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BAYESIAN ESTIMATION OF EXPONENTIATED PARETO DISTRIBUTION

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ABSTRACT

This paper develops Bayesian estimators of two unknown parameters of Exponentiated Pareto distribution under various loss functions. Bayesian estimators cannot be obtained in closed forms. Lindley's approximation is suggested to compute the approximate Bayesian estimates. Bayesian estimates are compared with their maximum likelihood counterparts via Monte Carlo simulation. A real data is analyzed for illustrative purposes.

KEYWORDS

Exponentiated Pareto distribution; Maximum Likelihood Estimators; Bayesian estimators; Lindley's approximation; LINEX loss function; Squared Error loss function; General Entropy loss function; Precautionary loss function; Monte Carlo simulation

1. INTRODUCTION

The Exponentiated Pareto distribution (EPD) was introduced by Gupta *et al.* (1998) as a life time model. EPD can be defined by raising the cumulative distribution function (CDF) of a Pareto distribution to a positive power. It is used in insurance–risk, occurrence of natural resources and business failures. The CDF of EPD is

$$F\left(x,\alpha,\theta\right)=\left[1-\left(1+x_i\right)^{-\alpha}\right]^{\theta}, x>0,\alpha,\theta>0, \tag{1}$$

where α, θ are two shape parameters. The corresponding probability density function (PDF) is

$$f\left(x,\alpha,\theta\right)=\left[1-\left(1+x_i\right)^{-\alpha}\right]^{\theta-1}\left(1+x_i\right)^{-(\alpha+1)}, x>0,\alpha,\theta>0. \tag{2}$$

Parameter estimation plays an important role in classical and Bayesian inference. Several estimation methods have been proposed to estimate the parameters of distributions. A method of estimation must be chosen which minimizes sampling errors. EPD has been studied quite extensively by several authors, e.g., Shawky and Abu-Zinadah (2009) and, Hasan and Basheikh (2012) studied different methods of estimations and concluded that maximum likelihood estimators (MLE) are best for estimating the parameters of EPD. Similarly, Sing *et al.* (2013) compared MLEs and corresponding Bayesian estimators of EPD in terms of their risks based on simulated samples for Progressive Type-II censored. Moreover, Ahmadi *et al.* (2010), Soliman (2008) and Ghafoori *et al.* (2011) considered Bayesian estimation using censored data.

The objective of this paper is to develop the Bayesian estimators using informative and non-informative priors considering different loss functions. Including this introduction section, the rest of the paper is arranged as follows. Section 2, comprises the derivation of MLEs and the observed Fisher information matrix. Bayesian estimators are developed under different loss functions by taking both informative and non-informative priors in Section 3. Simulation study is carried out in Section 4 and a real data set is analyzed in Section 5 for illustrative purposes. Finally, conclusion is given in Section 6.

2. MAXIMUM LIKELIHOOD ESTIMATORS

Suppose that $X_1, X_2, ..., X_n$ be the set of n random life times from EPD with parameters α and θ. The likelihood function of equation (2) is

$$L(x,\alpha,\theta)=\alpha^n\theta^n\prod_{i=1}^{n}\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\prod_{i=1}^{n}(1+x_i)^{-(\alpha+1)}, \tag{3}$$

The log-likelihood function is

$$\log L(x,\alpha,\theta)=n\log(\alpha)+n\log(\theta)+(\theta-1)\sum_{i=1}^{n}\log\left(1-(1+x_i)^{-\alpha}\right)$$
$$-(\alpha+1)\sum_{i=1}^{n}\log(1+x_i)^{-(\alpha+1)}, \tag{4}$$

from (4), we have

$$\frac{\partial\log L(x,\alpha,\theta)}{\partial\theta}=\frac{n}{\alpha}+(\theta-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}\log(1+x_i)}{1-(1+x_i)^{-\alpha}}-\sum_{i=1}^{n}\log(1+x_i), \tag{5}$$

$$\frac{\partial\log\mathrm{L}(x,\alpha,\theta)}{\partial\theta}=\frac{n}{\theta}+\sum^{n}\log\left(1-(1+x_i)\right)^{-\alpha}. \tag{6}$$

