Pak. J. Statist.
2017 Vol. 33(1), 15-25

CLASSICAL AND BAYESIAN INFERENCE OF PARETO DISTRIBUTION AND FUZZY LIFE TIMES

Muhammad Shafiq
Department of Economics, Kohat University of Science and Technology
Kohat, Pakistan
Email: shafiq@kust.edu.pk

ABSTRACT

Life time data are mainly used in medicine, public health, and engineering sciences. The classical statistics related to these fields are based on precise measurements. However, according to modern measurement science precise measurement of continuous phenomena is not possible. In addition the observed quantity is more or less precise.

In order to obtain realistic results, in addition to classical and Bayesian inference, the most up to date model for this are fuzzy number approaches which are more suitable and realistic.

In this study, I consider the classical and Bayesian inference criteria for the Pareto distribution to obtain more realistic results for fuzzy observations of life time.

In addition to stochastic variation the proposed generalized estimators cover fuzziness of the life times as well.

KEY WORDS

Bayesian inference; Characterizing function; Fuzzy number; Life time; Survival function.

1. INTRODUCTION

Pareto distribution has significant applications in economics, engineering, reliability, survival analysis, insurance, and life testing.

Its pdf is defined by

$$f(x) = \frac{\beta\theta_0^{\beta}}{x^{\beta+1}} \text{ with } \beta > 0, 0 < \theta_0 < x < \infty \tag{1}$$

For precise life time observations $x_1, x_2, \dots, x_n$ corresponding maximum likelihood estimators are given by

$$\hat{\theta}_0 = \min\{x_1, x_2, \dots, x_n\} \tag{2}$$

and

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} ln\left(\frac{x_i}{x_1}\right)} \tag{3}$$

where $x_1 = \min\{x_1, x_2, \dots, x_n\}$.

For details see (Howlader et al., 2007).

Survival function is defined as $S(x) = P(X > x)$, for the Pareto distribution it is given by

$$S(x) = \left(\frac{\theta_0}{x}\right)^{\beta} \text{ for } x \geq \theta_0 \tag{4}$$

In classical statistics for efficient estimation and testing of hypotheses, generally large data is required. But with the technological advancement, life times of the units are increased, and it is very time consuming and expensive to obtain a large number of observations. To get sophisticated results with small sample size, techniques are developed to use some prior information about the parameters, called *Bayesian inference*.

1.1 Bayesian Inference

Let X be a continuous random variable representing life time. In classical statistics this life time can be modeled as $X \sim f(\cdot\,|\theta)$, where θ is considered as fixed value over the parameter space $\Theta \subseteq (0\,,\infty]$. But in Bayesian statistics the parameter is also considered as random variable having a-priori density, denoted by $\pi(\theta)$.

For precise sample $\underline{x} = (x_1, x_2, \dots, x_n)$ the so-called *likelihood function* is written as:

$$l(\theta;\, \underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) \tag{5}$$

Based on the likelihood function and a-priori density the so-called *posterior density* of is θ denoted by $\pi(\theta|\underline{x}\,)$, and is obtained as

$$\pi(\theta|\underline{x}) = \frac{\pi(\theta)\cdot l(\theta;\underline{x})}{\int_{\Theta} \pi(\theta)\cdot l(\theta;\underline{x})d\theta} \tag{6}$$

or it can be simply written in a non-normalized form like

$$\pi(\theta|\underline{x}) \propto \pi(\theta)\cdot l(\theta;\, \underline{x}) \tag{7}$$

The parameter estimators obtained from the density mentioned in equation (6) are called Bayesian estimators (Bolstad, 2004).

Sophisticated stochastic models are developed to utilize all the available information in the best way for making suitable inference.

According to (Howlader et al., 2007) let the random variable X taking values in $(0\,,\infty]$, and denote the waiting time until the death/failure of a unit. Pareto distribution is considered as one of the most popular models.

The likelihood function is

$$l(\theta_0, \beta|\underline{x}) = \beta^n exp\left[\frac{(-\beta c_0)}{G}\right]^n \tag{8}$$

where $G = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ and $c_0 = ln\left({}^{G}/_{\theta_0}\right)$.

