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A POWER SERIES DISTRIBUTION INVOLVING THE DOUBLE FACTORIAL

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ABSTRACT

This work proposes a one–parameter distribution with a formulation that involves the double factorial. The new distribution is intended to be competitive with the Poisson, geometric and negative binomial distributions and, like these, belongs to the natural exponential family of distributions (EFD) and to power series distributions. The new distribution is over–dispersed and appears to be unimodal or multimodal depending of the value of its parameter. The behaviour of the rate of successive probabilities is different from that of traditional discrete distributions, in which this rate is always increasing or decreasing. Furthermore, the hazard rate function also presents unusual behaviour in comparison with the classical distribution types mentioned above. We establish the normal approximation to the distribution proposed here. Estimation methods are considered, and applications show that the distribution works well when the data sets considered present two or more modal values.

KEYWORDS

Estimation, Hazard, Natural Exponential Family, Over–dispersion, Power Series.

1. INTRODUCTION

A search of the scientific literature reveals that very little has been published on the subject of the double factorial (DF). The digital library JSTOR contains only a few papers in this respect. Some studies can be found in the web, but these are non–formal papers and contain little of interest. A more formal page has been developed by Weisstein (Double Factorial) in MathWorld. The only two journal–published papers concerning the term DF (also called semi–factorial; see Johnson and Kotz (2005, p. 2)) are those by Gould and Quaintance (2012) and by Meserve (1948). Nevertheless, these are published in divulgative journals and therefore not well known to mathematical researchers. References to the DF term also appear in specific pages of some books.

Merserve (1948) is the earliest reported use of the DF notation, which was used in order to simplify the expression of certain trigonometric integrals arising in the derivation of the Wallis product. Double factorials have many applications in number theory (in particular in enumerative combinatorics). In the present paper, initial consideration is given to the use of the DF operator in statistical distributions.

This operator is defined as follows: for an odd number $n \in N$

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

and for an even number $n \in N$

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n).$$

Furthermore, $(-1)!! = 0!! = 1$, by definition (see Arfken, 1985, p. 547). The DF is implemented in Mathematica as $n!!$ or $Factorial2[n]$. Although the DF can be extended to complex arguments, this subject is not addressed in the present paper.

Because the DF only involves about half the factors of the ordinary factorial, its value is not substantially larger than the square root of the factorial $n!$, and it is much smaller than the iterated factorial $(n!)!$. Two interesting series representations involving the DF, and which appear in Gould and Quaintance (2012), are:

$$\sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!!} = \exp\left(\frac{y^2}{2}\right), \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{y^{n}}{n!!} = \left[1 + \sqrt{\frac{\pi}{2}}(2\Phi(y)-1)\right]\exp\left(\frac{y^2}{2}\right), \tag{2}$$

where $\Phi(z)$ represents the cumulative distribution function of the standard normal distribution evaluated in z, i.e.

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{t^2}{2}\right) dt.$$

A significant finding is that series (2) contains the cumulative distribution function of the standard normal distribution.

This series representation is now used to build a one–parameter discrete distribution, the properties of which are examined in Section 2. As we will see, this distribution belongs to the natural EFD and is a power series distribution (PSD); it is unimodal or multimodal and has a closed–form expression for the moment–generating function, the mean, the variance and the moments about the mean. The normal approximation to this distribution is also established. Some estimation methods are then presented in Section 3. In particular the new distributions seems to work well when the data sets chosen present more than one modal value. This feature is characteristic in inventory control, in biological data and also in modeling the days to recover from the injuries caused by collisions caused by traffic accidents, among others. Numerical applications are then developed in Section 4 and, finally, some conclusions are drawn in Section 5.

2. THE PROPOSED DISTRIBUTION

Min and Czado (2010) observed that the Poisson distribution is too simple to capture over–dispersion (variance greater than the mean). In this sense, it is possible to obtain a distribution with this characteristic by using the above–mentioned second series.

This is so, first, from expressions (1) and (2), see Gould and Quaintance (2012) and the MathWorld web page (shown without a proof), the following result is obtained.

Theorem 1:

The equalities (1) and (2) are sustained.

Proof:

The first equality is obtained straightforwardly from the identity $(2x)!! = 2^x x!$. For the second equality, we have

$$\sum_{n=0}^{\infty} \frac{y^n}{n!!} = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!!} + \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!!}.$$

Here, the first summand is obviously (1). For the second summand, consider expression 4.1 in Albano et al. (2011), which establishes that

$$erf(u) = \frac{2}{\sqrt{\pi}} e^{-u^2} \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!!} u^{2k+1},$$

where $erf(u)$ is the error function. Now, because $erf\left(u/\sqrt{2}\right) = 1 - 2\Phi(-u)$ the result follows after some elementary algebra.

