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INVERTED KUMARASWAMY DISTRIBUTION: PROPERTIES AND ESTIMATION

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ABSTRACT

In this paper inverted Kumaraswamy distribution is introduced. Some of its properties are presented through, some models of stress strength, measures of central tendency and dispersion and order statistics. Some sub-models, limiting distributions and the relation between inverted Kumaraswamy and other distributions are derived. The maximum likelihood and Bayes estimators, confidence intervals for the parameters, the reliability and the hazard rate functions of the inverted Kumaraswamy distribution based on Type II censored samples are obtained. A numerical study is carried out to illustrate the theoretical results for both approaches, the maximum likelihood and the Bayesian estimation. Moreover, the results are applied on real data.

KEYWORDS

Kumaraswamy distribution; Stress-strength models; Maximum likelihood estimation; Type II censored; Bayesian estimation; Monte Carlo simulation; Markov Chain Monte Carlo (MCMC).

1. INTRODUCTION

The inverted distributions have a wide range of applications; in problems related to econometrics, biological sciences, survey sampling, engineering sciences, medical research and life testing problems. In addition, it is employed in financial literature, environmental studies, survival and reliability theory [See Abd EL-Kader (2013)].

Many researchers focused on the inverted distributions and its applications; for example, Calabria and Pulcini (1990) studied the inverse Weibull, AL-Dayian (1999) introduced the inverted Burr Type XII distribution, Abd EL-Kader *et al.* (2003) described the inverted Pareto Type I distribution, AL-Dayian (2004) discussed inverted Pareto Type II distribution, Prakash (2012) studied the inverted exponential model and Aljuaid (2013) presented exponentiated inverted Weibull distribution.

Kumaraswamy (1980) presented a distribution, which has many similarities to the beta distribution but it has a number of advantages of an invertible closed-form cumulative distribution function, particularly straightforward distribution and quantile functions which do not depend on special functions. This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures and hydrological data such as daily rain fall and daily stream flow. For more details [See Kumaraswamy (1980), Jones (2009), Golizadeh *et al.* (2011), Sindhu *et al.* (2013) and Sharaf EL-Deen *et al*. (2014)].

The inverted Kumaraswamy distribution can be derived from *Kumaraswamy* (Kum) distribution using the transformation $=\frac{1}{X}-1$, when X has a *Kum* distribution with *probability density function* (pdf)

$$f(x;\alpha,\beta)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\,, 0<x<1;\ \alpha,\beta>0 \tag{1}$$

Thus, the distribution of $T=\frac{1}{X}-1$ is referred to as the *inverse or inverted Kumaraswamy* (IKum) distribution and the domain of T is $(0,\infty)$. Using the previous transformation is better than using $T=\frac{1}{X}$, which let the pdf more flexible, also using this transformation is analogous to deriving inverted beta distribution (beta Type II) from beta distribution Type I.

Assuming T is a random variable which has IKum distribution with shape parameters, α and $\beta>0$, denoted by $T\sim$IKum (α,β), then the pdf and *cumulative distribution function* (cdf) are given, respectively, by

$$f(t;\alpha,\beta)=\alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{\beta-1}\,, t>0;\ \alpha,\beta>0\,, \tag{2}$$

and

$$F(t;\alpha,\beta)=(1-(1+t)^{-\alpha})^{\beta}\,, t>0;\ \alpha,\beta>0. \tag{3}$$

The Beta Burr XII distribution; a five parameter distribution, was introduced by Paranaiba *et al.* (2011) with the following pdf and cdf, respectively

$$f(t;a,b,c,s,k)=\frac{ck}{s^{c}B(a,b)}t^{c-1}\left[1+\left(\frac{t}{s}\right)^{c}\right]^{-(kb+1)}\left\{1-\left[1+\left(\frac{t}{s}\right)^{c}\right]^{-k}\right\}^{a-1},$$
$$t>0, a,b,s,c,k>0,$$

and

$$F(t;a,b,c,s,k)=\frac{1}{B(a,b)}\int_{0}^{\{1-[1+(\frac{t}{s})^{c}]^{-k}\}}\omega^{a-1}(1-\omega)^{b-1}\,d\omega.$$

If $b=s=c=1, k=\alpha\ and\ a=\beta$, then the pdf and cdf for IKum distribution given by (2) and (3) can be obtained.

The curves of the pdf and hrf shows that the IKum distribution exhibits a long right tail; compared with other commonly used distributions. Thus it will affect long term reliability predictions, producing optimistic predictions of rare events occurring in the right tail of the distribution compared with other distributions. Also the IKum distribution provides good fit to several data in literature.

This paper is organized as follows: in Section 2 the main properties of the IKum distribution are presented. Some sub-models, limiting distributions and the relation between inverted Kumaraswamy and other distributions are derived in Section 3. Maximum likelihood estimation for the parameters, reliability function and hazard rate function of the IKum distribution based on Type II censored samples are obtained, also

Bayesian estimation of the parameters, reliability function and hazard rate function of the IKum distribution based on Type II censored samples are developed in Section 4. A numerical study is presented in Section 5 to illustrate the application procedures of the various results developed in this paper.

2. THE MAIN PROPERTIES OF THE INVERTED KUMARASWAMY DISTRIBUTION

This section is devoted to illustrate the statistical properties of IKum distribution, through *reliability function* (rf), some models of the stress-strength, *hazard rate function* (hrf) and reversed hazard function (rhrf), measures of central tendency and dispersion, graphical and order statistics.

a. Reliability Function

$$R_1(t) = P(T > t) = 1 - F(t) = 1 - (1-(1+t)^{-\alpha})^{\beta}\ , t > 0. \tag{4}$$

b. Some Stress-Strength Models

1) Let T be the stress component subject strength Y, the random variables T and Y are independent distributions from IKum (α,β_1) and (α,β_2) respectively, then the rf is given by;

$$\begin{aligned} R_2(t) = P(T < Y) &= \int_0^{\infty}\int_0^{y} f(y)f(t)\,dt\,dy \\ &= \int_0^{\infty} f(y)F_T\,(y)dy \\ &= \int_0^{\infty} \alpha\beta_2(1+y)^{-(\alpha+1)}(1-(1+y)^{-\alpha})^{\beta_1+\beta_2-1}dy \\ &= \frac{\beta_2}{\beta_1+\beta_2}\,. \end{aligned} \tag{5}$$

2) Let T and Z be two independent random stress variables with known cdfs $H_T(t), G_Z(z)$, and follow IKum (α,β_1) and IKum (α,β_3), respectively, and let Y be independent of T and Z be a random strength variable with known cdf, $F_Y(y)$, and follows IKum (α,β_2). Then it is given by;

$$\begin{aligned} R_3(t) &= P(T<Y<Z) \\ &= \int_0^{\infty} H_T(y)\,dF_Y(y) - \int_0^{\infty} H_T(y)G_Z(y)\,dF_Y(y) \\ &= \int_0^{\infty}\left[(1-(1+y)^{-\alpha})^{\beta_1} - (1-(1+y)^{-\alpha})^{\beta_1+\beta_3}\right] f(y)dy \\ &= \int_0^{\infty} \alpha\beta_2(1+y)^{-(\alpha+1)}(1-(1+y)^{-\alpha})^{\beta_1+\beta_2-1}dy \\ &\quad - \int_0^{\infty} \alpha\beta_2(1+y)^{-(\alpha+1)}(1-(1+y)^{-\alpha})^{\beta_1+\beta_2+\beta_3-1}dy, \end{aligned}$$

Hence

$$R_3(t) = \frac{\beta_2}{\beta_1+\beta_2} - \frac{\beta_2}{\beta_1+\beta_2+\beta_3} = \frac{\beta_2\beta_3}{(\beta_1+\beta_2)(\beta_1+\beta_2+\beta_3)}\,. \tag{6}$$

Hazard and Reversed Hazard Rate Functions

The hrf denoted by $h_1(t)$ and rhrf denoted by $h_2(t)$ are given, respectively, by

$$h_1(t) = \frac{f(t)}{R(t)} = \frac{\alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{\beta-1}}{1-(1-(1+t)^{-\alpha})^{\beta}}\, , t > 0\, ;\ \alpha, \beta > 0\, , \tag{7}$$

and

$$h_2(t) = \frac{f(t)}{F(t)} = \alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{-1}\, , t > 0;\ \alpha, \beta > 0. \tag{8}$$

