^{\theta}\right\}^{\alpha-1} \left\{1 - \left[1 - e^{-(x)^2}\right]^{\theta}\right\}^{\beta-1}.$$

7.1 Rainfall Data Set

The first data consists of the mean of maximum daily rainfall for 30 years (1975-2004) at 35 stations in the middle and west of peninsular Malaysia. Peninsular Malaysia lies in the equatorial zone, situated in the northern latitude between 1 and $\mathbf{6^0}$ N and the eastern longitude from 100 and $\mathbf{103^0}$ E. Throughout the year, the peninsular has a wet and humid condition with daily temperature rangesfrom 25.5 to $\mathbf{35^0}$ C. It has a tropical climate due to its location with out of respect for equator and the effect of monsoon seasons. There are two monsoons that contribute to rainy seasons are the Southwest monsoon, occurring in May until September, and the Northeast monsoon which occurs from November until March. This data was recently studied by Khaleel et al. (2017a). The data are: 1.134, 1.196, 1.181, 1.178, 1.048, 1.077, 0.835, 1.163, 0.880, 1.056, 1.164, 0.914, 1.141, 1.068, 1.007, 1.027, 1.298, 0.842, 0.991, 0.955, 0.703, 0.953, 1.018, 1.003, 1.106, 1.110, 1.249, 1.092, 1.187, 1.047, 0.989, 0.955, 1.234, 0.937, 0.933.

7.2 Aircraft Windshield Data Set

The second data set correspond on failure time of 84 for a particular model aircraft windshield. This data was recently studied by Tahir et al. (2015). This data consists of 85 failed windshields, the unit for measurement is 1000 h. The data are: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699,1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

We estimated unknown parameters of the distribution by maximum likelihood method using R language, the (package: Adequacy Model) to find the best fit of the data, we compute the MLEs by using the Nelder-Mead (NM) developed by Pedro Rafael DinizMarinho, Cicero Rafael Barros Dias and Marcelo Bourguignon. It is freely available from http://cran.rroject.org/web/packages/AdequacyModel/AdequacyModel.pdf. We use some measures of goodness of fit, including Kolmogorov – Smirnov ($\boldsymbol{K-S}$), Akaike information criterion ($\boldsymbol{AIC}$), consistent Akaike information criterion ($\boldsymbol{CAIC}$), Bayesian information criterion ($\boldsymbol{BIC}$) and Hannan-Quinn information criterion ($\boldsymbol{HQIC}$) statistics and they are:

$$\boldsymbol{AIC = -2l + 2k, BIC = -2l + klog(n), CAIC = AIC + \frac{2k(k+1)}{n-k-1}}$$

$$\boldsymbol{HQIC = 2\log[\log(n)(k-2l)]}$$

where $\boldsymbol{k}$ is the number of parameters in the statistical model, $\boldsymbol{n}$the sample size and $\boldsymbol{l(.)}$ is the maximized value of the log-likelihood function under the considered model. Smaller values of these statistics indicate a better fit. Tables 2 and 3 compare the $\boldsymbol{WBX}$ distribution with the $\boldsymbol{BBX}, \boldsymbol{BBX1}, \boldsymbol{BX}$, and $\boldsymbol{BX1}$. Moreover, values of $\boldsymbol{l}, \boldsymbol{AIC}, \boldsymbol{CAIC}, \boldsymbol{BIC}$, and $\boldsymbol{HQIC}$ are listed in Tables 2 and 3.

According to the criteria $\boldsymbol{AIC}, \boldsymbol{CAIC}, \boldsymbol{BIC}$, and $\boldsymbol{HQIC}$ we found that $\boldsymbol{WBX}$ is the best fitted model than the sub model $\boldsymbol{BX}$ and $\boldsymbol{BX1}$ as well as the non-nested $\boldsymbol{BBX}$ and $\boldsymbol{BBX1}$ distributions for the rainfall data set and for the aircraft windshield data set and also the $\boldsymbol{K-S}$ is very small for the $\boldsymbol{WBX}$ distribution. So, the WBX model could be chosen as the best model and the values of $\boldsymbol{K - S}$ suggest that the $\boldsymbol{WBX}$ model yields a better fit to these data than other distributions. The histogram of two data sets and the estimated PDFs and CDFs for the fitted models are displayed in Figures (3, 4, 5, 6). It is clear from Tables 2 and 3 and Figures (3, 4, 5, 6) that the WBX provides a better _t to the data and therefore could be chosen as the best model for both data sets.