Now, it can readily be seen that

$$\Pr(X = x) = p_x = \kappa(\lambda) \frac{\lambda^x}{x!!}, \quad x = 0,1,\ldots,\ \lambda > 0, \tag{3}$$

where $\kappa(\lambda)$ is a normalization constant given by

$$\kappa(\lambda) = \sqrt{2\pi}\varphi(\lambda) \left\{ 1 - \sqrt{\frac{\pi}{2}} \left[1 - 2\Phi(\lambda) \right] \right\}^{-1},$$

and where $\varphi(u)$ and $\Phi(u)$, representing the probability density function and the cumulative distribution function of the standard normal distribution evaluated in u, respectively, define a genuine probability function (pf) for lattice data.

By using $(2x)!! = 2^x x!$, it is straightforward to see that from series (1) the Poisson distribution with parameter $\lambda > 0$ can be defined. For this reason, we shall now consider only series (2) and the pf (3) obtained from it.

Since pf (3) can be written as

$$p_x = q(x) \exp\left[-x\theta - \log\left(\vartheta(\theta) \right) \right],$$

where $q(x) = 1/x!!$, $\theta = -\log\lambda$ and $\vartheta(\theta) = \frac{1}{\kappa\left(\exp(-\theta)\right)}$, and where $-\infty < \theta < \infty$, it can be seen that the new distribution is a member of the natural EFD. Furthermore, pf (3) can also be rewritten as

$$p_x = \frac{a_x \lambda^x}{g(\lambda)},$$

where $a_x = 1/x!!$ and $g(\lambda) = \frac{1}{\kappa(\lambda)}$ and therefore it is also a PSD (see Johnson et al. (2005), p.75). Thus, we have a new distribution, together with the Bernoulli, binomial, geometric, negative binomial, Poisson and logarithmic series, within this interesting class of distributions.

The moment–generating function of a random variable following the pf (3) is given by

$$M_X(t) = \frac{\kappa(\lambda)}{\kappa\left(\lambda e^t\right)} = \frac{1-\sqrt{\pi/2}\left(1-2\Phi\left(\lambda e^t\right)\right)}{1-\sqrt{\pi/2}\left(1-2\Phi(\lambda)\right)} \exp\left[-\frac{1}{2}\lambda^2\left(1-e^{2t}\right)\right] \qquad (4)$$

and it is easy to see that the relation

$$\frac{p_x}{p_{x-1}} = \frac{\lambda x!}{(x!!)^2}, \quad x = 1, 2, \ldots, \qquad (5)$$

which involves the DF only once.

Using two expressions from Gould and Quaintance (2012), we obtain:

- When x is even, it is verified that

$$\frac{p_x}{p_{x-1}} = \frac{2\kappa(\lambda)}{\pi}\int_0^{\pi/2} \sin^x t\, dt = \frac{\kappa(\lambda)}{\pi}\frac{\Gamma\left((x+1)/2\right)}{\Gamma(x/2+1)}, \qquad (6)$$

  where π is the circular constant and $\Gamma(z)$ the gamma function.

- When x is odd, it is verified that

$$\frac{p_x}{p_{x-1}} = \kappa(\lambda)\pi\int_0^{\pi/2} \sin^x t\, dt = \frac{\kappa(\lambda)\sqrt{\pi}}{2}\frac{\Gamma\left((x+1)/2\right)}{\Gamma(x/2+1)}. \qquad (7)$$

It is not difficult to see that both (6) and (7) are decreasing functions on x. Nevertheless, the behaviour of (5) is different from that of this rate in traditional discrete distributions, where p_x / p_{x-1} is always increasing or decreasing. This rate is constant for the geometric distribution, decreasing for the Poisson distribution and can be increasing or decreasing for the negative binomial distribution according to whether the dispersion parameter is smaller or larger than 1, respectively. Figure 0 shows (5) for selected values of λ. It can be seen that the rate is always decreasing for even values of x and also for odd values of x, but when we consider all the values of x the rate changes, to be alternatively increasing and decreasing.

