Pak. J. Statist.
2017 Vol. 33(2), 129-134

ON CHARACTERIZATIONS OF TRANSMUTED GEOMETRIC-G FAMILY OF DISTRIBUTIONS

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ABSTRACT

Afify et al. (2016) consider the transmuted geometric-G family of distributions and study certain properties and applications of this family of distributions. The present short note is intended to complete, in some way, the work of Afify et al. via establishing certain characterizations of this family in three directions.

1. INTRODUCTION

The problem of characterizing a distribution is an important problem which can help the investigator to see if their model is the correct one. This short note deals with various characterizations of transmuted geometric-G (TG-G) family of distributions to complement the work of Afify et al. (2016). These characterizations are presented in three directions: (i) based on the ratio of two truncated moments; (ii) in terms of the hazard function; (iii) in terms of the reverse hazard function. It should be noted that characterization (i) can be employed also when the cdf (cumulative distribution function) does not have a closed form.

Afify et al. (2016) introduced TG-G distribution with cdf and pdf (probability density function) given, respectively, by

$$F(x;\theta,\lambda,\phi)=\frac{\theta G_1(x;\phi)}{1+(\theta-1)G_1(x;\phi)}\left[1+\frac{\lambda\overline{G}_1(x;\phi)}{1+(\theta-1)G_1(x;\phi)}\right],\quad x\in\mathbb{R}, \tag{1}$$

and

$$f(x;\theta,\lambda,\phi)=\frac{\theta g_1(x;\phi)}{[1+(\theta-1)G_1(x;\phi)]^2}\left[1+\lambda-\frac{2\lambda\theta\overline{G}_1(x;\phi)}{1+(\theta-1)G_1(x;\phi)}\right],\ x\in\mathbb{R}, \tag{2}$$

where $\theta>0$, λ ($|\lambda|\le 1$) are parameters and $G_1(x;\phi)$ is the baseline cdf, depending on the parameter ϕ , with corresponding pdf $g_1(x;\phi)$.

2. CHARACTERIZATIONS OF TG-G DISTRIBUTION

We present our characterizations $(i)-(iii)$ in three subsections.

2.1 Characterizations based on two Truncated Moments

This subsection deals with the characterizations of TG-G distribution based on the ratio of two truncated moments. Our first characterization employs a theorem of Glänzel [2], see Theorem 1 of the Appendix A. The result, however, holds also when the interval H is not closed. As shown in [3], this characterization is stable in the sense of weak convergence.

Proposition 1.

Let $X:\Omega \to \mathbb{R}$ be a continuous random variable. Let $\theta \neq 1$,

$$h(x) = \left[1+\lambda - \frac{2\lambda\theta\overline{G}_1(x;\phi)}{1+(\theta-1)G_1(x;\phi)}\right]^{-1}$$

and

$$g(x) = h(x)[1+(\theta-1)G_1(x;\phi)]^{-1}$$

for $x \in \mathbb{R}$. Then, the random variable X has pdf (2) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2}\left[\frac{1}{1+(\theta-1)G_1(x;\phi)} + \frac{1}{\theta}\right], \quad x \in \mathbb{R}.$$

Proof.

Suppose the random variable X has (2), then

$$\big(1-F(x)\big)E[h(X) \mid X \geq x] = \frac{\theta}{\theta-1}\left[\frac{1}{1+(\theta-1)G_1(x;\phi)} - \frac{1}{\theta}\right], \quad x \in \mathbb{R},$$

and

$$\big(1-F(x)\big)E[g(X) \mid X \geq x] = \frac{\theta}{2(\theta-1)}\left[\frac{1}{[1+(\theta-1)G_1(x;\phi)]^2} - \frac{1}{\theta^2}\right], \ x \in \mathbb{R}.$$

Further,

$$\xi(x)h(x) - g(x) = \frac{1}{2}h(x)\left[\frac{1}{\theta} - \frac{1}{1+(\theta-1)G_1(x;\phi)}\right] \neq 0 \quad for \ \ x \in \mathbb{R}.$$

Conversely, if ξ is of the above form, then

$$s'(x) = \frac{\xi'(x)h(x)}{\xi(x)h(x)-g(x)} = \frac{\theta g_1(x;\phi)}{\overline{G}_1(x;\phi)[1+(\theta-1)G_1(x;\phi)]} \quad x \in \mathbb{R},$$

and consequently

$$s(x) = \log\left\{\frac{\overline{G}_1(x;\phi)}{1+(\theta-1)G_1(x;\phi)}\right\}, \quad x \in \mathbb{R}.$$

Now, according to Theorem 1, X has density (2).

Corollary 1.