Plots of pdf, hrf and rhrf of IKum are given, respectively in Figures 1-3. The plots, in Figures 1 and 2, indicate that the curves of the pdf and hrf are monotone decreasing at the values $\alpha = 2.36$ and $\beta = 0.5$, the curves of the pdf and hrf are increasing and then decreasing at the values $\alpha = 5.18$ and $\beta = 5.18$ and when $\alpha = 0.77$ and $\beta = 0.39$ the curves of the pdf and hrf are right skewed. From Figure 3 one can observe that the curves of the rhrf at all the values are decreasing and then constant.

c. The Mode of the Inverted Kumaraswamy Distribution

The mode of the IKum distribution is given by

$$\text{Mode} = t_{mode} = \left[\left(\frac{\alpha+1}{\alpha\beta+1}\right)^{-\frac{1}{\alpha}} - 1\right]. \tag{9}$$

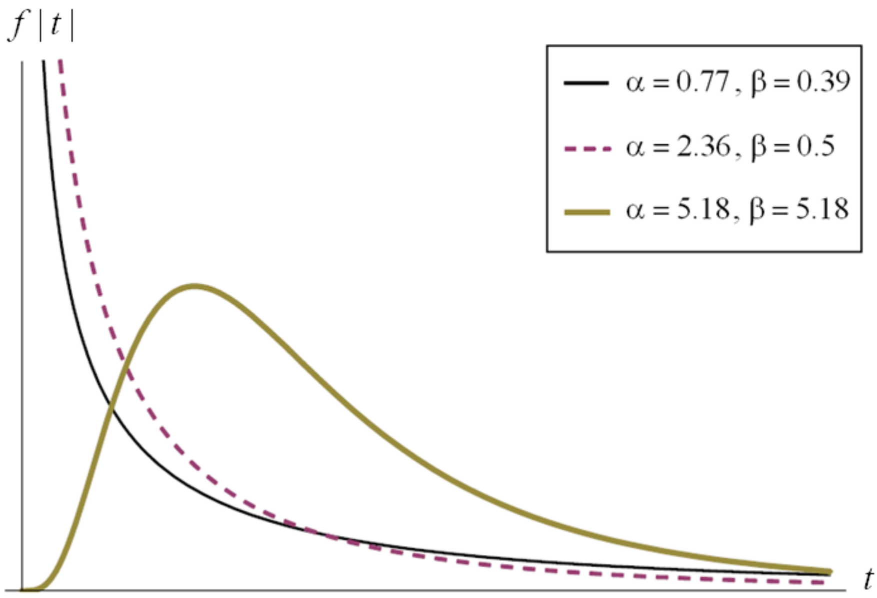

Figure 1: The Plots of the Probability Density Function

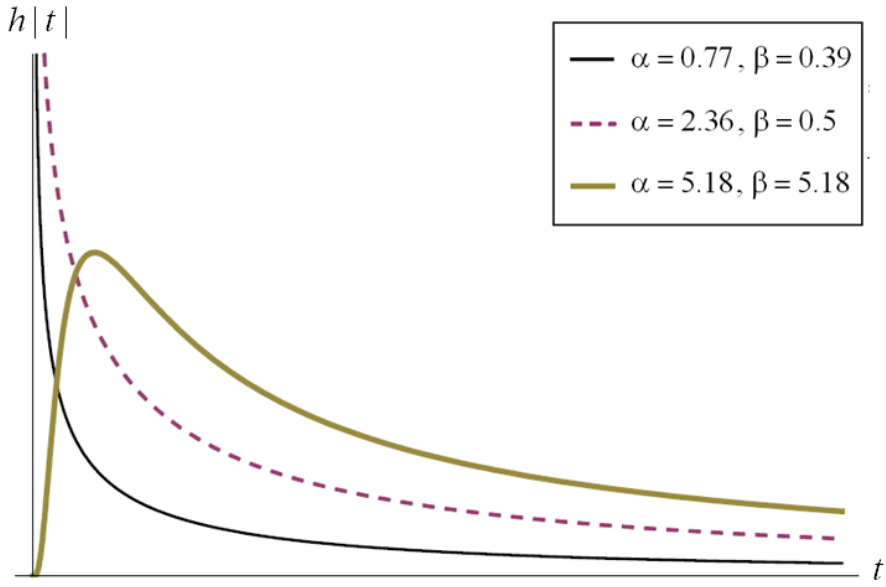

Figure 2: The Plots of the Hazard Rate Function

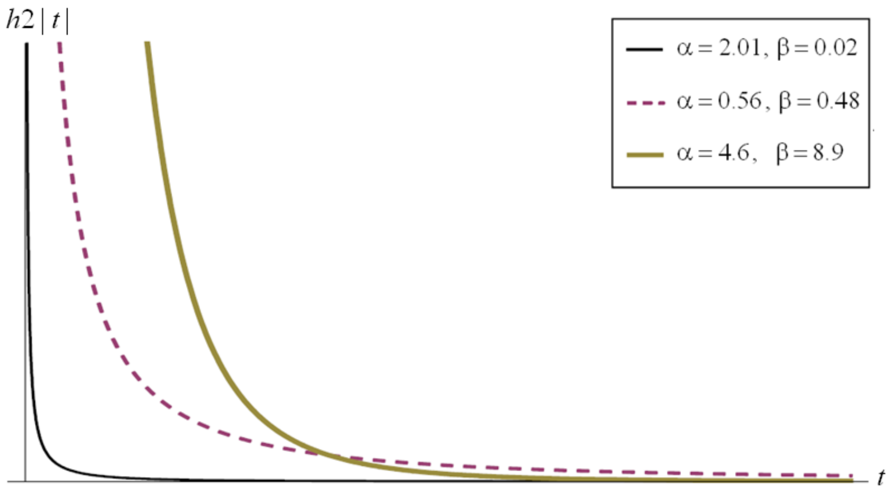

Figure 3: The Plots of the Reversed Hazard Rate Function

d. Quantiles of the Inverted Kumaraswamy Distribution

The quantile function of the IKum is given by;

$$t_q = \left[\left(1-(q)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1\right], 0 < q < 1. \tag{10}$$

Special cases can be obtained using (10) such as the second quartile (median), when $q = 0.5$,

$$\text{Median} = t_{median} = \left[\left(1-(0.5)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1\right], \tag{11}$$

and the inter-quantile range can be expressed as

$$IQR(T) = \left(1-\left(\frac{3}{4}\right)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - \left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}. \tag{12}$$

e. The Central and Non-Central Moments

The r^{th} non-central moment of the IKum (α, β) distribution is given by

$$\acute{\mu}_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \beta \, \mathrm{B}\left(1-\frac{j}{\alpha}, \beta\right), r = 1,2,\dots, \alpha > j, j = 0,\dots, r, \tag{13}$$

where B (., .) is the beta function.

Thus, the mean and the variance of IKum are given by:

$$\mu = \beta \mathrm{B}\left(1-\frac{1}{\alpha}, \beta\right) - 1, \alpha > 1, \tag{14}$$

and

$$\mu_2 = \beta \mathrm{B}\left(1-\frac{2}{\alpha}, \beta\right) - \left[\beta \mathrm{B}\left(1-\frac{1}{\alpha}, \beta\right)\right]^2, \alpha > 2. \tag{15}$$

It is noticed that the r^{th} moment depends on beta function, therefore, it is suitable to find the approximate mean and the approximate variance.

f. The Approximate Mean and Variance

If W~Exponential (β), with $\mu = \mathrm{E(W)} = \frac{1}{\beta}$ and $\sigma^2 = \mathrm{V(W)} = \frac{1}{\beta^2}$ then the random variable $T = g(w) = \left[(1-e^{-w})^{-\frac{1}{\alpha}} - 1\right]$ ~IKum(α, β), [See Appendix B]. Using this relation and applying Taylor series to obtain the approximate formula for the mean and the variance of T, one obtains:

$$E(T) \approx g(\mu) + \frac{1}{2}\sigma^2 g''(\mu), \tag{16}$$

and

$$V(T) \approx \sigma^2 (g'(\mu))^2, \tag{17}$$

where

$$g(\mu) = \left[(1-e^{-\mu})^{-\frac{1}{\alpha}} - 1\right], \tag{18}$$

$$g'(\mu) = -\frac{1}{\alpha}(1-e^{-\mu})^{-(\frac{1}{\alpha}+1)}e^{-\mu}, \tag{19}$$

and

$$g''(\mu) = \frac{1}{\alpha^2}e^{-2\mu}(1-e^{-\mu})^{-(\frac{1}{\alpha}+2)}[\alpha e^{\mu}+1], \tag{20}$$

(For more details, see Casella and Berger (2002)).