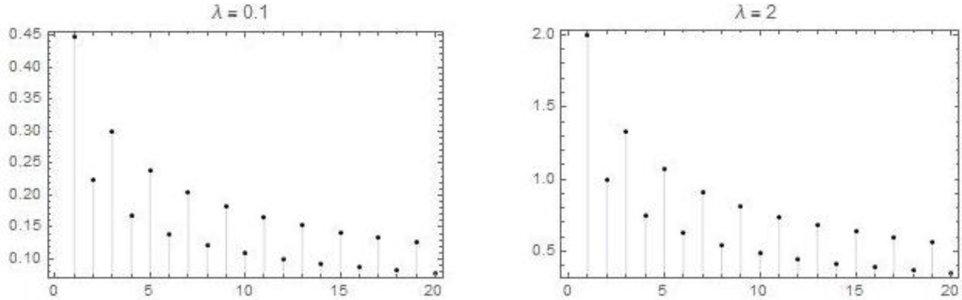

Figure 1: Plot of $\{p_x / p_{x-1}\}_{x\geq 1}$ for Selected Values of the Parameter λ

Since $\kappa(\lambda) < e^{-\lambda}$ the distribution takes a lower value in $x = 0$ than the Poisson distribution with parameter λ.

Bardwell (1960) – see also Amidi (1973) – discussed discrete probability density functions $\Pr(X = x;\lambda)$ which fit the relation

$$\frac{d\,\Pr(X = x;\lambda)}{d\lambda} = B(\lambda)\left[x - D(\lambda)\right]\Pr(X = x;\lambda). \tag{8}$$

It is shown that in this case the mean is $\mu = D(\lambda)$ and the variance is $\mu_2 = (d\mu / d\lambda)(1/B(\lambda))$. It is also shown that in this case

$$\mu_i = \mu_2\left[\frac{d\mu_{i-1}}{d\lambda}\frac{1}{d\mu/d\lambda} + (i-1)\mu_{i-2}\right], \tag{9}$$

where μ_i is the i th moment about the mean.

Expression (8) is verified for the pf (3). In this case we have

$$B(\lambda) = \frac{1}{\lambda}, \qquad D(\lambda) = -\frac{\lambda\,\kappa'(\lambda)}{\kappa(\lambda)}.$$

Thus, the mean of the variate following (3) is given by

$$\mu = E(X) = -\frac{\lambda\,\kappa'(\lambda)}{\kappa(\lambda)} = \lambda\left(\lambda + \kappa(\lambda)\right), \tag{10}$$

and the variance is

$$\mu_2 = var(X) = \lambda\left[\lambda + \kappa(\lambda) + \lambda\left(1 - \kappa(\lambda)\right)\left(\lambda + \kappa(\lambda)\right)\right], \tag{11}$$

Additionally, see Noack (1950) and Johnson et al. (2005), the following recurrence relations between the moments about the origin and the central moments (about the mean), respectively, are satisfied,

$$\mu'_{r+1} = \theta \frac{d\mu'_r}{d\theta} + \mu'_1 \mu'_r,$$

$$\mu_{r+1} = \theta \left(\frac{d\mu_r}{d\theta} + r \frac{d\mu'_1}{d\theta} \mu_{r-1} \right).$$

Furthermore, relation connecting the cumulants $k_{[r]}$ and the moments μ_r about the origin can be obtained using expression (8) in Noack (1950). Relations between factorial–cumulants and cumulants can also be given using results in Khatri (1959). See also Johnson et al. (2005), p.77.

Observe that because

$$\frac{\mu_2}{\mu} = 1 + \frac{\lambda\left[1 - \kappa(\lambda)(\lambda + \kappa(\lambda)\right]}{\lambda + \kappa(\lambda)} > 0, \tag{12}$$

the pf (3) is over–dispersed. Thus, it can be seen that the new distribution is an alternative of equi–dispersed Poisson distribution, but only to an over dispersed distribution.

In Figure 1 the mean and variance of the distribution are shown as a function of λ. Both are increasing functions on this parameter, and the difference between μ_2 and μ increases when λ increases.

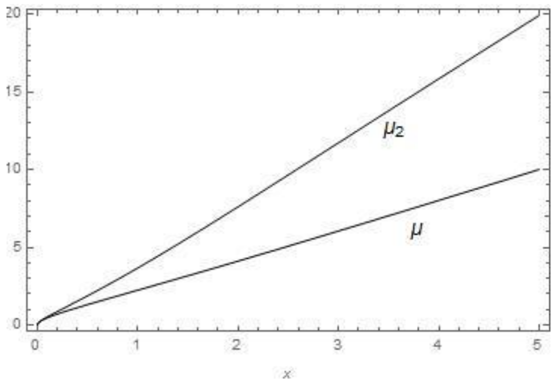

Figure 2: Mean and Variance as a Function of λ

Since pf (3) is a member of the family discussed by Bardwell (1960) we have the following result.