In case of θ_0 is given the prior distribution is assumed to be

$$\pi(\beta) \propto \beta^{c-1} exp(-p\beta) \text{ with } p, c, \beta > 0 \tag{9}$$

where c and p are hyper parameters.

For the posterior density we obtain

$$\pi\big(\beta\big|\theta_0,\underline{x}\big) = \frac{(nc_0+p)^{(c+n)}}{\Gamma(c+n)}\beta^{c+n-1}\exp\{-\beta(nc_0+p)\} \qquad (10)$$

The posterior density obtained in equation (10) is a gamma distribution, i.e. $G(c+n\,,nc_0+p)$. The Bayesian estimate of parameter β with squared loss function is given by

$$\hat{\beta}_B = \frac{c+n}{nc_0+p}. \qquad (11)$$

With the advancement in measurement science, and in practical applications it is observed that exact measurement of continuous quantities is not possible. Many apparatuses are developed to quantify continuous quantities precisely, yet the problem of exact measurement is unsolved and decisions are based on approximate numbers.

Correspondingly in the real world of measurements one cannot get a precise measurement of irregular phenomena, for example depth of a river because of its water level fluctuation. In the same way one cannot find a precise criterion between high or low blood pressure, high or low temperature, effective or ineffective teacher, etc. From such type of situations it is concluded that there are actually two types of uncertainty in measurements: variation among the observations and the imprecision of individual observations, called *fuzziness* (Viertl, 2006).

All classical and Bayesian stochastic models cover variation among the observations, but ignore the imprecision of individual observations. This imprecision may contain some useful information, and by neglecting it we may lose information and get misleading results.

To integrate imprecision of individual observations the generalization of *classical sets* to *fuzzy sets* is useful (Zadeh, 1965).

For classical sets, a two-valued characteristic function mentioned in equation (12) is called *indicator function*. It is used to represent whether or not an element ω belongs to a subset Ω of a universal set U.

$$I_\Omega(\omega) = \begin{Bmatrix} 1 \; if \;\omega \;\in \Omega \\ 0 \; if \;\omega \;\notin \Omega \end{Bmatrix} \forall\,\omega\;\in\;\Omega. \qquad (12)$$

Fuzzy sets are the generalization of classical sets, therefore, the indicator function is generalized to the so-called *membership function* $\mu_{\Omega^*}$of a fuzzy subset Ω^* of U, given in equation (13) below:

$$\mu_{\Omega^*}(\omega)\begin{Bmatrix} 1\; if\;\omega\;\in\; core\;(\Omega^*) \\ \delta\in(0\,,1]\; if\;\omega\; belongs\; to\;\Omega^* \\ to\; some\; degree\;\delta \\ 0\; if\;\omega\;\notin \text{to}\;\Omega^* \end{Bmatrix}\;\forall\omega\in U \qquad (13)$$

The set of all points ω in U such that $\mu_{\Omega^*}(\omega)=1$ is called core of Ω^*, in symbols $core\;(\Omega^*)$ Szeliga, 2004).

According to (Viertl, 2011) some special concepts of fuzzy theory are explained below.

1.2 Fuzzy Numbers

Let x^* represent a so-called *fuzzy number* which is determined by its so-called *characterizing function* $\xi(\cdot)$, a real function of one real variable satisfying the below conditions 1-3:

1. $\xi : \mathbb{R} \rightarrow [0\,,1]$
2. $\xi(\cdot)$ has a bounded support, i.e. $\mathrm{supp}[\xi(\cdot)] :=\{ x \in \mathbb{R} : \xi(x) > 0\} \subseteq [a\,, b]$.
3. For all $\delta \in (0\,,1]$ the so-called δ-cut $C_\delta(x^*) := \{x \in \mathbb{R} : \xi(x) \geq \delta\}$ is a finite union of non-empty and compact intervals, i.e.
   $C_\delta(x^*) = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}\,, b_{\delta,j}] \neq \emptyset\, \forall\, \delta \in (0\,,1]$.

If all δ-cuts of a fuzzy number x^* are non-empty closed bounded intervals, the corresponding fuzzy number is called *fuzzy interval*.