If $\mu = \frac{1}{\beta}$ and $\sigma^2 = \frac{1}{\beta^2}$ and substituting (18), (19) and (20) in (16) and (17) it can be shown that the approximate mean and variance of T are

$$E(T) \approx \frac{1}{2\alpha^2\beta^2}\left[e^{-\frac{2}{\beta}}(1-e^{-\frac{1}{\beta}})^{-(\frac{1}{\alpha}+2)}(\alpha e^{\frac{1}{\beta}}+1)\right] + [(1-e^{-\frac{1}{\beta}})^{-\frac{1}{\alpha}} - 1] \,, \tag{21}$$

and

$$V(T) \approx \frac{1}{\alpha^2\beta^2}e^{-\frac{2}{\beta}}(1-e^{-\frac{1}{\beta}})^{-2(\frac{1}{\alpha}+1)}. \tag{22}$$

g. Order Statistics

The i^{th} order statistic, $t_{(i)}$ of a random sample of size n from the IKum (α,β) distribution has the following density

$$g\big(t_{(i)}\big) = \frac{n!}{(i-1)!(n-i)!}\alpha\beta(1+t_{(i)})^{-(\alpha+1)}\Big(1-\big(1+t_{(i)}\big)^{-\alpha}\Big)^{\beta(i-1)} * \Big(1-\big(1-(1+t_{(i)})^{-\alpha}\big)^{\beta}\Big)^{(n-i)}, t_{(i)} > 0 \tag{23}$$

Special Cases

i) When $i = 1$, one obtains the pdf of the first order statistic

$$g\big(t_{(1)}\big) = n\alpha\beta(1+t_{(1)})^{-(\alpha+1)}\big[1-(1-(1+t_{(1)})^{-\alpha})^{\beta}\big]^{(n-1)}, t_{(1)} > 0. \tag{24}$$

ii) When $i = n$, the pdf of the largest order statistic can be obtained

$$g\big(t_{(n)}\big) = n\alpha\beta(1+t_{(n)})^{-(\alpha+1)}\big(1-(1+t_{(n)})^{-\alpha}\big)^{\beta(n-1)}, t_{(n)} > 0\,. \tag{25}$$

iii) The median distribution of a random sample of size n from the IKum (α, β) distribution (when n is odd) has the density

$$g\left(t_{\left(\frac{n+1}{2}\right)}\right) = \frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\alpha\beta(1+t_{\left(\frac{n+1}{2}\right)})^{-(\alpha+1)}\left(1-(1+t_{\left(\frac{n+1}{2}\right)})^{-\alpha}\right)^{\beta\left(\frac{n-1}{2}\right)} * \left(1-\left(1-(1+t_{\left(\frac{n+1}{2}\right)})^{-\alpha}\right)^{\beta}\right)^{\left(\frac{n-1}{2}\right)}, \; t_{\left(\frac{n+1}{2}\right)} > 0. \tag{26}$$

3. RELATED DISTRIBUTIONS

This section discussed some sub-models of IKum distribution, some relations of the IKum distribution to other distributions and limiting distributions.

3.1 Some sub-models

i) **Lomax (Pareto Type II) distribution**
The Lomax distribution is a special case from IKum distribution, when $\beta = 1$ in (2) with the following pdf

$$f(t;\alpha) = \frac{\alpha}{(1+t)^{\alpha+1}}\ , t > 0, \alpha > 0. \tag{27}$$

ii) **Beta Type II (inverted beta) distribution**
The inverted beta Type II $(\beta, 1)$ is a special case from IKum distribution, when $\alpha = 1$ in (2) as follows

$$f(t;\beta) = \frac{1}{B(\beta,1)} t^{\beta-1}(1+t)^{-(\beta+1)}\ , t > 0\ , \beta > 0. \tag{28}$$

Also, when $Y1 = k(1-(1+T)^{-1})^{-1}$ then $Y1 \sim$ Pareto TypeI(β, k),

and, when $Y2 = \ln((1-T)^{-1})^{-1}$ then $Y2 \sim$ exponential (β).

iii) **The log-logistic (Fisk) distribution**
The log-logistic (Fisk) distribution is a special case from IKum distribution, when $\alpha = \beta = 1$ in (2), with the following form

$$f(t) = \frac{1}{(1+t)^2}\ , 0 < t < \infty. \tag{29}$$

3.2 Some Relations between the Inverted Kumaraswamy Distribution and other Distributions

The IKum distribution can be transformed to several distributions using appropriate transformations such as exponentiated Weibull (exponentiated exponential, Weibull, Burr Type X, exponential and Rayleigh), generalized uniform (beta Type I, inverted generalized Pareto Type I and uniform (0, 1)), left truncated exponentiated exponential (left truncated exponential and exponential), exponentiated Burr Type XII (Burr Type XII, generalized Lomax, beta Type II and F- distribution), Kum-Dagum (Dagum, Kum-Burr Type III, Burr Type III and log logistic) and Kum-inverse Weibull (Kum-inverse exponential and inverse exponential). Table 1 summarizes the transformations from IKum distribution to other distributions [See Appendix A].

3.3 Limiting Distributions

i) If $T \sim$IKum (α,β) and $Y_1 = \beta^{-\frac{1}{\alpha}}(1+T)$ on $\left(\beta^{-\frac{1}{\alpha}}, \infty\right)$, then the pdf of y_1 is $f(y_1;\alpha,\beta) = \alpha {y_1}^{-(\alpha+1)}\left(1-\frac{{y_1}^{-\alpha}}{\beta}\right)^{\beta-1}$. As $\beta \to \infty$, the pdf of y_1 tends to $f(y_1;\alpha) = \alpha {y_1}^{-(\alpha+1)} e^{-{y_1}^{-\alpha}}, y_1 > 0$, which is the pdf of the inverted Weibull distribution.

ii) If $T \sim$IKum (α,β) and $Y_2 = \alpha(1-(1+T)^{-1})$ on $(0,\alpha)$, then the pdf of y_2 is $f(y_2;\alpha,\beta) = \beta\left(1-\frac{y_2}{\alpha}\right)^{\alpha-1}\left(1-\left(1-\frac{y_2}{\alpha}\right)^{\alpha}\right)^{\beta-1}$. As $\alpha \to \infty$ the pdf of y_2 tends to $f(y_2;\beta) = \beta e^{-y_2}(1-e^{-y_2})^{\beta-1}, y_2 > 0$, which is the pdf of the generalized exponential distribution.

iii) If $T \sim \text{IKum}(\alpha, \beta)$ and $Y_3 = \alpha\left(1 - \beta^{\frac{1}{\alpha}}(1+T)^{-1}\right)$ on $\left(\alpha\left(1-\beta^{\frac{1}{\alpha}}\right), \alpha\right)$, then the pdf of y_3 is $f(y_3; \alpha, \beta) = \left(1 - \frac{y_3}{\alpha}\right)^{\alpha-1}\left(1 - \frac{\left(1-\frac{y_3}{\alpha}\right)^{\alpha}}{\beta}\right)^{\beta-1}$. As both $\beta \to \infty$ and $\alpha \to \infty$, the pdf of y_3 tends to $f(y_3) = e^{-y_3} exp[-e^{-y_3}]$, $y_3 > 0$, which is the pdf of the standard extreme value distribution of the first Type. [For proofs, See Appendix B].

4. ESTIMATION BASED ON TYPE II CENSORED DATA

This section develops the estimation of the shape parameters, rf and hrf based on Type II censored samples using maximum likelihood and Bayesian approaches. Also the confidence intervals of the shape parameters, rf and hrf are obtained.

4.1 Maximum Likelihood Estimation

In this subsection, the *maximum likelihood* (ML) estimation is used to estimate the parameters, rf and hrf of IKum based on Type II censored samples.