Theorem 2

If X follows the pf (3), then it is verified that:

1. *Mean deviation* $= E|X-\mu| = -2\lambda\left(1-\lambda^2\right)\frac{\partial}{\partial\lambda}F\left([\mu],\lambda\right)$.

2. $\Pr(X = x;\lambda) = \eta \exp\left\{\int \frac{x-\mu}{\mu_2}\frac{d\mu}{d\lambda}d\lambda\right\}$.

Here, $F\left([\mu],\lambda\right) = \sum_{x=0}^{[\mu]}\Pr(X = x;\lambda)$, where $[\cdot]$ represents the integer part and η is a constant of normalization.

Proof:

The results follow directly, using Theorems 2 and 3 in Bardwell (1960).

Examples of pf (3) for special cases of parameter λ are shown in Figure 2. This Figure also shows that the new pf is, as the Poisson distribution, versatile in the sense that different values of the parameter λ provide a different value of the modal value. It seems that for $\lambda < 0$ the pf has a zero vertex moving the mode(s) to the right as λ increases.

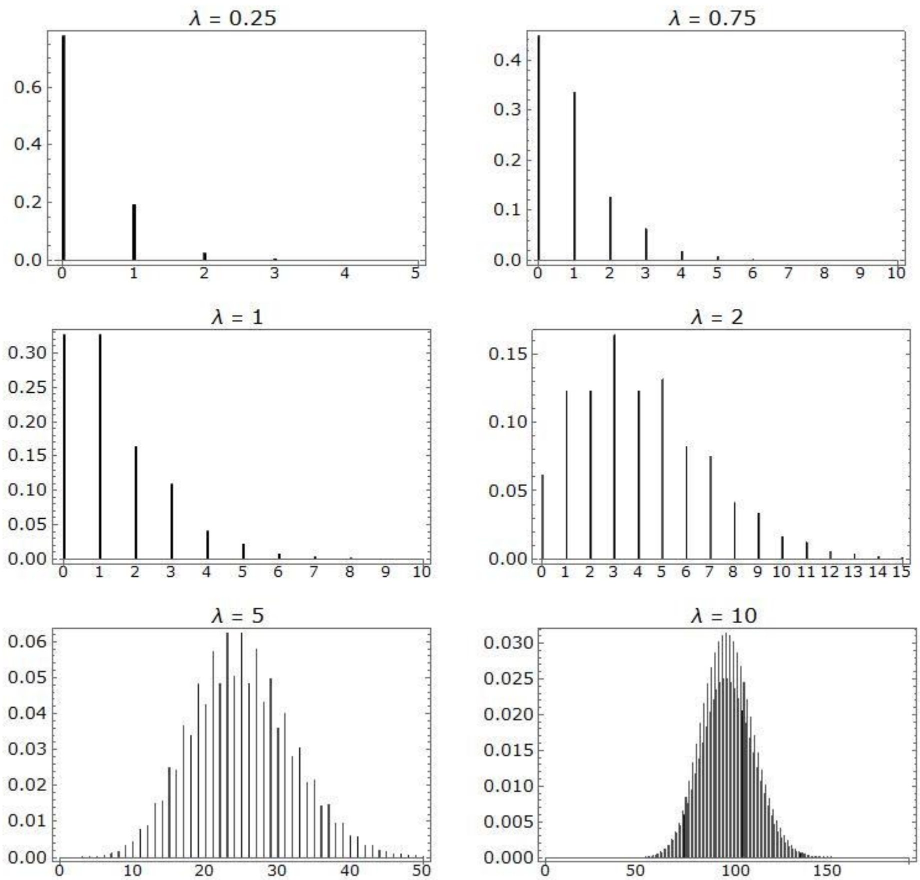

Figure 3: Examples of pf (3) for Special Cases of Parameter λ

Thus, the proposed distribution can be unimodal or multimodal.

According to Bardwell (1960), a characteristic of the family of functions satisfying (8) is that each has a unique maximum. That is, the modal value is achieved at $x_{mode} = [\mu]-1$. The behaviour of the value of the probabilities also seems to be completely different from that found in classical discrete distributions belonging to the natural EFD and to PSD. We can see that, as the parameter λ increases, the shape rapidly tends towards the familiar bell of the normal distribution. This is established in the following result.

Theorem 3

The limiting distribution of the pf (3) as λ tends to infinity is normal.