Table 2
The ML Estimates, $(-\boldsymbol{l}), \boldsymbol{K-S}, \boldsymbol{AIC}, \boldsymbol{CAIC}, \boldsymbol{BIC}$ and $\boldsymbol{HQIC}$ for rainfall Dataset

$\boldsymbol{Model}$	$\boldsymbol{MLEs}.$	$-\boldsymbol{l}$	$\boldsymbol{K-S}$	$\boldsymbol{AIC}$	$\boldsymbol{BIC}$	$\boldsymbol{CAIC}$	$\boldsymbol{HQIC}$
$\boldsymbol{WBX}$	$\hat{\boldsymbol{\alpha}}$ = 10.64(18.85) $\hat{\boldsymbol{\beta}}$= 1.400(1.1) $\hat{\boldsymbol{\lambda}}$ = 1.019(-) $\hat{\boldsymbol{\theta}}$ = 5.550(6.29)	-24	0.0662	-41.9	-37.3	-41.2	-40.3
$\boldsymbol{BBX}$	$\hat{\boldsymbol{\alpha}}$ = 0.613(0.632) $\hat{\boldsymbol{\beta}}$= 34.75(55.33) $\hat{\boldsymbol{\lambda}}$= 1.019(-) $\hat{\boldsymbol{\theta}}$ = 12.44(9.166)	-22.4	0.0678	-38.8	-38.1	-34.2	-37.2
$\boldsymbol{BBX1}$	$\hat{\boldsymbol{\alpha}}$ = 0.637(0.656) $\hat{\boldsymbol{\beta}}$= 37.13(59.51) $\hat{\boldsymbol{\theta}}$ = 11.81(8.626)	-22.4	0.0679	-38.8	-38.1	-34.2	-37.2
$\boldsymbol{BX}$	$\hat{\boldsymbol{\lambda}}$ = 1.95(0.125) $\hat{\boldsymbol{\theta}}$ = 12.44 (9.166)	-19.7	0.0817	-35.4	-35	-32.3	-34.2
$\boldsymbol{BX1}$	$\hat{\boldsymbol{\theta}}$ = 2.35(0.397)	3.61	0.397	9.23	9.35	10.8	9.77