1.3 Lemma

For any characterizing function $\xi(\cdot)$ of a fuzzy numberx^*, the following holds:

$$\xi(x) = \max\{\delta\cdot I_{C_\delta(x^*)}(x) : \delta \in [0\,,1]\}\ \forall x \in \mathbb{R}$$

For the proof see (Viertl, 2011).

1.4 Remark

It should be noted that not all families $(A_\delta\,;\, \delta \in [0\,,1])$ of nested finite unions of compact intervals are the δ-cuts of a fuzzy number. But the following construction lemma holds:

1.5 Construction Lemma

Let $(A_\delta\,;\, \delta \in [0\,,1])$ with $A_\delta = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}\,, b_{\delta,j}]$ be a nested family of non-empty uniformly bounded subsets of $\mathbb{R}$. Then the characterizing function of the generated fuzzy number is given by

$$\xi(x) = \sup\{\delta\cdot I_{C_\delta(x^*)}(x) : \delta \in (0\,,1]\}\ \forall x \in \mathbb{R}.$$

For the proof see (Viertl and Hareter, 2006).

1.6 Fuzzy Vector

A m-dimensional fuzzy vector $\underline{x}^*$ is determined by its so-called *vector-characterizing function* $\zeta(.\,,\ldots,.)$, which is a real function of m real variables $x_1, x_2, \ldots, x_m$ obeying the following conditions 1-3:

1. $\zeta : \mathbb{R}^m \rightarrow [0\,,1]$
2. The support of the characterizing function $\zeta(.\,,\ldots,.)$, i.e. $supp[\zeta(.\,,\ldots,.)]$ is a bounded set.
3. For all $\delta \in [0\,,1]$ the so-called δ-cut $C_\delta(\underline{x}^*) = \{\underline{x} \in \mathbb{R}^m : \zeta(\underline{x}) \geq \delta\}$ is a finite union of non-empty, bounded and a simply connected and closed sets.

Therefore, dealing with life time analysis instead of precise number models fuzzy models are more appropriate to make inference based on fuzzy life time data. Keeping in view the reality of fuzziness in the data, some work in the literature has been done like, Modeling future life time as fuzzy variable (Shapairo, 2013), Reliability estimation in Rayleigh distribution based on fuzzy lifetime data (Pak et al., 2013), On reliability estimation based on fuzzy lifetime data (Viertl, 2009), Statistical confidence intervals for fuzzy data (Wu, 2009), first course on fuzzy theory and applications (Lee, 2006), Fundamentals of statistics with fuzzy data (Nguyen and Wu, 2006), Estimation of Weibull parameters using fuzzy least squares method (Hung and Liu, 2004).

In this study parameter estimation for the Pareto distribution is generalized for fuzzy life time data.

2. PARETO DISTRIBUTION AND FUZZY LIFE TIMES

Let $x_1^*, x_2^*, \dots, x_n^*$ represent fuzzy life times with δ-cuts $C_\delta(x_i^*) = \left[\underline{x}_{i,\delta}\, , \overline{x}_{i,\delta}\right]$, $i = 1(1)n, \forall \delta \in [0, 1]$, then the corresponding fuzzy parameter estimates are denoted by $\hat{\theta}_0^*$ and $\hat{\beta}^*$, respectively.

The corresponding δ-cuts of the fuzzy parameter estimates are denoted by

$$C_\delta\left(\hat{\theta}_0^*\right) = \left[\underline{\theta}_{0,\delta}\, , \overline{\theta}_{0,\delta}\right] \forall\, \delta \in (0\,, 1]$$

and

$$C_\delta\left(\hat{\beta}^*\right) = \left[\underline{\beta}_\delta\, , \overline{\beta}_\delta\right] \; \forall\, \delta \in (0\,, 1]$$

Using the parameter estimator mentioned in equation (2), lower and upper ends of the generating family of intervals for the fuzzy parameter estimate $\hat{\theta}_0^*$ are obtained in the following way:

$$\underline{\theta}_{0,\delta} = \inf\left\{\underline{x}_{i,\delta} \,:\, i = 1(1)n\right\} \forall\, \delta \in (0\,, 1]$$

and

$$\overline{\theta}_{0,\delta} = \inf\left\{\overline{x}_{i,\delta} \,:\, i = 1(1)n\right\} \forall\, \delta \in (0\,, 1]$$

From this generating family of intervals $\left(A_\delta\left(\hat{\theta}_0^*\right) = \left[\underline{\theta}_{0,\delta}\, , \overline{\theta}_{0,\delta}\right] \forall\, \delta \in (0\,, 1]\right)$ the characterizing function of $\hat{\theta}_0^*$ is obtained by the construction lemma mentioned in section 1.5.