Obviously, the above equations cannot be written in a closed form. BFGS (Broyden Fletcher Gold Farb Shanno, Battiti and Masulli (1990)) quasi-Newton optimization method is applied to compute the MLEs. The observed Fisher information matrix is obtained by taking the second and mixed partial derivatives with respect to α and θ respectively. Therefore, the observed Fisher information matrix may be written as

$$I(\alpha,\theta)=\begin{pmatrix}\frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\alpha^2} & \frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\alpha\,\partial\theta}\\ \frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\theta\,\partial\alpha} & \frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\theta^2}\end{pmatrix},$$

where

$$\frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\alpha^2}=-\frac{n}{\alpha^2}-(\theta-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\alpha}\right)^2},$$

$$\frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\theta^2}=-\frac{n}{\theta^2},\frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\alpha\partial\theta}=\frac{\partial^2 \log L(x,\alpha,\theta)}{\partial\theta\partial\alpha}$$

$$=\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}\log(1+x_i)}{1-(1+x_i)}.$$

3. BAYESIAN ESTIMATION

In Bayesian estimation, we consider four types of loss functions. The first is the LINEX (linear exponential) loss function which is asymmetric. It has been introduced by Varian (1975) and this loss function has been used by Abu-Zinadah (2010) and Afify (2010) in different estimation problems. The second is the generalization of the entropy loss function which is discussed by Preda *et al.* (2010). Furthermore, the third one is squared error loss function which is discussed by Nadar *et al.* (2012). Moreover, fourth one is Precautionary loss function which is used by Pandey and Rao (2009) as a symmetric loss function. For Bayesian estimation, we need prior distributions of α and θ. Assuming that α and θ each have independent Gamma (a_1,b_1) and Gamma (a_2,b_2) priors respectively for $a_1,b_1,a_2,b_2>0$, i.e.,

$$\pi_1(\alpha)\propto\alpha^{a_1-1}e^{b_1\alpha}\ \text{ and }\ \pi_2(\theta)\propto\alpha^{a_2-1}e^{b_2\alpha}.$$

And for Levy prior α and $\theta>0$ i.e.,

$$\pi_1(\alpha)\propto\alpha^{\frac{1}{2}}e^{-\frac{\alpha}{2}}\ \text{ and }\ \pi_2(\theta)\propto\alpha^{-\frac{3}{2}}e^{-\frac{1}{2\theta}}.$$

The joint posterior density of α and θ can be written as

$$\pi(\alpha,\theta)=\frac{L(x,\alpha,\theta)\times\pi_1(\alpha)\times\pi_2(\theta)}{\int_0^\infty\int_0^\infty L(x,\alpha,\theta)\times\pi_1(\alpha)\times\pi_2(\theta)\,d\alpha d\theta}. \tag{7}$$

Therefore, the Bayesian estimators of any function of α and θ, say $g(\alpha,\ \theta)$, under any loss function is

$$\pi(\alpha,\theta)=\frac{\int_0^\infty\int_0^\infty g(\alpha,\theta)\times L(x,\alpha,\theta)\times\pi_1(\alpha)\times\pi_2(\theta)\,d\alpha d\theta}{\int_0^\infty\int_0^\infty L(x,\alpha,\theta)\times\pi_1(\alpha)\times\pi_2(\theta)\,d\alpha d\theta}. \tag{8}$$

It is not possible for (8) to have a close form. Therefore, we use Lindley's approximation (1980) to approximate the ratio of the two integrals such as (8), which can be written as

$$\hat{g}=\hat{g}\left(\hat{\alpha},\hat{\theta}\right)+\frac{1}{2}\left[\sum_{i=1}^{2}\sum_{j=1}^{2}v_{ij}\tau_{ij}+l_{30}B_{12}+l_{03}B_{21}+l_{21}C_{12}+l_{12}C_{21}\right]+w_1A_{12}+w_2A_{21}, \quad (9)$$

$$l_{ij}=\frac{\partial^{i+j}L(\alpha,\theta)}{\partial\alpha^i\partial\theta^j}, i,j=0,1,2,3 \text{ where } i+j=3,$$

$$w_1=\frac{\partial\log\pi(\alpha,\theta)}{\partial\alpha}, w_2=\frac{\partial\log\pi(\alpha,\theta)}{\partial\theta}, v_{12}=\frac{\partial^2 g(\alpha,\theta)}{\partial\alpha\partial\theta}, v_{21}=\frac{\partial^2 g(\alpha,\theta)}{\partial\theta\partial\alpha},$$

$$v_{11}=\frac{\partial^2 g(\alpha,\theta)}{\partial\alpha^2}, v_{22}=\frac{\partial^2 g(\alpha,\theta)}{\partial\theta^2}, v_1=\frac{\partial g(\alpha,\theta)}{\partial\alpha}, v_2=\frac{\partial g(\alpha,\theta)}{\partial\theta},$$

$$B_{ij}=\left(v_i\tau_{ii}+v_j\tau_{ij}\right)\tau_{ii}, A_{ij}=v_i\tau_{ii}+v_j\tau_{ji}; i,j=1,2.$$