Proof:

Let X follow the pf (3). From (4)

$$\lim_{\lambda\to\infty} E\left[\exp\left(\frac{t(X-\mu)}{\sqrt{\mu_2}}\right)\right] = \exp\left(-\frac{t\mu}{\sqrt{\mu_2}}\right)\lim_{\lambda\to\infty} E\left\{\exp\left(-\frac{tX}{\sqrt{\mu_2}}\right)\right\}$$
$$= \exp\left(-\frac{t\mu}{\sqrt{\mu_2}}\right)\lim_{\lambda\to\infty} \cdot \exp\left[\frac{1}{2}\lambda^2\left(1-e^{2t/\sqrt{\mu_2}}\right)\right],$$

since

$$\lim_{\lambda\to\infty} \frac{1-\sqrt{\pi/2}\left(1-2\Phi\left(\lambda e^{t/\sqrt{\mu_2}}\right)\right)}{1-\sqrt{\pi/2}\left(1-2\Phi(\lambda)\right)} = 1.$$

Then, after some algebra the above limit can be written as

$$\lim_{\lambda\to\infty} \exp\left[-\frac{t\lambda\kappa(\lambda)}{\sqrt{\mu_2}}+\frac{(t\lambda)^2}{\mu_2}+\frac{2t^3\lambda^2}{3\mu_2^{3/2}}+\ldots\right]. \quad (13)$$

Finally, taking into account that

$$\lim_{\lambda\to\infty}\kappa(\lambda) = 0, \quad \lim_{\lambda\to\infty}\frac{\lambda\kappa(\lambda)}{\sqrt{\mu_2}} = 0,$$
$$\lim_{\lambda\to\infty}\frac{\lambda^2}{\mu_2} = \frac{1}{2}, \quad \lim_{\lambda\to\infty}\frac{\lambda^2}{\mu_2^{3/2}} = 0,$$

and that the other summands in (13) are also zero, we obtain

$$\lim_{\lambda\to\infty} E\left[\exp\left(\frac{t(X-\mu)}{\sqrt{\mu_2}}\right)\right] = \lim_{\lambda\to\infty} E\left[\exp\left(\frac{t^2}{2}\right)\right],$$

which is the moment generating function of a standard normal distribution and therefore the theorem is proved.

Additionally, the entropy is found to be

$$H_\lambda = -\log \kappa(\lambda) - \lambda\left(\lambda + \kappa(\lambda)\right)\log\lambda - \kappa(\lambda)\sum_{x=0}^{\infty} \frac{\lambda^x \log x!!}{x!!}.$$

To end this section, observe that since the pf (3) belongs to the EFD, there is a conjugate family of priors, which is given by

$$g(\theta) = \exp\left\{-\theta x_0 - n_0 \log\left(\vartheta(\theta)\right) + \log d\left(n_0, x_0\right)\right\},$$

where the normalization constant is given by

$$d\left(n_0, x_0\right) = \int_{-\infty}^{\infty} \exp\left\{-\theta x_0 - n_0 \log\left(\vartheta(\theta)\right)\right\} d\theta. \tag{14}$$

In terms of parameter λ this prior distribution can be rewritten as

$$\pi(\lambda) \propto \int_0^{\infty} \left\{1 - \sqrt{\frac{\pi}{2}}\left[1 - 2\Phi(\lambda)\right]\right\}^{-1} \lambda^{x_0 - 1} e^{-n_0 \lambda^2/2}\, d\lambda,$$

which has a non–closed form. Nevertheless, the prior distribution is reminiscent of the gamma distribution.

2.1 Skewness, kurtosis and hazard rate

Some important indices of the shape of the distribution, apart from the mean and the variance, are the skewness $\left(\sqrt{\beta_1} = \mu_3 / (\mu_2)^{3/2}\right)$ and the kurtosis $\left(\beta_2 = \mu_4 / (\mu_2)^2\right)$. Expressions for these two indices can be given in closed form using (9) but they are very large and therefore not presented here. In Figure 3, these two indices are used to show the skewness and kurtosis of the proposed distribution versus that of the Poisson distribution. It can be seen that for the proposed distribution both values can be smaller or larger those for the Poisson distribution.