Similarly the parameter estimate mentioned in equation (3) is generalized for fuzzy life time observations in the following way:

$$\underline{\beta}_\delta = \frac{n}{\sum_{i=1}^n ln\left(\frac{\overline{x}_{i,\delta}}{\underline{x}_{1,\delta}}\right)} \; \forall\, \delta \in (0\,, 1]$$

and

$$\overline{\beta}_\delta = \frac{n}{\sum_{i=1}^n ln\left(\frac{\underline{x}_{i,\delta}}{\overline{x}_{1,\delta}}\right)} \forall\, \delta \in (0\,, 1]$$

The characterizing function of $\hat{\beta}^*$is obtained from the generating family of intervals $\left(A_\delta\left(\hat{\beta}^*\right) = \left[\underline{\beta}_\delta\, , \overline{\beta}_\delta\right] \; \forall\, \delta \in (0\,, 1]\right)$ using the mentioned construction lemma.

Example 1:
Characterizing functions of fuzzy life time observations, and proposed estimators are depicted in Figure 1, Figure 2, and Figure 3 respectively.

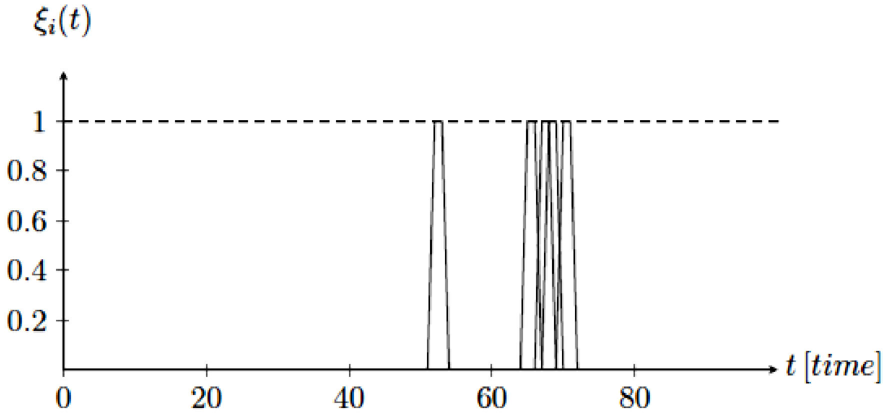

Figure 1: Characterrizing Function of Fuzzy Life Times

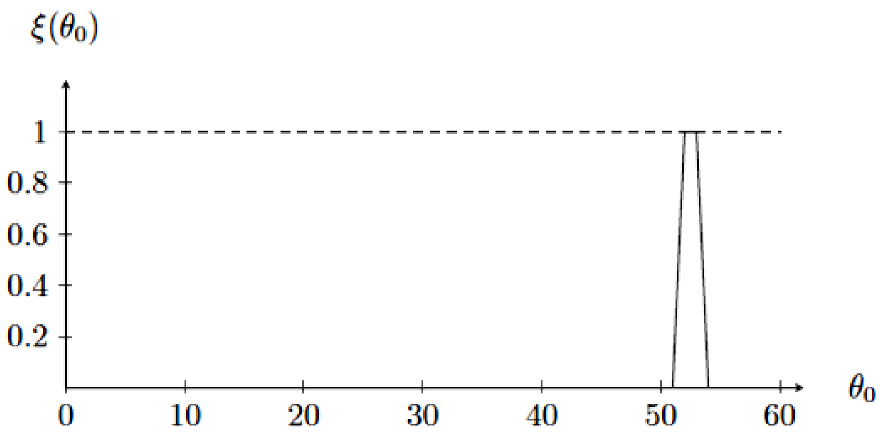

Figure 2: Characterrizing Function of the Fuzzy Estimator $\widehat{\theta}_0^*$

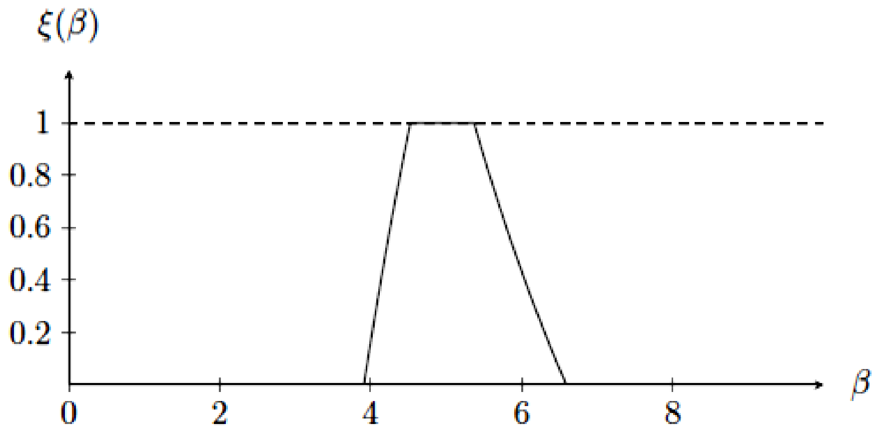

Figure 3: Characterrizing Function of the Fuzzy Estimator $\widehat{\boldsymbol{\beta}}^*$