Suppose that $(T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(r)})$ is a censored sample of size r obtained from a life test on n items, the likelihood function based on Type II censored sample is given by

$$L(\alpha, \beta; \underline{t}) \propto [\textstyle\prod_{i=1}^{r} f(t_i, \alpha, \beta)][R(t_r, \alpha, \beta)]^{(n-r)}$$

$$\propto \alpha^r \beta^r \textstyle\prod_{i=1}^{r}\left(1 + t_{(i)}\right)^{-(\alpha+1)}$$

$$* \textstyle\prod_{i=1}^{r}\left(1 - \left(1 + t_{(i)}\right)^{-\alpha}\right)^{\beta-1}\left[1 - \left(1 - \left(1 + t_{(r)}\right)^{-\alpha}\right)^{\beta}\right]^{n-r}. \quad (30)$$

The natural logarithm of the likelihood function is given by

$$\begin{aligned} \ell &= \ln L(\alpha, \beta; \underline{t}) \\ &= r \ln \alpha + r \ln \beta - (\alpha+1) \textstyle\sum_{i=1}^{r} \ln\left(1 + t_{(i)}\right) \\ &\quad +(\beta - 1) \textstyle\sum_{i=1}^{r} \ln\left(1 - \left(1 + t_{(i)}\right)^{-\alpha}\right) \\ &\quad +(n-r) \ln\left[1 - \left(1 - (1 + t_{(r)})^{-\alpha}\right)^{\beta}\right]. \end{aligned} \quad (31)$$

Considering the two parameters α and β are unknown and differentiating the log likelihood function in (31) with respect to α and β as follows:

$$\frac{\partial \ell}{\partial \beta} = \frac{r}{\beta} + \textstyle\sum_{i=1}^{r} \ln(1 - (1 + t_{(i)})^{-\alpha}) - \frac{(n-r)\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta} \ln\left[\left(1-(1+t_{(r)})^{-\alpha}\right)\right]}{1-\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}}, \quad (32)$$

and

$$\frac{\partial \ell}{\partial \alpha} = \frac{r}{\alpha} - \textstyle\sum_{i=1}^{r} \ln\left(1 + t_{(i)}\right) + (\beta - 1) \textstyle\sum_{i=1}^{r} \frac{(1+t_{(i)})^{-\alpha} \ln(1+t_{(i)})}{\left(1-(1+t_{(i)})^{-\alpha}\right)}$$

$$- \frac{(n-r)\beta\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta-1}(1+t_{(r)})^{-\alpha} \ln(1+t_{(r)})}{1-\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}}. \quad (33)$$

Equating the two nonlinear Equations (32) and (33) to zero, and solving numerically one obtains the *maximum likelihood estimates* (MLEs) of α and β.

The MLEs of the $R_1(t)$, $R_2(t)$, $R_3(t)$, $h_1(t)$ and $h_2(t)$ can be obtained, using the invariance property of ML estimation, by replacing the parameters α and β in (4) - (8) by their ML estimators. The ML estimators of $R_1(t)$, $R_2(t)$, $R_3(t)$, $h_1(t)$ and $h_2(t)$ are given, respectively by

$$\hat{R}_1(t) = 1-\left(1-(1+t)^{-\hat{\alpha}}\right)^{\hat{\beta}}, t>0, \tag{34}$$

$$\hat{R}_2(t) = \frac{\hat{\beta}_2}{\hat{\beta}_1+\hat{\beta}_2}, \tag{35}$$

$$\hat{R}_3(t) = \frac{\hat{\beta}_2\hat{\beta}_3}{(\hat{\beta}_1+\hat{\beta}_2)(\hat{\beta}_1+\hat{\beta}_2+\hat{\beta}_3)}, \tag{36}$$

$$\hat{h}_1(t) = \frac{\hat{\alpha}\hat{\beta}t^{-(\hat{\alpha}+1)}\left(1-(1+t)^{-\hat{\alpha}}\right)^{\hat{\beta}-1}}{1-\left(1-(1+t)^{-\hat{\alpha}}\right)^{\hat{\beta}}}, t>0, \tag{37}$$

and

$$\hat{h}_2(t) = \hat{\alpha}\hat{\beta}(1+t)^{-(\hat{\alpha}+1)}\left(1-(1+t)^{-\hat{\alpha}}\right)^{-1}, t>0. \tag{38}$$

Asymptotic Variance–Covariance Matrix of Maximum Likelihood Estimators

The asymptotic variance-covariance matrix of the ML estimators for the two parameters α and β is the inverse of the observed Fisher information matrix as follows

$$\tilde{I}^{-1} \approx \begin{bmatrix} \widetilde{var}(\hat{\alpha}) & \widetilde{cov}(\hat{\alpha},\hat{\beta}) \\ \widetilde{cov}(\hat{\alpha},\hat{\beta}) & \widetilde{var}(\hat{\beta}) \end{bmatrix} \approx \frac{1}{|I|}\left.\begin{bmatrix} -\frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} \\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\alpha^2} \end{bmatrix}\right|_{\hat{\alpha},\hat{\beta}},$$

with

$$\frac{\partial^2\ell}{\partial\beta^2} = \frac{-r}{\beta^2} - \frac{(n-r)\left[1-(1+t_{(r)})^{-\alpha}\right]^{2\beta}\ln\left[1-(1+t_{(r)})^{-\alpha}\right]^2}{\left[\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}-1\right]^2} + \frac{(n-r)\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}\ln\left[1-(1+t_{(r)})^{-\alpha}\right]^2}{\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}-1}, \tag{39}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\alpha^2} = \frac{-r}{\alpha^2} - (\beta-1)\sum_{i=1}^{r}\left[\frac{(1+t_{(i)})^{-2\alpha}ln(1+t_{(i)})^2}{\left((1+t_{(i)})^{-\alpha}-1\right)^2} - \frac{(1+t_{(i)})^{-\alpha}\ln(1+t_{(i)})^2}{(1+t_{(i)})^{-\alpha}-1}\right] \\ + \frac{\beta(n-r)(\beta-1)(1+t_{(r)})^{-2\alpha}\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta-2}\ln(1+t_{(r)})^2}{\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}-1} \\ - \frac{\beta^2(n-r)(1+t_{(r)})^{-2\alpha}\left(1-(1+t_{(r)})^{-\alpha}\right)^{2\beta-2}\ln(1+t_{(r)})^2}{\left[\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}-1\right]^2} \\ - \frac{\beta(n-r)(1+t_{(r)})^{-\alpha}\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta-1}\ln(1+t_{(r)})^2}{\left(1-(1+t_{(r)})^{-\alpha}\right)^{\beta}-1},\end{aligned} \tag{40}$$

and

$$\frac{\partial^2 \ell}{\partial\beta\partial\alpha} = \frac{(n-r)\beta\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta-1}\ln\left(1+t_{(r)}\right)\ln\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)}{\left(1+t_{(r)}\right)^{\alpha}\left[\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta}-1\right]}$$
$$-\sum_{i=1}^{r}\frac{\left(1+t_{(i)}\right)^{-\alpha}\ln\left(1+t_{(i)}\right)}{\left[\left(1+t_{(i)}\right)^{-\alpha}-1\right]} - \frac{(n-r)\left(1+t_{(r)}\right)^{-\alpha}\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta}\ln\left(1+t_{(r)}\right)}{\left(\left(1+t_{(r)}\right)^{-\alpha}-1\right)\left[\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta}-1\right]}$$
$$-\frac{(n-r)\beta\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta-1}\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta}\ln\left(1+t_{(r)}\right)\ln\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)}{\left(1+t_{(r)}\right)^{\alpha}\left[\left(1-\left(1+t_{(r)}\right)^{-\alpha}\right)^{\beta}-1\right]^{2}}. \quad (41)$$

The asymptotic normality of ML estimation can be used to compute the asymptotic 100(1- ω)% confidence intervals for α and β as follows

$$\hat{\alpha} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\widetilde{var}(\hat{\alpha})} \text{ and } \hat{\beta} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\widetilde{var}(\hat{\beta})}. \quad (42)$$

Also, the asymptotic 100(1- ω)% confidence intervals for rf and hrf are given by

$$\hat{R}_1(t) \pm Z_{(1-\frac{\omega}{2})}\sqrt{\widetilde{var}(\hat{R}_1(t))} \text{ and } \hat{h}_1(t) \pm Z_{(1-\frac{\omega}{2})}\sqrt{\widetilde{var}\left(\hat{h}_1(t)\right)}, \quad (43)$$

where $Z_{\left(1-\frac{\omega}{2}\right)}$ is standard normal and $(1-\omega)$ is the confidence coefficient.

4.2 Bayesian estimation

In this subsection, the Bayesian approach is considered, under *squared error* (SE) loss function to estimate the parameters, rf and hrf of the IKum distribution based on Type II censored samples, using the non-informative prior. Also credible intervals for the parameters, rf and hrf are obtained.