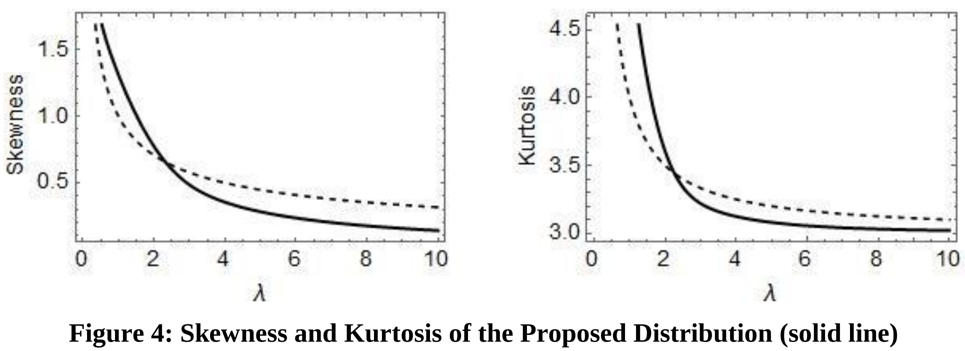

Figure 4: Skewness and Kurtosis of the Proposed Distribution (solid line) and POISSON Distribution (dashed line)

The cumulative distribution function, $F(x) = \Pr(X \leq x)$, which is not presented here, is obtained in closed form but it is too large. The survival function is obtained from the cumulative distribution function, by $\bar{F}(x) = 1 - F(x-1)$, from which we obtain the failure rate given by $r(x) = p_x / \bar{F}(x)$. This function also presents a strange behaviour pattern, being simultaneously increasing and decreasing.

3. METHODS OF ESTIMATION

Assume a sample of n independent observations given by $x_1, x_2, \ldots, x_n$ from the pf (3). Expression (10) can be used estimate the parameter λ by the method of moments and it is obvious that the solution of the equation involved cannot be expressed in closed form. A numerical search of the equation can be performed by directly solving the equations involved using Mathematica. Since expression (10) is an increasing function from 0 to infinity, a unique solution is guaranteed for $\bar{x} > 0$, where $\bar{x}$ is the sample mean. Furthermore, the moment estimator of λ can also be obtained directly from Table 0, where the exact value of the mean is shown for different values of the parameter λ.

Table 1
Mean Value of the pf for Selected Values of the Parameter

λ	μ	λ	μ
0.1	0.482434	2.0	4.123240
0.2	0.725114	3.0	6.054560
0.3	0.936829	4.0	8.023050
0.4	1.135280	5.0	10.009500
0.5	1.326860	6.0	12.003800
0.6	1.514640	7.0	14.001500
0.7	1.700300	8.0	16.000600
0.8	1.884860	9.0	18.000200
0.9	2.068960	10.0	20.000100
1.0	2.253020	20.0	40.000000

A numerical interpolation method can then be used to derive, in an exact form, the estimator of λ.

However, instead of moment estimation we can also use the fact that

$$E\left[\frac{X!!}{(X-1)!!}\right] = \lambda,$$

a result which is obtained by taking into account that from the PSD it is verified (see Papathanasiou (1993)) that

$$E\left(\frac{a_{X-1}}{a_X}\right) = \lambda.$$

Then, if we can use the sample value $\tilde{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i!!}{(x_i-1)!!}$ the estimator of the parameter λ is just $\tilde{p}$.

Let us now consider the maximum likelihood method. The log–likelihood equation is proportional to

$$\ell\left(x_1, x_2, \ldots, x_n; \lambda\right) \propto n \log \kappa(\lambda) - \frac{n\lambda^2}{2} + n\bar{x}\log(\lambda), \tag{15}$$

where $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$ is the sample mean. By deriving (15) we obtain the equation

$$-\frac{\lambda\,\kappa'(\lambda)}{\kappa(\lambda)} + \bar{x} = 0. \tag{16}$$

Hence, the maximum likelihood estimator of λ coincides with the moment estimator and therefore is also unique. This was established by Patil (1962). Since the new distribution belongs to the family of PSD, it is verified (see Johnson et al. (2005) that the maximum likelihood estimator of λ is a function of $T = \sum_{i=1}^{n} x_i$ and therefore T is sufficient for λ.

Some algebra provides

$$I(\lambda) = E\left(-\frac{\partial^2 \ell}{\partial\lambda^2}\right) = n + \left(\lambda + \kappa(\lambda)\right)\left(n\left(1+\kappa(\lambda)\right) + \frac{1}{\lambda}\right).$$

Proposition 1

The unique maximum likelihood estimator $\hat{\lambda}$ *of* λ *is consistent and asymptotically normal and therefore*

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow{d} N\left(0, I^{-1}(\lambda)\right),$$

where $I(\lambda)$ is the Fisher information about λ.