Based on fuzzy life times and fuzzy parameter estimates $\hat{\theta}_0^*$ and $\hat{\beta}^*$ the generalized estimate of the survival function is denoted by $S^*(x)$. The corresponding lower δ-level curves and upper δ-level curves are denoted by $\underline{S}_\delta(x)$, and $\overline{S}_\delta(x)$ respectively, where

$$\underline{S}_\delta(x) = \left(\frac{\underline{\theta}_{0,\delta}}{\overline{x}_\delta}\right)^{\underline{\beta}_\delta} \forall\, \delta \in (0\,,1]$$

and

$$\overline{S}_\delta(x) = \left(\frac{\overline{\theta}_{0,\delta}}{\overline{x}_\delta}\right)^{\overline{\beta}_\delta} \forall\, \delta \in (0\,,1]$$

Some lower and upper δ-level curves of the fuzzy estimate of the survival function are given in the following Figure 4.

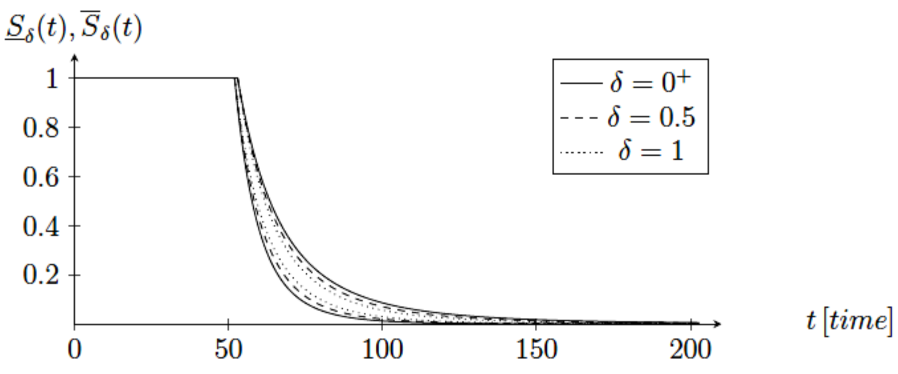

Figure 4: Some Lower and Upper $\boldsymbol{\delta}$-level of the Fuzzy Estimate of the Survival Function

The curves show the boundaries of the supports and the ends of the δ-cuts of the corresponding characterizing functions of the fuzzy estimate of the survival function for some values of δ.

Keeping in view the importance of Bayesian inference, (Viertl, 1987) proposed the use of Bayesian inference for fuzzy data.