Assuming that both of the parameters α and β are unknown and independent distributions, the joint non-informative prior on α and β is

$$\pi(\alpha,\beta) \propto \frac{1}{\alpha\beta}, \ \alpha,\beta > 0. \quad (44)$$

The joint posterior distribution of α and β can be obtained using (30) and (44) as follows

$$\pi(\alpha,\beta|\underline{t}) \propto L(\alpha,\beta|\underline{t})\,\pi(\alpha,\beta) \quad (45)$$

$$\propto \alpha^{r-1}\beta^{r-1}e^{-\alpha\sum_{i=1}^{r}\ln(1+t_{(i)})}e^{(\beta-1)\sum_{i=1}^{r}\ln(u_i)}e^{(n-r)\ln(1-u^{\beta})} \quad (46)$$

$$= \rho\alpha^{r-1}\beta^{r-1}e^{-\alpha\sum_{i=1}^{r}\ln(1+t_{(i)})}e^{(\beta-1)\sum_{i=1}^{r}\ln(u_i)}e^{(n-r)\ln(1-u^{\beta})}, \quad (47)$$

where

$$u = \left(1-\left(1+t_{(r)}\right)^{-\alpha}\right) \ and \ u_i = \left(1-\left(1+t_{(i)}\right)^{-\alpha}\right),$$

and

$$\rho^{-1} = \int_0^{\infty}\int_0^{\infty}\alpha^{r-1}\beta^{r-1}e^{-\alpha\sum_{i=1}^{r}\ln(1+t_{(i)})}e^{(\beta-1)\sum_{i=1}^{r}\ln(u_i)}e^{(n-r)\ln(1-u^{\beta})}d\alpha d\beta,$$

which is a normalizing constant.

Considering the SE loss function as a symmetric loss function, then the Bayes estimators of the parameters, based on Type II censoring, is the mean of the posterior density and using the joint non-informative prior, one can obtain the Bayes estimators for the parameters, which are given by their marginal posterior expectations using (47) as follows

$$\alpha^* = E\left(\alpha\middle|\underline{t}\right)$$
$$= \int_0^\infty \int_0^\infty \rho\alpha^r \beta^{r-1} e^{-\alpha\sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1)\sum_{i=1}^r \ln(u_i)} e^{(n-r)\ln\left(1-u^\beta\right)}\, d\beta d\alpha\,, \qquad (48)$$

and

$$\beta^* = E\left(\beta\middle|\underline{t}\right)$$
$$= \int_0^\infty \int_0^\infty \rho\alpha^{r-1} \beta^{r} e^{-\alpha\sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1)\sum_{i=1}^r \ln(u_i)} e^{(n-r)\ln\left(1-u^\beta\right)} d\alpha d\beta\,, \qquad (49)$$

Also, the Bayes estimators of the rf and hrf under SE loss function can be obtained using (4), (7) and (47) as follows

$$R_1^*(t) = E\left(R_1(t)\middle|\underline{t}\right) = 1 - \int_0^\infty \int_0^\infty (1-(1+t)^{-\alpha})^\beta\, \rho\alpha^{r-1}\beta^{r-1}$$
$$* \, e^{-\alpha\sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1)\sum_{i=1}^r \ln(u_i)} e^{(n-r)\ln\left(1-u^\beta\right)} d\alpha\, d\beta, \qquad (50)$$

and

$$h_1^*(t) = E\left(h_1(t)\middle|\underline{t}\right) = \int_0^\infty \int_0^\infty \frac{\alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{\beta-1}}{1-(1-(1+t)^{-\alpha})^\beta}$$
$$* \, \rho\alpha^{r-1}\beta^{r-1} e^{-\alpha\sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1)\sum_{i=1}^r \ln(u_i)} e^{(n-r)\ln\left(1-u^\beta\right)} d\alpha\, d\beta. \qquad (51)$$

To obtain the Bayes estimates of the parameters, rf and hrf, the Equations (48)-(51) should be solved numerically.

The credible intervals for the Bayes estimators of α and β can be obtained from IKum distribution based on Type II censored data. In general, $L(\underline{t})$, $U(\underline{t})$ is a 100(1-ω) % credible interval for θ if

$$P\left[L(\underline{t}) < \theta < U(\underline{t})\right] = \int_{L(\underline{t})}^{U(\underline{t})} \pi(\theta|\underline{t})\, d\theta = 1-\omega\,,$$

where $L(\underline{t})$ and $U(\underline{t})$ are the *lower limit* (LL) and *upper limit* (UL).

Since, the posterior distribution is given by (47), then a 100 (1- ω) % credible interval for α is $\left(L_1(\underline{t}), U_1(\underline{t})\right)$,

where

$$P\left[\alpha > L_1(\underline{t})\middle|\underline{t}\right],= \int_{L_1(\underline{t})}^\infty \int_0^\infty \rho\alpha^{r-1}\beta^{r-1} e^{-\alpha\sum_{i=1}^r \ln\left(1+t_{(i)}\right)}$$
$$* \, e^{(\beta-1)\sum_{i=1}^r \ln(u_i)} e^{(n-r)\ln\left(1-u^\beta\right)} d\beta d\alpha = 1-\frac{\omega}{2}\,, \qquad (52)$$

and

$$P\left[\alpha > U_1(\underline{t})\middle|\underline{t}\right] = \int_{U_1(\underline{t})}^\infty \int_0^\infty \rho\alpha^{r-1}\beta^{r-1} e^{-\alpha\sum_{i=1}^r \ln\left(1+t_{(i)}\right)}$$
$$* \, e^{(\beta-1)\sum_{i=1}^r \ln(u_i)} e^{(n-r)\ln\left(1-u^\beta\right)} d\beta d\alpha = \frac{\omega}{2}. \qquad (53)$$

Also, a 100 (1- ω) % credible interval for parameter β is $\left(L_2(\underline{t}), U_2(\underline{t})\right)$

where

$$P[\,\beta > L_2(\underline{t})|\underline{t}] = \int_{L_2(\underline{t})}^{\infty}\int_0^{\infty} \rho\alpha^{r-1}\beta^{r-1}e^{-\alpha\sum_{i=1}^{r}\ln(1+t_{(i)})} * e^{(\beta-1)\sum_{i=1}^{r}\ln(u_i)}e^{(n-r)\ln(1-u^{\beta})}d\alpha d\beta = 1-\frac{\omega}{2}, \quad (54)$$

and

$$P[\,\beta > U_2(\underline{t})|\underline{t}] = \int_{U_2(\underline{t})}^{\infty}\int_0^{\infty} \rho\alpha^{r-1}\beta^{r-1}e^{-\alpha\sum_{i=1}^{r}\ln(1+t_{(i)})} * e^{(\beta-1)\sum_{i=1}^{r}\ln(u_i)}e^{(n-r)\ln(1-u^{\beta})}d\alpha d\beta = \frac{\omega}{2}. \quad (55)$$

5. NUMERICAL ILLUSTRATION

5.1 Simulation Study

a. In this section, a simulation study is presented to illustrate the application of the various theoretical results developed in the previous section on basis of generated data from IKum (α,β) distribution, for different sample sizes (n=20, 30, 60 and 100) using number of replications NR=10000. The computations are performed using Mathematica 9.

b. The *relative absolute biases* (RABs), *relative mean square errors* (RMSEs), variances and *estimated risks* (ERs) of ML and Bayes estimates of the shape parameters, rf and hrf are computed as follows:
   1) RABs (estimator) = $\frac{|\text{bias (estimator)}|}{\text{true value}}$,
   2) Relative mean square errors (RMSEs) = $\frac{\text{MSE(estimator)}}{\text{true value}}$,
   3) Variances (estimator) = $\text{MSE(estimator)} - \text{bias}^2\text{(estimator)}$,
   4) Estimated risk (estimator)= $\frac{\sum_{i=1}^{NR}(estimator-true\ value)^2}{\text{NR}}$.

c. Table 2 displays the (RABs), (RMSEs) and variances of MLEs and 95% *confidence intervals* (CIs) where the population parameter values are $\alpha = 0.4$, $\beta = 0.8$ based on three levels of Type II censoring 70%, 90% and 100%. Table 3 displays the same computational results but for different population parameter values $\alpha = 1,\ \beta = 0.8$ and for $\alpha = 0.4,\ \beta = 1$ in Table 4.

d. Table 5 presents RABs, RMSEs of the MLEs and 95% CIs of the rf and hrf at different mission time t_0 where ($t_0 = 0.3$, 0.6 and 1) from IKum distribution for different sample sizes where (n=20, 60 and 100) and level of censoring 90% and repetitions NR = 10000.

e. Table 6 shows the Bayes averages of the parameters, rf and hrf and their ERs, *standard deviations* (Sds) and CIs based on Type II censoring using the joint non-informative prior. The computations are performed using *Markov Chain Monte Carlo* (MCMC) method.

Concluding Remarks

i) From Tables 2, 3 and 4, one can observe that the RABs, variances and RMSEs of the MLEs of the shape parameters α and β decrease when the sample size n increases. Also, it is observed that as the level of censoring decreases the RABs, variances and RMSEs of the estimates decrease. The lengths of the CI becomes narrower as the sample size increases.

ii) Table 5 indicates that the rf decreases and the hrf increases when the mission time t_0 increases, also the RABs and RMSEs of the MLEs of the rf and the hrf decrease when the sample size increases.

iii) It is clear from Table 6 that the MSEs and Sds of the Bayes averages of the parameters, rf and hrf performs better and the lengths of the CIs get shorter when the sample size increases.

iv) These results are expected since decreasing the level of censoring means that more information is provided by the sample and hence increase the accuracy of the estimates.

v) In general, when $r = n$, all the results obtained for Type II censored sample reduce to those of the complete sample.