Proof:

The distribution satisfies the regularity conditions provided in Lehmann and Casella (1998, p. 449) and Krishna and Pundir (2009), under which the unique maximum likelihood estimator $\hat{\lambda}$ of λ is consistent and asymptotically normal. These conditions are verified as follows. Firstly, the parameter space $(0,\infty)$ is a subset of the real line and the range of x is independent of λ. Now, it is easy to show that $E\left(\frac{\partial \log p_x}{\partial\lambda}\right) = 0$ and due to the uniqueness of the maximum likelihood estimator, $\left.\frac{\partial^2 \ell}{\partial \lambda^2}\right|_{\lambda=\hat{\lambda}} < 0$, the Fisher information is positive. Some algebra provides that

$$\frac{\partial^3 \ell}{\partial \lambda^3} = n\left[\frac{\kappa'''(\lambda)}{\kappa(\lambda)} - \frac{\kappa''(\lambda)\kappa'(\lambda)}{[\kappa(\lambda)]^2} - 2\kappa'(\lambda)\kappa(\lambda)\left(\lambda+\kappa(\lambda)\right)\left(\lambda+2\kappa(\lambda)\right) + \frac{2\bar{x}}{\lambda^3}\right], \qquad (17)$$

and, since $\kappa(\lambda) > 0, \kappa'(\lambda) < 0, \kappa''(\lambda) > 0$ and $\kappa'''(\lambda) < 0$, by taking $M(x)$ the positive summands of (17) we have that $\left|\frac{\partial^3 \log p_x}{\partial \lambda^3}\right| \leq M(x)$, where obviously $E\left(M(x)\right)$ is finite. Thus, the proposition is proved.

In conclusion, by using Corollary 3.11 in Lehmann and Casella (1998, p.450), we conclude that the unique root of equation (16) is asymptotically efficient.

4. APPLICATIONS

In this section, the distribution described above is illustrated by three sets of data. Two of them in the biological field and the other in inventory control. The first data set concerns distributions of Microcolanus nauplii in samples of marine plankton, considered by Bliss and Fisher (1953), and the second refers to haemacytometer yeast cell counts, discussed in Bardwell and Crow (1964). Biological data is generally concerned with plant or animal counts obtained for each of a set of equal units of space or time. These kind of count data can be fitted initially by the Poisson distribution, but in practice they present over–dispersion phenomena. Ross and Preece (1985) pointed out that over–dispersion arises when the organisms are clumped, clustered or aggregated in space or time. Similar arguments are employed by Clapham (1936). The third set of data appears in (Cardós et al. (2013) and corresponds to the daily demand of a class of item. The Poisson distribution with parameter $\theta > 0$, the negative binomial (Poisson–Gamma distribution) with paramters $r > 0$ and $0 < p < 1$ and the geometric distribution (a special case of the Poisson–Gamma distribution) with parameter $r > 0$ are also fitted to the empirical distributions for comparison. The maximum likelihood estimates were calculated using Mathematica package. The data and the fitted values are shown in Tables 1, 2 and 3, together with the rest of distributions, fitted by maximum likelihood. In parenthesis appear the standard errors.

Table 2
Observed and Fitted Distributions of Microcolanusnauplii in Samples of Marine Plankton (Bliss and Fisher (1953)).

Counts		Observed		Fitted	
		Poisson	Geometric	Negative Binomial	New Distribution
0	0	0.01	14.15	0.14	0.55
1	2	0.09	12.81	0.76	1.70
2	4	0.46	11.60	2.14	2.64
3	3	1.49	10.51	4.33	5.45
4	5	3.59	9.52	7.09	6.33
5	8	6.90	8.62	9.95	10.46
6	16	11.04	7.81	12.38	10.13
7	13	15.14	7.07	14.03	14.34
8	12	18.17	6.40	14.72	12.14
9	13	19.38	5.80	14.49	15.27
10	15	18.61	5.25	13.51	11.64
11	15	16.24	4.75	12.02	13.31
12	9	12.99	4.31	10.26	9.30
13	9	9.59	3.90	8.45	9.82
14	7	6.58	3.53	6.75	6.37
15	4	4.21	3.20	5.23	6.27
16	4	2.52	2.90	3.91	3.81
17	6	1.42	2.62	2.92	3.54
18	2	0.76	2.37	2.11	2.03
19	0	0.38	2.15	1.50	1.78
20	2	0.18	1.95	1.05	0.97
21	1	0.08	1.76	0.72	0.81
22	0	0.03	1.60	0.48	0.42
AIC		888.259	998.649	860.307	856.474
χ^2		> 40	> 40	7.78	8.54
d.f.		13	13	13	13
p-value		0.00%	0.00%	80.19%	80.65%
$\hat{r}$			0.094 (0.07)	11.128 (2.890)	
$\hat{p}$				0.536 (0.065)	
$\hat{\theta}$		9.60 (0.252)			
$\hat{\lambda}$					3.096 (0.057)