$$C_{ij}=3v_i\tau_{ii}\tau_{ij}+v_j\left(\tau_{ii}\tau_{jj}+2\tau_{ij}^2\right); i,j=1,2,$$

$$p=-\frac{\partial^2\log L(x,\alpha,\theta)}{\partial\alpha^2}, q=-\frac{\partial^2\log L(x,\alpha,\theta)}{\partial\theta^2},$$

and $$r=-\frac{\partial^2\log L}{\partial\alpha\partial\theta}=-\frac{\partial^2\log L}{\partial\theta\partial\alpha}, \tau_{11}=\frac{q}{pq-r^2}, \tau_{22}=\frac{p}{pq-r^2}, \tau_{12}=\tau_{21}=\frac{r}{pq-r^2},$$

where L(.) is the log-likelihood function of the observed data, τ_{ij} is the (i, j) th element of the inverse of Fisher information matrix. Therefore, the approximate Bayesian estimators of α and θ under different loss functions are presented from 10 to 33.

$$\begin{aligned}\hat{\alpha}_{BLGP}=&-\frac{1}{k}\log\Bigg[e^{-k\hat{\alpha}}+\frac{1}{2}\Bigg\{\left(k^2e^{-k\hat{\alpha}}\right)\tau_{11}\\&-\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{11}^2\\&-\left(\frac{2n}{\theta^3}\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{21}\tau_{22}+3\tau_{11}\tau_{12}\left(ke^{-k\hat{\alpha}}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\Bigg\}\\&-\left(\frac{a_1-1}{\hat{\alpha}}-b_1\right)\left(ke^{-\hat{\alpha}}\right)\tau_{11}-\left(\frac{a_2-1}{\hat{\theta}}-b_2\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{12}\Bigg],\end{aligned} \quad (10)$$

$$\hat{\theta}_{BLGP} = -\frac{1}{k}\log\left[e^{-k\hat{\theta}} + \frac{1}{2}\left\{\left(k^2 e^{-k\hat{\theta}}\right)\tau_{22}\right.\right.$$

$$-\left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{11}\tau_{12}$$

$$\left.-\left(\frac{2n}{\hat{\theta}^3}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{22}{}^2 + \left(\tau_{11}\tau_{12} + 2\tau_{12}{}^2\right)\left(ke^{-k\hat{\theta}}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\right\}$$

$$\left.-\left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)\left(ke^{-k\hat{\theta}}\right)\tau_{21} - \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)\left(ke^{-k\hat{\theta}}\right)\tau_{22}\right]. \quad (11)$$

$$\hat{\alpha}_{BLLP} = -\frac{1}{k}\log\left[e^{-k\hat{\alpha}} + \frac{1}{2}\left\{\left(k^2 e^{-k\hat{\alpha}}\right)\tau_{11}\right.\right.$$

$$-\left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{11}{}^2$$

$$\left.-\left(\frac{2n}{\theta^3}\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{21}\tau_{22} + 3\tau_{11}\tau_{12}\left(ke^{-k\hat{\alpha}}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\right\}$$

$$\left.-\frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\hat{\theta}}\right)\left(ke^{-\hat{\alpha}}\right)\tau_{11} - \left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^2}\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{12}\right], \quad (12)$$

$$\hat{\theta}_{BLLP} = -\frac{1}{k}\log\left[e^{-k\hat{\theta}} + \frac{1}{2}\left\{\left(k^2 e^{-k\hat{\theta}}\right)\tau_{22}\right.\right.$$

$$-\left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{11}\tau_{12}$$

$$\left.-\left(\frac{2n}{\hat{\theta}^3}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{22}{}^2 + \left(\tau_{11}\tau_{12} + 2\tau_{12}{}^2\right)\left(ke^{-k\hat{\theta}}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\right\}$$

$$\left.-\frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\hat{\theta}}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{21} - \frac{1}{2}\left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^2}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{22}\right]. \quad (13)$$

$$\hat{\alpha}_{BLUP} = -\frac{1}{k}\log\Bigg[e^{-k\hat{\alpha}} + \frac{1}{2}\Bigg\{ \left(k^2 e^{-k\hat{\alpha}}\right)\tau_{11} - \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right)\left(ke^{-k\hat{\alpha}}\right)\tau_{11}^{2} - \left(\frac{2n}{\theta^3}\right)\left(ke^{-k\hat{\alpha}}\right)\tau_{21}\tau_{22} + 3\tau_{11}\tau_{12}\left(ke^{-k\hat{\alpha}}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \Bigg\}\Bigg], \quad (14)$$