5.2 Applications

In this subsection, three applications of real data sets are provided to illustrate the importance of the IKum distribution. To check the validity of the fitted model, Kolmogorov- Smirnov goodness of fit test is performed for each data set and the p values in each case indicates that the model fits the data very well.

Table 7 shows ML averages of the parameters, rf, hrf and their ERs, for the real data based on Type II censoring.

Table 8 displays the Bayes averages of the parameters and their ERs based on Type II censoring using the joint non-informative prior.

i) The first application is a real data set given by Hinkley (1977). It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data is 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

ii) The second application is given by Murthy *et al.* (2004). The data refers to the time between failures for repairable items. The data is 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

iii) The third application is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used by Bhaumik *et al.* (2009). The data is 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

Concluding Remark

The Bayes estimates of the parameters, rf and hrf based on Type II censoring, under non-informative prior and SE loss function, have ERs smaller than the corresponding ERs of the ML estimates.

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APPENDIX A

Table 1
Summary of Some Transformations Applied to the Inverted Kumaraswamy and the Resulting Distributions

Transformation	The resulting Distribution	Pdf
$[\ln(1+T)]^{\frac{1}{\theta}}$ **Special cases:**	Exponentiated Weibull (α,β,θ)	$f(y_1)=\alpha\beta\theta {y_1}^{\theta-1} * e^{-\alpha {y_1}^{\theta}}\left(1-e^{-\alpha {y_1}^{\theta}}\right)^{\beta-1}$, $0<y_1<\infty$, $\alpha,\beta,\theta>0.$
i) $\theta=1$	Exponentiated exponential (α,β)	
ii) $\beta=1$	Weibull (α,θ)	
iii) $\theta=2$	Burr Type X (generalized Rayleigh) (α,β)	
iv) $\beta=\theta=1$	Exponential (α)	
v) $\beta=1$ and $\theta=2$	Rayleigh (α)	
$w^{-\frac{1}{\alpha}}(1+T)^{-1}+c$ **Special cases:**	Generalized uniform (α,β,w,c)	$f(y_2)=\dfrac{\alpha\beta w(y_2-c)^{\alpha-1}}{(1-w(y_2-c)^{\alpha})^{-\beta+1}}$, $c<y_2<c+w^{-\frac{1}{\alpha}}$, $\alpha,\beta,w,c>0.$
i) $\alpha=w=1$ and c = 0	Beta Type I (1, β)	
ii) c =0	Inverted generalized Pareto Type I (α,β,w)	
iii) $\alpha=\beta=w=1$	Uniform (0,1)	
$\ln(1+T)^{\theta}+b$ **Special cases:**	Left truncated exponentiated exponential (α,β,θ,b)	$f(y_3)=\dfrac{\alpha\beta}{\theta}e^{-\alpha\left(\frac{y_3-b}{\theta}\right)} * \left(1-e^{-\alpha\left(\frac{y_3-b}{\theta}\right)}\right)^{\beta-1}$, $b<y_3<\infty.$
i) $\beta=1$	Left truncated exponential (α,θ,b)	
ii) $\beta=1$ and b = 0	Exponential (α,θ)	

Table 1 (Continued)

Transformation	The resulting Distribution	Pdf
$sT^{\frac{1}{c}}$ **Special cases:**	Exponentiated Burr Type XII (α,β,s,c)	$f(y_4)=\frac{\alpha\beta c}{s^c}y_4^{c+1}\left(1+\left(\frac{y_4}{s}\right)^c\right)^{-(\alpha+1)} * \left(1-(1+\left(\frac{y_4}{s}\right)^c)^{-\alpha}\right)^{\beta-1}$, $y_4>0;\ \alpha,\beta,s,c>0.$
i) $\beta=1$	Burr Type XII (α,s,c)	
ii) $\beta=c=1$	Generalized Lomax (Pareto Type II) (α,s)	
iii) $\beta=c=1$ and s = 1	Beta Type II $(1,\alpha)$	
iv) $\beta=c=1$ and $s=\alpha$	F-distribution $(2,2\alpha)$	
$\left(\frac{T}{\lambda}\right)^{-\frac{1}{s}}$ **Special cases:**	Kumaraswamy-Dagum (α,β,λ,s)	$f(y_5)=\frac{\alpha\beta\lambda s}{y_5^{s+1}}(1+\lambda y_5{}^{-s})^{-(\alpha+1)} * (1-(1+\lambda y_5{}^{-s})^{-\alpha})^{\beta-1}$, $y_5>0;\ \alpha,\beta,\lambda,s>0.$
i) $\beta=1$	Dagum (α,λ,s)	
ii) $\lambda=1$	Kumaraswamy-Burr Type III (α,β,s)	
iii) $\beta=\lambda=1$	Burr Type III (α,s)	
iv) $\alpha=1$	Kumaraswamy-Fisk (log logistic) (λ,β,s)	
$\left(\frac{\ln(1+T)}{\theta}\right)^{-\frac{1}{b}}$ **Special cases:**	Kumaraswamy-inverse Weibull (α,β,θ,b)	$f(y_6)=\frac{\alpha\beta\theta b}{y_6^{b+1}}e^{-\frac{\alpha\theta}{y_6{}^b}}\left(1-e^{-\frac{\alpha\theta}{y_6{}^b}}\right)^{\beta-1}$, $y_6>0$.
i) b = 1	Kumaraswamy-inverse exponential (α,β,θ)	
ii) b= $\beta=1$	Inverse exponential (α,θ)	

Table 2
Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the shape parameters α and β from IKum for different sample size n, censoring size r and replications NR = 10000 ($\alpha = 0.4$, $\beta = 0.8$)

n	r	Estimator	RAB	RMSE	Variance	LL	UL	Length
30	21	$\hat{\alpha}$	0.0019	0.0323	0.0327	0.1204	0.8296	0.7092
		$\hat{\beta}$	0.0044	0.0799	0.0818	0.3545	1.4755	1.1209
	27	$\hat{\alpha}$	0.0007	0.0172	0.0182	0.1818	0.7119	0.5301
		$\hat{\beta}$	0.0025	0.0585	0.0619	0.3986	1.3745	0.9759
	30	$\hat{\alpha}$	0.0004	0.0127	0.0138	0.2044	0.6656	0.4613
		$\hat{\beta}$	0.0018	0.0489	0.0527	0.4227	1.3229	0.9002
60	42	$\hat{\alpha}$	0.0004	0.0120	0.0132	0.2098	0.6595	0.4496
		$\hat{\beta}$	0.0009	0.0269	0.0292	0.5174	1.1875	0.6701
	54	$\hat{\alpha}$	0.0002	0.0069	0.0077	0.2506	0.5953	0.3447
		$\hat{\beta}$	0.0007	0.0208	0.0228	0.5466	1.1389	0.5923
	60	$\hat{\alpha}$	0.0001	0.0056	0.0063	0.2634	0.5735	0.3099
		$\hat{\beta}$	0.0004	0.0188	0.0212	0.5487	1.1196	0.5709
100	70	$\hat{\alpha}$	0.0001	0.0062	0.0071	0.2546	0.5839	0.3293
		$\hat{\beta}$	0.0003	0.0134	0.0151	0.5879	1.0692	0.4813
	90	$\hat{\alpha}$	0.0001	0.0037	0.0043	0.2849	0.5409	0.2561
		$\hat{\beta}$	0.0002	0.0114	0.0130	0.5991	1.0463	0.4472
	100	$\hat{\alpha}$	0.0001	0.0029	0.0034	0.2964	0.5238	0.2274
		$\hat{\beta}$	0.0001	0.0099	0.0114	0.6103	1.0284	0.4181