3. BAYESIAN INFERENCE FOR FUZZY DATA

After this proposal some Bayesian work has been done dealing with fuzzy observations like, On Bayes theorem for fuzzy data (Viertl and Hule, 1991), On fuzzy Bayesian inference (Frühwirth-Schnatter, 1993), Fundamentals and Bayesian analyses of decision problems with fuzzy-valued utilities (Gil and Lopez-Diaz, 1996}, Fuzzy Bayesian inference (Yang, 1997), A Bayesian approach to fuzzy hypotheses testing (Taheri and Behboodian, 2001), generalized Bayes' theorem for non-precise a-priori distribution (Viertl and Hareter, 2004), Bayesian reliability analysis for fuzzy lifetime data (Huang et al., 2006), Fuzzy Bayesian inference (Viertl and Sunanta, 2013}, but still in most of the papers dealing with life times fuzziness is ignored.

According to (Viertl, 2011) some preliminary concepts for Bayesian inference for fuzzy data are presented.

Based on fuzzy observations $x_1^*, x_2^*, \dots, x_n^*$, let $f^*(x)$, $l^*(\theta\,,\underline{x})$, and $\pi^*(\theta)$ represent fuzzy density function, fuzzy likelihood function, and fuzzy a-priori density function respectively.

Their corresponding lower and upper \$\delta\$-level curves are denoted as

$$C_\delta[f^*(x)] = \left[\underline{f}_\delta(x)\,,\overline{f}_\delta(x)\right] \forall\ \delta \in (0\,,1]\ (14) \tag{14}$$

$$C_\delta\left[l^*\left(\theta\,,\underline{x}\right)\right] = \left[\underline{l}_\delta\left(x\theta\,,\underline{x}\right),\overline{l}_\delta(\theta\,,\underline{x})\right]\forall\ \delta \in (0\,,1] \tag{15}$$

$$C_\delta[\pi^*(\theta)] = \left[\underline{\pi}_\delta(\theta)\,,\overline{\pi}_\delta(\theta)\right]\forall\ \delta \in (0\,,1] \tag{16}$$

Using fuzzy likelihood and fuzzy a-priori density, the generalized fuzzy Bayes theorem is written as

$$\pi^*\left(\theta\middle|\underline{x}\right) = \frac{\pi^*(\theta)\cdot l^*\left(\theta;\underline{x}\right)}{\int_\Theta^{\cdot} \pi^*(\theta)\cdot l^*\left(\theta;\underline{x}\right)d\theta} \tag{17}$$

having lower and upper δ-level curves $\underline{\pi}_\delta(\theta|\underline{x})$ and $\overline{\pi}_\delta(\theta|\underline{x})$ respectively.

Where

$$\underline{\pi}_\delta\left(\theta\middle|\underline{x}\right) = \frac{\underline{\pi}(\theta)\cdot\underline{l}\left(\theta;\underline{x}\right)}{\int_\Theta^{\cdot}\overline{\pi}(\theta)\cdot\overline{l}\left(\theta;\underline{x}\right)d\theta}\ \forall\ \delta \in (0\,,1] \tag{18}$$

and

$$\overline{\pi}_\delta\left(\theta\middle|\underline{x}\right) = \frac{\overline{\pi}(\theta)\cdot\overline{l}\left(\theta;\underline{x}\right)}{\int_\Theta^{\cdot}\underline{\pi}(\theta)\cdot\underline{l}\left(\theta;\underline{x}\right)d\theta}\ \forall\ \delta \in (0\,,1] \tag{19}$$

Unfortunately lower and upper δ-level curves defined in equations (18) and (19) do not satisfy the condition of sequential updating, therefore these are modified as

$$\underline{\pi}_{\delta}(\theta|\underline{x}) = \frac{\underline{\pi}(\theta)\cdot\underline{l}(\theta;\underline{x})}{\int_{\Theta}\frac{\underline{\pi}(\theta)+\overline{\pi}(\theta)}{2}\cdot\overline{l}(\theta;\underline{x})d\theta}\ \forall\ \delta\in(0\,,1] \quad (20)$$

and

$$\overline{\pi}_{\delta}(\theta|\underline{x}) = \frac{\overline{\pi}(\theta)\cdot\overline{l}(\theta;\underline{x})}{\int_{\Theta}\frac{\underline{\pi}(\theta)+\overline{\pi}(\theta)}{2}\cdot\underline{l}(\theta;\underline{x})d\theta}\ \forall\ \delta\in(0\,,1] \quad (21)$$

Based on the fuzzy life time observations $x_1^*, x_2^*, \ldots, x_n^*$, the fuzzy probability density function of the Pareto distribution is denoted by $f^*(x)$, and fuzzy a-priori density is $G(c^*\,,p^*)$, denoted by $\pi^*(\beta)$.