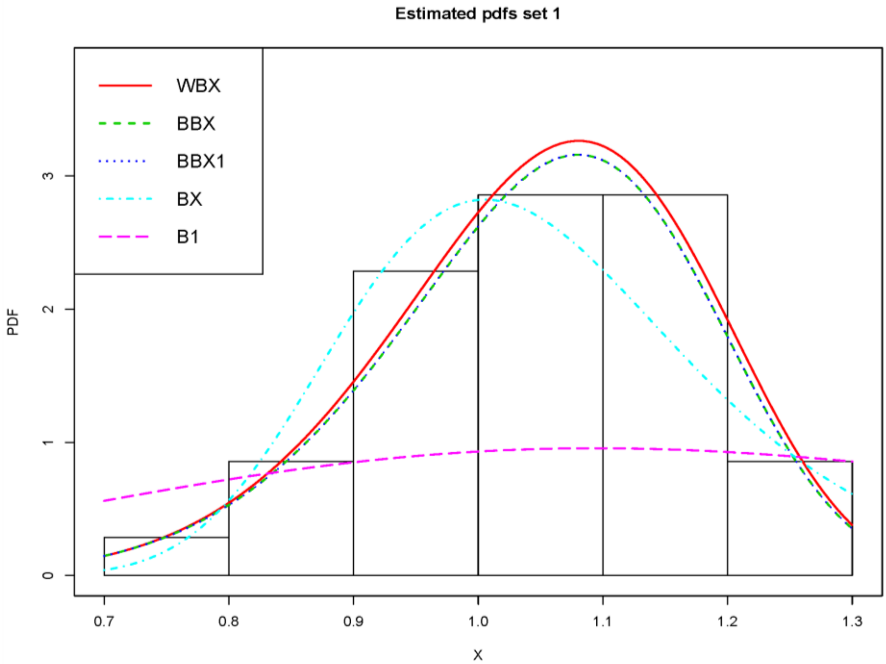

Figure 3: Estimated PDFs for the Rainfall Data Set

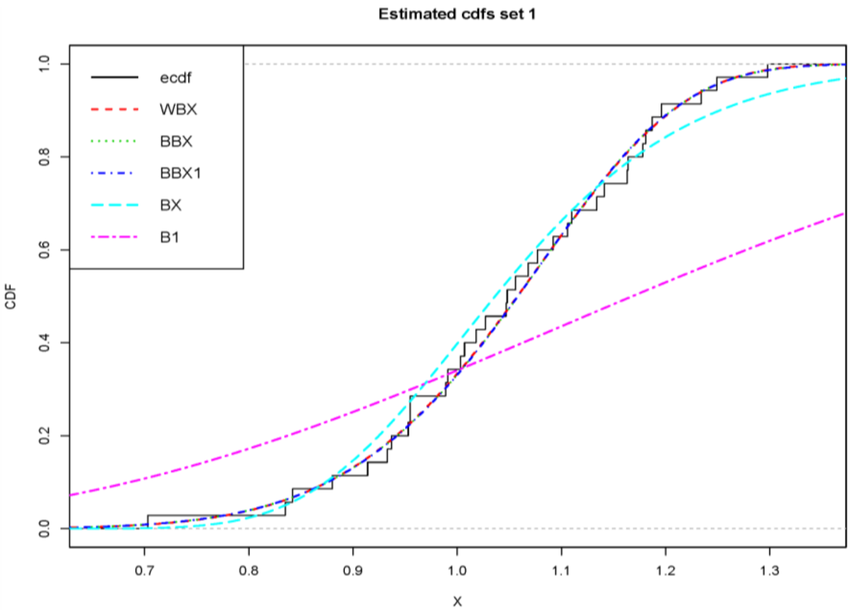

Figure 4: Estimated CDFs for the Rainfall Data Set

Table 3
The ML Estimates, $(-l), K-S, AIC, CAIC, BIC$ and $HQIC$ for Aircraft Windshield Dataset

$Model$	$MLEs.$	$-l$	$K-S$	AIC	BIC	$CAIC$	$HQIC$
WBX	$\hat{\alpha}$= 0.506 (0.838) $\hat{\beta}$= 1.628 (0.348) $\hat{\lambda}$=0.810 (-) $\hat{\theta}$= 0.371 (0.263)	128	0.0568	263	263	270	266
BBX	$\hat{\alpha}$= 14.08 (41.10) $\hat{\beta}$= 0.512 (0.065) $\hat{\lambda}$= 0.521 (-) $\hat{\theta}$ = 0.072 (0.212)	132	0.0638	270	270	277	273
$BBX1$	$\hat{\alpha}$= 0.752 (0.157) $\hat{\beta}$= 0.128 (0.015) $\hat{\theta}$ = 1.530 (-)	134	0.0797	272	272	277	274
BX	$\hat{\lambda}$= 0.378 (0.025) $\hat{\theta}$ = 40.81 (18.69)	132	0.0804	269	269	274	271
$BX1$	$\hat{\theta}$ = 5.33 (0.578)	463	0.60	928	928	930	929