Table 3

Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the shape parameters α and β from IKum for different sample size n, censoring size r and repetitions NR = 10000 ($\alpha = 1$, $\beta = 0.8$)

n	r	Estimator	RAB	RMSE	Variance	LL	UL	Length
30	21	$\hat{\alpha}$	0.0169	0.1403	0.1787	0.3282	1.9853	1.6571
		$\hat{\beta}$	0.0072	0.0622	0.0797	0.3484	1.4552	1.1068
	27	$\hat{\alpha}$	0.0077	0.0841	0.1105	0.4541	1.7574	1.3032
		$\hat{\beta}$	0.0042	0.0439	0.0574	0.4083	1.3480	0.9397
	30	$\hat{\alpha}$	0.0057	0.0660	0.0872	0.5127	1.6704	1.1576
		$\hat{\beta}$	0.0037	0.0397	0.0522	0.4257	1.3211	0.8953
60	42	$\hat{\alpha}$	0.0058	0.0618	0.0810	0.5342	1.6501	1.1158
		$\hat{\beta}$	0.0023	0.0246	0.0322	0.5070	1.2104	0.7034
	54	$\hat{\alpha}$	0.0021	0.0330	0.0448	0.6403	1.4700	0.8297
		$\hat{\beta}$	0.0011	0.0168	0.0228	0.5446	1.1367	0.5920
	60	$\hat{\alpha}$	0.0013	0.0271	0.0373	0.6658	1.4233	0.7574
		$\hat{\beta}$	0.0005	0.0140	0.0196	0.5534	1.1025	0.5491
100	70	$\hat{\alpha}$	0.0028	0.0332	0.0440	0.6524	1.4753	0.8229
		$\hat{\beta}$	0.0007	0.0114	0.0155	0.5893	1.0775	0.4881
	90	$\hat{\alpha}$	0.0010	0.0198	0.0271	0.7165	1.3630	0.6465
		$\hat{\beta}$	0.0004	0.0093	0.0130	0.6009	1.0479	0.4470
	100	$\hat{\alpha}$	0.0003	0.0149	0.0211	0.7382	1.3081	0.5698
		$\hat{\beta}$	0.0001	0.0077	0.0109	0.6106	1.0201	0.4095

Table 4
Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the shape parameters α *and* β from IKum for different sample size n,censoring size r and repetitions NR = 10000 ($\alpha = 0.4$, $\beta = 1$)

n	r	Estimator	RAB	RMSE	Variance	LL	UL	Length
30	21	$\hat{\alpha}$	0.0012	0.0256	0.0251	0.1456	0.7678	0.6222
		$\hat{\beta}$	0.0059	0.1275	0.1249	0.4360	1.8217	1.3856
	27	$\hat{\alpha}$	0.0005	0.0149	0.0152	0.1952	0.6791	0.4839
		$\hat{\beta}$	0.0042	0.0976	0.0969	0.4989	1.7179	1.2189
	30	$\hat{\alpha}$	0.0004	0.0122	0.0126	0.2110	0.6525	0.4405
		$\hat{\beta}$	0.0035	0.0861	0.0860	0.5231	1.6729	1.14987
60	42	$\hat{\alpha}$	0.00024	0.0104	0.0109	0.2216	0.6308	0.4092
		$\hat{\beta}$	0.0015	0.0479	0.0491	0.6299	1.4982	0.8682
	54	$\hat{\alpha}$	0.0001	0.0063	0.0066	0.2585	0.5779	0.3194
		$\hat{\beta}$	0.0011	0.0383	0.0396	0.6642	1.4439	0.7797
	60	$\hat{\alpha}$	0.0001	0.0049	0.0053	0.2715	0.5576	0.2861
		$\hat{\beta}$	0.0007	0.0325	0.0341	0.6815	1.4061	0.7246
100	70	$\hat{\alpha}$	0.0000	0.0057	0.0061	0.2628	0.5680	0.3058
		$\hat{\beta}$	0.0006	0.0253	0.0267	0.7177	1.3582	0.6406
	90	$\hat{\alpha}$	0.0000	0.0033	0.0035	0.2941	0.5288	0.2346
		$\hat{\beta}$	0.0004	0.0190	0.0201	0.7541	1.3097	0.5556
	100	$\hat{\alpha}$	0.0000	0.0029	0.0031	0.3006	0.5189	0.2183
		$\hat{\beta}$	0.0003	0.0180	0.0192	0.7566	1.2998	0.5431

Table 5
Maximum likelihood averages, relative absolute biases, relative mean square errors of ML estimates and 95% confidence intervals of the reliability and hazard rate at $t_0 = (0.3, 0.6, 1)$, from IKum for different sample size n, censoring size r =90% and repetitions NR= 10000

n	r	t_0	Estimator	Average	RAB	RMSE	LL	UL	Length
30	27	0.3	$\hat{R}(t_0)$	0.9929	0.0031	0.0081	0.9514	1.0321	0.0820
			$\hat{h}(t_0)$	0.0241	0.0289	0.0763	0.0000	0.1526	0.2570
		0.6	$\hat{R}(t_0)$	0.9783	0.0074	0.0208	0.8761	1.0803	0.2042
			$\hat{h}(t_0)$	0.1217	0.0054	0.1352	0.0000	0.8726	1.5017
		1	$\hat{R}(t_0)$	0.8888	0.0052	0.0267	0.6250	1.1526	0. 5276
			$\hat{h}(t_0)$	0.4780	0.0234	0.3617	0.0000	1.7176	2.4794
60	54	0.3	$\hat{R}(t_0)$	0.9985	0.0033	0.0085	0.9672	1.0250	0.0587
			$\hat{h}(t_0)$	0.0115	0.0316	0.0811	0.0000	0.1061	0.1891
		0.6	$\hat{R}(t_0)$	0.9916	0.0088	0.0226	0.9386	1.0450	0.1059
			$\hat{h}(t_0)$	0.0496	0.0133	0.0463	0.0000	0.2963	0.4932
		1	$\hat{R}(t_0)$	0.9245	0.0084	0.0265	0.7576	1.0913	0.3336
			$\hat{h}(t_0)$	0.3548	0.0073	0.1333	0.0000	1.1187	1.5272
100	90	0.3	$\hat{R}(t_0)$	0.9989	0.0035	0.0088	0.9822	1.0157	0.0334
			$\hat{h}(t_0)$	0.0034	0.0333	0.0840	0.0000	0.0567	0.1066
		0.6	$\hat{R}(t_0)$	0.9960	0.0093	0.0234	0.9634	1.0285	0.0651
			$\hat{h}(t_0)$	0.0308	0.0159	0.0427	0.0000	0.1472	0.2327
		1	$\hat{R}(t_0)$	0.9427	0.0104	0.0291	0.8158	1.0690	0.2538
			$\hat{h}(t_0)$	0.2906	0.0024	0.0753	0.0000	0.8830	1.1841

Table 6
Bayes averages of the parameters, reliability and hazard rate at $t_0 = 0.5$ and their mean, standard deviations, squared errors, and credible intervals based on Type II censoring

n	r	Estimator	Sd	ER	Average	LL	UL	CI
20	14	α^*	0.0052	5.132E-5	0.9947	0.9808	0.9999	0.0191
		β^*	0.0051	4.946E-5	0.9949	0.9809	0.9999	0.0190
		$R_1^*(t_0)$	0.0023	2.374E-5	0.6662	0.6607	0.6707	0.0100
		$h_1^*(t_0)$	0.0041	4.159E-5	0.6654	0.6555	0.6727	0.0172
	18	α^*	0.0033	3.048E-5	0.9966	0.9876	0.9999	0.0123
		β^*	0.0029	2.964E-5	0.997	0.9891	0.9999	0.0108
		$R_1^*(t_0)$	0.0014	1.329E-5	0.6665	0.6633	0.6693	0.0060
		$h_1^*(t_0)$	0.0025	2.336E-5	0.6657	0.6594	0.6701	0.0107
30	21	α^*	0.0024	2.496E-5	0.9976	0.9913	0.9999	0.0086
		β^*	0.0021	2.108E-5	0.9978	0.992	0.9999	0.0079
		$R_1^*(t_0)$	0.0010	9.658E-6	0.6665	0.6642	0.6686	0.0044
		$h_1^*(t)$	0.0018	1.799E-5	0.666	0.6615	0.6691	0.0076
	27	α^*	0.0015	1.447E-5	0.9985	0.9944	1.0000	0.0056
		β^*	0.0013	1.295E-5	0.9987	0.9951	1.0000	0.0049
		$R_1^*(t_0)$	6.445E-4	6.252E-6	0.6666	0.6652	0.6679	0.0027
		$h_1^*(t_0)$	0.0011	1.13E-5	0.6662	0.6633	0.6682	0.0049
60	42	α^*	6.599E-4	6.842E-6	0.9993	0.9976	1.0000	0.0024
		β^*	4.893E-4	4.660E-6	0.9995	0.9982	1.0000	0.0018
		$R^*{}_1(t_0)$	2.525E-4	2.449E-6	0.6667	0.6661	0.6672	0.0011
		$h^*{}_1(t)$	4.873E-4	4.886E-6	0.6664	0.6652	0.6672	0.0020
	54	α^*	3.739E-4	3.694E-6	0.9996	0.9986	1.0000	0.0014
		β^*	3.314E-4	3.154E-6	0.9997	0.9988	1.0000	0.0012
		$R_1^*(t_0)$	1.575E-4	1.569E-6	0.6666	0.6663	0.6670	0.0007
		$h_1^*(t_0)$	2.867E-4	2.880E-6	0.6666	0.6659	0.6671	0.0012
100	70	α^*	2.407E-4	2.474E-6	0.9998	0.9991	1.0000	0.0009
		β^*	1.697E-4	1.759E-6	0.9999	0.9994	1.0000	0.0006
		$R_1^*(t_0)$	8.990E-5	9.803E-7	0.6667	0.6665	0.6669	0.0004
		$h_1^*(t_0)$	1.764E-4	1.893E-6	0.6666	0.6661	0.6669	0.0008
	80	α^*	1.313E-4	1.432E-6	0.999	0.999	1.0000	0.0010
		β^*	1.157E-4	1.230E-6	0.999	0.999	1.0000	0.0010
		$R_1^*(t_0)$	5.534E-5	6.043E-7	0.6667	0.6664	0.6666	0.0002
		$h_1^*(t_0)$	1.009E-4	1.109E-6	0.6666	0.6665	0.6666	0.0001