Where c^* = [2, 2.1, 2.2, 2.3] and p^*= [4, 4.1, 4.2, 4.3] are trapezoidal fuzzy hyper parameters.

Some lower and upper δ-level curves of the fuzzy posterior density obtained by equations (21) and (22) are depicted in Figure 5.

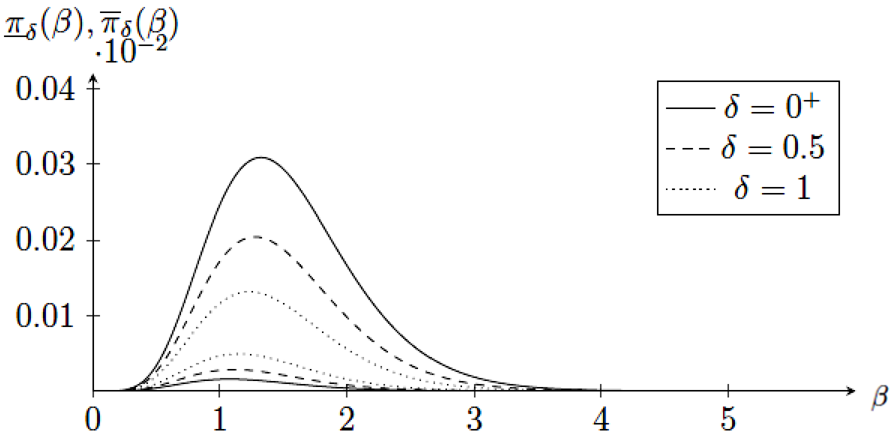

Figure 5: Upper and Lower δ-Level of the Fuzzy Posterior Density

Keeping $\hat{\theta}_0^*$ constant the fuzzy Bayesian estimator is denoted by $\hat{\beta}_B^*$ with δ-cuts

$$C_{\delta}[\hat{\beta}_B^*] = \left[\underline{\beta}_{B,\delta}\,,\overline{\beta}_{B,\delta}\right]\ \forall\ \delta\in(0\,,1]$$

where

$$\underline{\beta}_{B,\delta} = \frac{\underline{c}+n}{n\overline{c}_0+\overline{p}}\ \forall\ \delta\in(0\,,1]$$

and

$$\overline{\beta}_{B,\delta} = \frac{\overline{c}+n}{n\underline{c}_0+\underline{p}}\ \forall\ \delta\in(0\,,1].$$

From this generating family of intervals $\left(A_{\delta}(\hat{\beta}_B^*) = \left[\underline{\beta}_{B,\delta}\,,\overline{\beta}_{B,\delta}\right]\ \forall\ \delta\in(0\,,1]\right)$ the characterizing function of the fuzzy parameter estimate $\hat{\beta}_B^*$ is obtained through discussed construction lemma and is depicted in Figure 6.

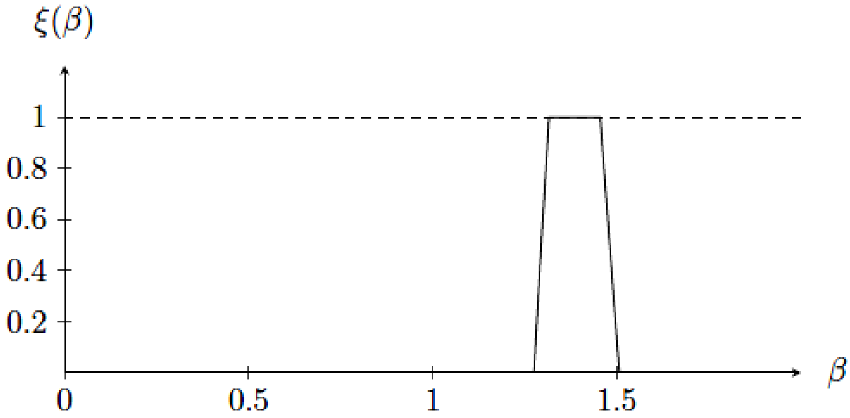

Figure 6: Characterizing Function of the Fuzzy Parameter Estimate $\widehat{\boldsymbol{\beta}}_{\boldsymbol{B}}^{*}$

4. CONCLUSION

Precise measurement of a continuous phenomenon always remained an issue in research. Classical statistical inference is mainly based on precise measurement, but in fact all continuous measurements are more or less imprecise, i.e. fuzzy.