Table 3
Observed and Fitted Yeast in 400 Squares of Haemacytometer
(Bardwell and Crow (1964))

Counts		**Observed**		**Fitted**	
		Poisson	**Geometric**	**Negative Binomial**	**New Distribution**
0	213	202.14	237.74	214.15	212.94
1	128	137.96	96.44	122.79	128.14
2	37	47.08	39.12	45.019	38.55
3	18	10.71	15.86	13.40	15.47
4	3	1.83	6.43	3.53	3.49
5	1	0.25	2.61	0.85	1.12
AIC		904.999	914.857	905.046	896.385
χ^2		10.20	16.12	3.26	0.56
d.f.		3	3	3	3
p -value		1.69%	0.10%	19.53%	90.60%
$\hat{r}$				3.586 (1.750)	
$\hat{p}$			0.594 (0.019)	0.840 (0.066)	
$\hat{\theta}$		0.682 (0.041)			
$\hat{\lambda}$					0.601 (0.033)

Table 4
Observed and Fitted Distributions of Daily Demand of a class of item (Cardós et al. (2013))

Counts		Observed		Fitted	
		Poisson	Geometric	Negative Binomial	New Distribution
0	18	2.36	130.98	15.86	21.36
1	53	14.06	112.15	45.65	51.87
2	81	41.88	96.02	77.51	63.00
3	95	83.13	82.22	101.08	102.00
4	88	123.77	70.39	111.91	92.88
5	105	147.40	60.27	110.68	120.31
6	107	146.30	51.61	100.75	91.31
7	92	124.46	44.18	86.05	101.37
8	76	92.64	37.83	69.85	67.31
9	60	61.30	32.39	54.42	66.43
10	42	36.50	27.73	40.96	39.70
11	37	19.76	23.75	29.96	35.62
12	21	9.80	20.33	21.37	19.51
13	11	4.49	17.41	14.92	16.16
14	8	1.91	14.90	10.22	8.22
15	8	0.75	12.76	6.89	6.35
16	3	0.28	10.92	4.57	3.03
17	3	0.09	9.35	3.00	2.20
18	0	0.03	8.01	1.94	0.99
19	1	0.01	6.86	1.24	0.68
20	0	0.00	5.83	0.79	0.29
21	2	0.00	5.02	0.49	0.19
22	0	0.00	4.30	0.31	0.07
23	0	0.00	3.68	0.19	0.05
24	0	0.00	3.15	0.11	0.02
AIC		5034.32	5224.75	4759.77	4754.50
χ^2		>100	>100	13.75	16.35
d.f.		15	15	15	15
p-value		0.00%	0.00%	46.81%	35.91%
$\hat{r}$			0.143 (0.004)	5.574 (0.531)	
$\hat{p}$				0.483 (0.024)	
$\hat{\theta}$		5.955 (0.080)			
$\hat{\lambda}$					2.428 (0.023)

These Tables also show the estimators (obtained by the maximum likelihood method), the χ^2 statistics, the p–values and the AIC (Akaike Information Criteria). With respect to all examples considered, it should be emphasized that this new and more versatile distribution allows us to accommodate the change of monotony in the value of the probabilities. The proposed pf provides a good fit in the three cases presented, on the basis of the AIC, χ^2 statistics and the corresponding p–values.

In order to complete this work, we have fitted the previous models to the data included in Ayuso et al. (2013) and corresponding to the days to recover from the injuries caused by collisions caused by traffic accidents. The results obtained are similar to the previous ones, and are available by the authors on request.

5. CONCLUSIONS

This paper offers a new discrete distribution, belonging to the EFD and also to the PSD. The new distribution is competitive with the Poisson distribution, with respect to which it is over–dispersed, which is a feature of most data sets to be found in the literature. A novel contribution of this distribution is that the expression of the pf involves the DF function, which has not previously been considered in this setting. The behaviour of the rate of successive probabilities is different from that of traditional discrete distributions, where this rate is always increasing or decreasing. As occurs with the Poisson distribution, the proposed distribution is approximately normal when the parameter tends to infinity. Moment and maximum likelihood estimators are easy to compute. We conclude that the proposed distribution is better than the Poisson distribution for fitting discrete data sets.

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