Table 7
ML Averages of the parameters, rf, hrf and their estimated risks and relative absolute biases for the real data based on Type II censoring

Real Data	n	r	Estimator	Average	ER	RAB
I	30	21	$\hat{\alpha}$ $\hat{\beta}$ $\hat{R}(t_0)$ $\hat{h}(t_0)$	0.8957 1.7429 0.7391 0.3200	0.2457 0.2948 0.0062 0.0233	0.0745 0.1673 0.0001 0.0044
II	30	21	$\hat{\alpha}$ $\hat{\beta}$ $\hat{R}(t_0)$ $\hat{h}(t_0)$	0.8274 1.5091 0.7137 0.3232	0.1827 0.0956 0.0108 0.0242	0.0597 0.0312 0.0035 0.0079
III	34	24	$\hat{\alpha}$ $\hat{\beta}$ $\hat{R}(t_0)$ $\hat{h}(t_0)$	0.6719 1.1095 0.6658 0.3153	0.0739 0.0082 0.0230 0.0219	0.0224 0.0036 0.0026 0.0040

Table 8
Bayes Averages of the parameters rf, hrf and their estimated risks and credible intervals under (non-informative prior) for the real data based on Type II censoring

Real Data	n	r	Estimator	Average	ER	LL	UL	CI
I	30	21	α^* β^* $R_1^*(t_0)$ $h_1^*(t_0)$	0.1009 0.7004 0.8994 0.5982	0.0895 0.0897 9.11e-05 0.0078	0.0969 0.6980 0.8981 0.5987	0.0995 0.7004 0.9003 0.6026	0.0026 0.0024 0.0022 0.0039
II	30	21	α^* β^* $R_1^*(t_0)$ $h_1^*(t_0)$	0.0987 0.6991 0.8988 0.5996	0.0907 0.0895 0.0001 0.0080	0.0973 0.6983 0.8974 0.5978	0.0996 0.7001 0.8999 0.6006	0.0023 0.0018 0.0025 0.0028
III	34	24	α^* β^* $R^*{}_1(t)$ $h^*{}_1(t)$	0.1000 0.7000 0.9011 0.6000	3.99e-02 1.19e-07 6.21e-05 0.00807	0.0992 0.6995 0.8998 0.5991	0.1005 0.7011 0.9039 0.6015	0.0013 0.0016 0.0042 0.0024

APPENDIX B

1) If W has exponential pdf with parameter (β), then the random variable $T=\left[(1-e^{-w})^{-\frac{1}{\alpha}}-1\right]$ has IKum distribution with parameters (α,β) and given in Equation (2).

Proof:

Let $f(w)$ be the pdf of exponential (β) distribution, given by

$$f(w;\beta)=\beta e^{-\beta w}\ ,w>0\ .$$

Assuming $T=\left[(1-e^{-w})^{-\frac{1}{\alpha}}-1\right]$, then $w=-\ln(1-(1+t)^{-\alpha})$,

and $\left|\frac{dw}{dt}\right|=\frac{\alpha(1+t)^{-(\alpha+1)}}{(1-(1+t)^{-\alpha})}$,

hence, the pdf of T is given by

$$f_T(t;\alpha)=f_W(-\ln(1-(1+t)^{-\alpha})\left|\frac{dw}{dt}\right|,$$

then

$$f(t;\alpha,\beta)=\alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{\beta-1}\ ,t>0;\ \alpha,\beta>0,$$

which is the pdf for IKum distribution (α,β).

2) **Limiting distributions**

a) Let $f_T(t)$ be the pdf of IKum(α,β) distribution, given by (2)

Considering $Y_1=\beta^{-\frac{1}{\alpha}}(1+T)$,

Then the interval (0,∞) is transformed to $(\beta^{-\frac{1}{\alpha}},\infty)$,

and $t=\beta^{\frac{1}{\alpha}}y_1-1$, $\left|\frac{dt}{dy_1}\right|=\beta^{\frac{1}{\alpha}}$.

Therefore, the pdf of y_1 is given by

$$f_{Y_1}(y_1;\alpha,\beta)=f_T\left(\beta^{\frac{1}{\alpha}}y_1-1\right)\left|\frac{dt}{dy_1}\right|,$$

hence

$$f(y_1;\alpha,\beta)=\alpha {y_1}^{-(\alpha+1)}\left(1-\frac{{y_1}^{-\alpha}}{\beta}\right)^{\beta-1}.$$

As $\beta\rightarrow\infty$, the pdf of y_1 tends to

$$f(y_1;\alpha)=\alpha {y_1}^{-(\alpha+1)}e^{-{y_1}^{-\alpha}},y_1>0,$$

which is the pdf of the inverted Weibull distribution.

b) Let $f_T(t)$ be the pdf of IKum(α,β) distribution, given by (2)

Considering $Y_2 = \alpha(1-(1+T)^{-1})$,

then the interval (0,∞) is transformed to $(0,\alpha)$,

and $t = \left(1-\frac{y_2}{\alpha}\right)^{-1} - 1$, $\left|\frac{dt}{dy_2}\right| = \frac{1}{\alpha}\left(1-\frac{y_2}{\alpha}\right)^{-2}$.

Therefore, the pdf of y_2 is given by

$$f_{Y_2}(y_2;\alpha) = f_T(\left(1-\frac{y_2}{\alpha}\right)^{-1} - 1)\left|\frac{dt}{dy_2}\right|,$$

then $f(y_2;\alpha,\beta) = \beta\left(1-\frac{y_2}{\alpha}\right)^{\alpha-1}\left(1-\left(1-\frac{y_2}{\alpha}\right)^{\alpha}\right)^{\beta-1}$.

As $\alpha\to\infty$ the pdf of y_2 tends to

$$f(y_2;\beta) = \beta e^{-y_2}(1-e^{-y_2})^{\beta-1}, y_2 > 0,$$

which is the pdf of the generalized exponential distribution.

c) Let $f_T(t)$ be the pdf of IKum(α,β) distribution, given by (2)

Let $Y_3 = \alpha\left(1-\beta^{\frac{1}{\alpha}}(1+T)^{-1}\right)$, then the interval (0,∞) is transformed to

$(\alpha\left(1-\beta^{\frac{1}{\alpha}}\right),\alpha)$ and $t = \beta^{\frac{1}{\alpha}}\left(1-\frac{y_3}{\alpha}\right)^{-1} - 1$, $\left|\frac{dt}{dy_3}\right| = \frac{\beta^{\frac{1}{\alpha}}}{\alpha}\left(1-\frac{y_3}{\alpha}\right)^{-2}$.

Therefore, the pdf of y_3 is given by $f_{Y_3}(y_3;\alpha,\beta) = f_T\left(\beta^{\frac{1}{\alpha}}\left(1-\frac{y_3}{\alpha}\right)^{-1} - 1\right)\left|\frac{dt}{dy_3}\right|$,

hence, the pdf of $y_3$is $f(y_3;\alpha,\beta) = \left(1-\frac{y_3}{\alpha}\right)^{\alpha-1}\left(1-\frac{\left(1-\frac{y_3}{\alpha}\right)^{\alpha}}{\beta}\right)^{\beta-1}$.

As both $\beta\to\infty$ and $\alpha\to\infty$, the pdf of $y_3$tends to

$$f(y_3) = e^{-y_3}exp[-e^{-y_3}], y_3 > 0,$$

which is the pdf of the standard extreme value distribution of the first Type.