In order to utilize all the available information, propagation of fuzziness in statistical inference is required.

In this study classical and Bayesian estimation of Pareto distribution is generalized in such a way that in addition to random variation, the fuzziness of single observations is also propagated.

For practical applications, the proposed estimators are more realistic because in addition of stochastic variation the fuzziness of observations is also integrated.

REFERENCES

1. Howlader, H.A., Hossain, A.M. and Makhmin, O. (2007). Bayesian prediction and estimation from Pareto distribution of the first kind. *International Journal of Statistical Sciences*, 6, 137-150.
2. Huang, H.-Z., Zuo, M.J. and Sun, Z.-Q. (2006). Bayesian reliability analysis for fuzzy lifetime data. *Fuzzy Sets and Systems*, 157(12), 1674-1686.
3. Hung, W.-L. and Liu, Y.-C. (2004). Estimation of Weibull parameters using a fuzzy least-squares method. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12(05), 701-711.
4. Lee, K.H. (2006). *First Course on Fuzzy Theory and Applications*. Springer, London.
5. Nguyen, H.T. and Wu, B. (2006). *Fundamentals of Statistics with Fuzzy Data*. Springer, New York.

6. Pak, A., Parham, G. and Saraj, M. (2013). Reliability estimation in Rayleigh distribution based on fuzzy lifetime data. *International Journal of System Assurance Engineering and Management*, 5(4), 487-494.
7. Shapiro, A.F. (2013). Modeling future lifetime as a fuzzy random variable. *Insurance: Mathematics and Economics*, 53(3), 864-870.
8. Szeliga, E. (2004). Structural reliability-fuzzy sets theory approach. *Journal of Theoretical and Applied Mechanics*, 42(3), 651-666.
9. Taheri, S. and Behboodian, J. (2001). A Bayesian approach to fuzzy hypotheses testing. *Fuzzy Sets and Systems*, 123(1), 39-48.
10. Viertl, R. (1987). Is it necessary to develop a fuzzy Bayesian inference? In Viertl, R., editor, *Probability and Bayesian Statistics*, pages 471-475. Springer, New York.
11. Viertl, R. (2006). Univariate statistical analysis with fuzzy data. *Computational Statistics & Data Analysis,* 51(1), 133-147.
12. Viertl, R. (2009). On reliability estimation based on fuzzy lifetime data. *Journal of Statistical Planning & Inference*, 139(5), 1750-1755.
13. Viertl, R. (2011). *Statistical Methods for Fuzzy Data*. Wiley, Chichester.
14. Viertl, R. and Hareter, D. (2004). Generalized Bayes theorem for non-precisea-priori distribution. *Metrika*, 59(3), 263-273.
15. Viertl, R. and Hareter, D. (2006). *Beschreibung und Analyse unscharfer Information: Statistische Methoden für unscharfe Daten*. Springer, Wien.
16. Viertl, R. and Hule, H. (1991). On Bayes theorem for fuzzy data. *Statistical Papers*, 32(1), 115-122.
17. Viertl, R. and Sunanta, O. (2013). Fuzzy Bayesian inference. *Metron*, 71(3), 207-216.
18. Wu, H.-C. (2009). Statistical confidence intervals for fuzzy data. *Expert Systems with Applications*, 36(2), 2670-2676.
19. Yang, C.C. (1997). Fuzzy Bayesian inference. In *Proceeding of Systems, Man, and Cybernetics, IEEE International Conference*, 3, 2707-2712.
20. Zadeh, L. (1965). Fuzzy sets. *Information and Control,* 8(3), 338-353.