Let $X: \Omega \to \mathbb{R}$ be a continuous random variable and let $h(x)$ be as in Proposition 1. The random variable X has pdf (2) if and only if there exist functions g and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)h(x)}{\xi(x)h(x)-g(x)} = \frac{\theta g_1(x;\phi)}{\overline{G}_1(x;\phi)[1+(\theta-1)G_1(x;\phi)]}, \quad x \in \mathbb{R}.$$

The general solution of the above differential equation is

$$\xi(x) = \left(\frac{1+(\theta-1)G_1(x;\phi)}{\overline{G}_1(x;\phi)}\right) \left[-\int \frac{\theta g_1(x;\phi)}{[1+(\theta-1)G_1(x;\phi)]^2}\left(h(x)\right)^{-1} g(x) + D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. Clearly, there are other triplets (h, g, ξ) which satisfy conditions of Theorem 1.

Remark 1.

For $= 1$, we will have $h(x) = [1+\lambda-2\lambda G_1(x;\phi)]^{-1}$, $g(x) = h(x)G_1(x;\phi)$ and $\xi(x) = \frac{1}{2}\left(1+G_1(x;\phi)\right)$ for $x \in \mathbb{R}$ with $D = 0$ in the general solution.

2.2 Characterization in terms of Hazard Function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x). \tag{3}$$

It should be mentioned that for many univariate continuous distributions, equation (3) is the only differential equation available in terms of the hazard function. The following characterization presents a non-trivial characterization of TG-G distribution, for $\theta = 1$, in terms of the hazard function, which is not of the trivial form given in (3).

Proposition 2.

Let $X: \Omega \to \mathbb{R}$ be a continuous random variable. For $\theta = 1$,the random variable X has pdf (2) if and only if its hazard function $h_F(x)$ satisfies the following differential equation

$$h'_F(x) - \frac{g'_1(x;\phi)}{g_1(x;\phi)} h_F(x) = \left(g_1(x;\phi)\right)^2 \left\{ \frac{1}{\left(\overline{G}_1(x;\phi)\right)^2} + \frac{\lambda^2}{\left(1 - \lambda G_1(x;\phi)\right)^2} \right\}, \tag{4}$$

with the initial condition $\lim_{x\to-\infty} h_F(x) = (1+\lambda) \lim_{x\to-\infty} g_1(x;\phi)$.

Proof.

It is clear that (4) holds if X has (2). Conversely, if (4) holds, then

$$\frac{d}{dx}\left\{\left(g_1(x;\phi)\right)^{-1} h_F(x)\right\} = \frac{d}{dx}\left\{\frac{1}{\overline{G}_1(x;\phi)} + \frac{\lambda}{1 - \lambda G_1(x;\phi)}\right\},$$

or

$$h_F(x) = g_1(x;\phi)\left\{\frac{1}{\overline{G}_1(x;\phi)} + \frac{\lambda}{1 - \lambda G_1(x;\phi)}\right\},$$

which is the hazard function of the TG-G distribution for $\theta = 1$.

2.3 Characterization in terms of the reverse (or reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, x \in support\ of\ F.$$

Proposition 3.

Let $X: \Omega \to \mathbb{R}$ be a continuous random variable. For $\theta = 1$, the random variable X has pdf (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r'_F(x) - \frac{g'_1(x;\phi)}{g_1(x;\phi)} r_F(x) = -(g_1(x;\phi))^2 \left\{ \frac{1}{\left(G_1(x;\phi)\right)^2} + \frac{\lambda^2}{(1 + \lambda \overline{G}_1(x;\phi))^2} \right\}. \tag{5}$$

Proof.

Clearly, (5) holds if X has (2). If (5) holds, then

$$\frac{d}{du}\left\{\left(g_1(x;\phi)\right)^{-1} r_F(x)\right\} = \frac{d}{dx}\left\{\frac{1}{G_1(x;\phi)} - \frac{\lambda}{1 + \lambda \overline{G}_1(x;\phi)}\right\},$$

or

$$r_F(x) = g_1(x;\phi)\left\{\frac{1}{G_1(x;\phi)} - \frac{\lambda}{1+\lambda\overline{G}_1(x;\phi)}\right\}$$
$$= \frac{g_1(x;\phi)\{1+\lambda-2\lambda G_1(x;\phi)\}}{G_1(x;\phi)\left[1+\lambda\overline{G}_1(x;\phi)\right]},$$

which is the reverse hazard function of the TG-G distribution.

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APPENDIX A

Theorem 1.

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$, $e = \infty$ might as well be allowed). Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\boldsymbol{E}[g(X) \mid X \geq x] = \boldsymbol{E}[h(X) \mid X \geq x]\xi(x), \quad x \in H,$$

is defined with some real function η. Assume that g, $h \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and ξ, particularly

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u)h(u) - g(u)} \right| \exp\left(-s(u)\right) \, du \ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi' \, h}{\xi \, h \, - \, g}$ and C is the normalization constant, such that $\int_H d\,F = 1$.