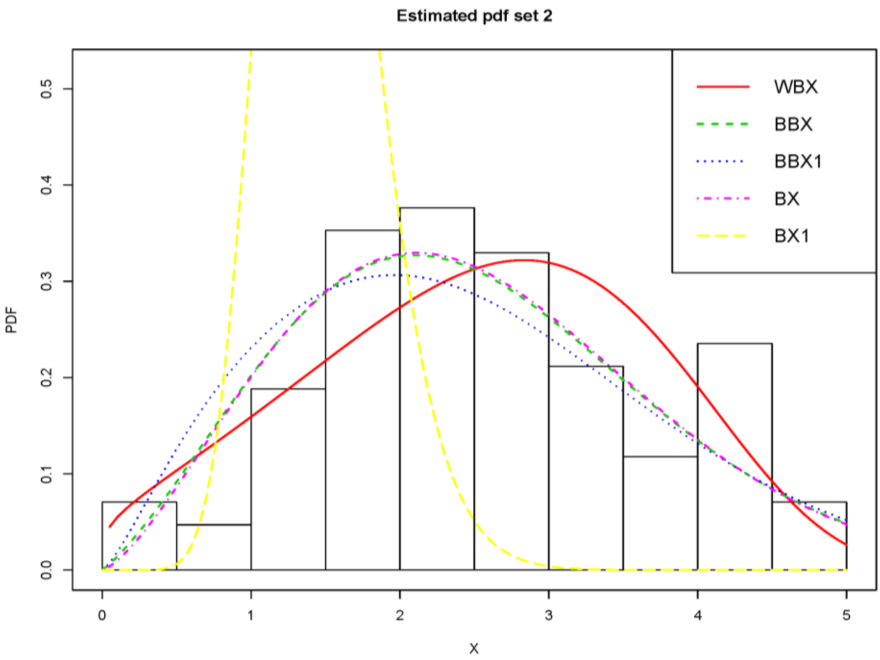

Figure 5: Estimated PDFs for the Aircraft Windshield Data Set

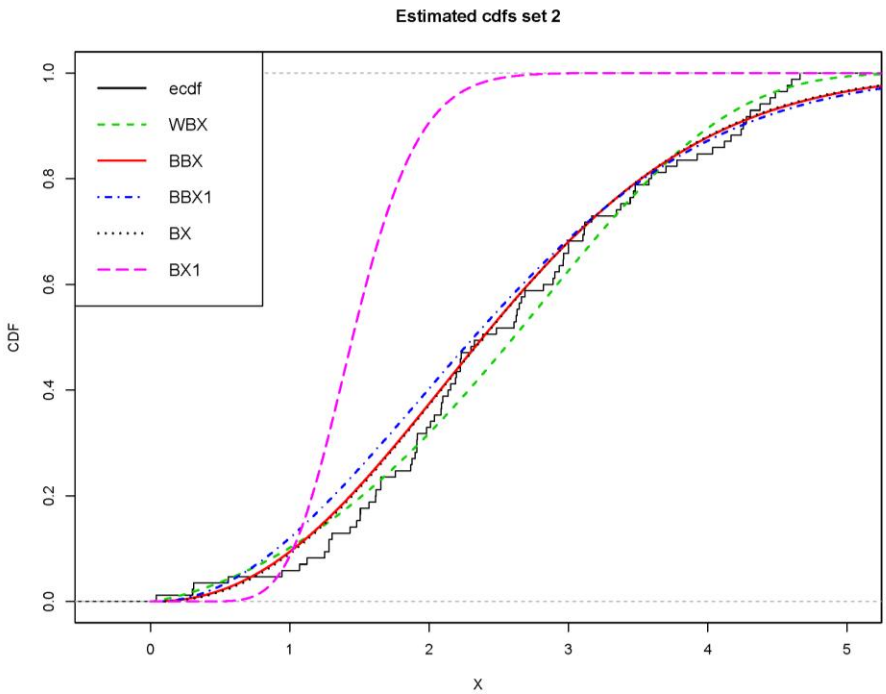

Figure 6: Estimated CDFs for the Aircraft Windshield Data Set

8. CONCLUDING

In this paper, we propose a new four-parameter model, called the Weibull Burr type X (WBX) distribution, which extends the Burr type X distribution. We provide some of its mathematical properties. The density function of the WBX can be expressed as an infinite linear combination of BX densities. The hazard function has various shapes such as increasing, decreasing and bathtub. We derive explicit expressions for quantile function, moments, moment generating function and Renyi entropy. We obtain the density function of order statistics. We estimate the unknown parameter by using some methods such as least square estimate, weighted least square estimate and maximum likelihood methods. We employ the Monte Carlo simulation approach to evaluate the parameter of WBX distribution. From the simulation results, the estimates of the parameters are quite stable and close to the true values as we increase the sample size. The usefulness of the new model is illustrated by two real data sets and the new model provides a better fit than others sub models and non-nested models.

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