$$\hat{\theta}_{BLUP} = -\frac{1}{k}\log\Bigg[e^{-k\hat{\theta}} + \frac{1}{2}\Bigg\{ \left(k^2 e^{-k\hat{\theta}}\right)\tau_{22} - \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right)\left(ke^{-k\hat{\theta}}\right)\tau_{11}\tau_{12} - \left(\frac{2n}{\hat{\theta}^3}\right)\left(ke^{-k\hat{\theta}}\right)\tau_{22}^{2} + \left(\tau_{11}\tau_{12} + 2\tau_{12}^{2}\right)\left(ke^{-k\hat{\theta}}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \Bigg\}\Bigg]. \quad (15)$$

$$\hat{\alpha}_{BGEGP} = \Bigg[\hat{\alpha}^{-k} + \frac{1}{2}\Bigg\{ \left(k(k+1)\hat{\alpha}^{-(k+2)}\right)\tau_{11} - \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{11}^{2} - \left(\frac{2n}{\hat{\theta}^3}\right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{21}\tau_{22} + 3\tau_{11}\tau_{12}\left(k\hat{\alpha}^{-(k+1)}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \Bigg\} - \left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{11} - \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{12} \Bigg]^{-\frac{1}{k}}, \quad (16)$$

$$\hat{\theta}_{BGEGP} = \left[\hat{\theta}^{-k} + \frac{1}{2} \left\{ k(k+1)\left(\hat{\theta}^{-(k+2)}\right)\tau_{22} \right.\right.$$

$$- \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right)\left(k\hat{\theta}^{-(k+1)}\right)\tau_{11}\tau_{12}$$

$$\left. - \left(\frac{2n}{\hat{\theta}^3}\right)\left(k\hat{\theta}^{-(k+1)}\right)\tau_{22}{}^2 + \left(\tau_{11}\tau_{12} + 2\tau_{12}{}^2\right)\left(k\hat{\theta}^{-(k+1)}\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \right\}$$

$$\left. - \left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)\left(k\hat{\theta}^{-(k+1)}\right)\tau_{21} - \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)\left(k\hat{\theta}^{-(k+1)}\right)\tau_{22} \right]^{-\frac{1}{k}}. \qquad (17)$$

$$\hat{\alpha}_{BGELP} = \left[\hat{\alpha}^{-K} + \frac{1}{2} \left\{ \left(k(k+1)\hat{\alpha}^{-(k+2)}\right)\tau_{11} \right.\right.$$

$$- \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{11}{}^2$$

$$\left. - \left(\frac{2n}{\hat{\theta}^3}\right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{21}\tau_{22} + 3\tau_{11}\tau_{12}\left(k\hat{\alpha}^{-(k+1)}\right)\sum_{i=1}^{n} \frac{(1+x_i)^{-\alpha}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \right\}$$

$$\left. - \left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\hat{\theta}}\right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{11} - \left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^2}\right)\left(k\hat{\alpha}^{-(k+1)}\right)\tau_{12} \right]^{-\frac{1}{k}}, \qquad (18)$$

$$\hat{\theta}_{BGELP} = \left[\hat{\theta}^{-K} + \frac{1}{2} \left\{ \left(k\left(k+1\right)\hat{\theta}^{-(k+2)} \right) \tau_{22} \right. \right.$$

$$- \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right) \sum_{i=1}^{n} \frac{\left(1+x_i\right)^{-\hat{\alpha}} \left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right) \left(\log\left(1+x_i\right)\right)^3}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^3} \right) \left(k\hat{\theta}^{-(k+1)}\right) \tau_{11}\tau_{12}$$

$$\left. - \left(\frac{2n}{\hat{\theta}^3}\right) \left(k\hat{\theta}^{-(k+1)}\right) \tau_{22}{}^2 + \left(\tau_{11}\tau_{12} + 2\tau_{12}{}^2\right) \left(k\hat{\theta}^{-(k+1)}\right) \sum_{i=1}^{n} \frac{\left(1+x_i\right)^{-\hat{\alpha}} \left(\log\left(1+x_i\right)\right)^2}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^2} \right\}$$

$$\left. - \frac{1}{2} \left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\hat{\theta}} \right) \left(k\hat{\theta}^{-(k+1)}\right) \tau_{21} - \left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^2} \right) \left(k\hat{\theta}^{-(k+1)}\right) \tau_{22} \right]^{-\frac{1}{k}}. \tag{19}$$

$$\hat{\alpha}_{BGEUP} = \left[\hat{\alpha}^{-K} + \frac{1}{2} \left\{ \left(k\left(k+1\right)\hat{\alpha}^{-(k+2)} \right) \tau_{11} \right. \right.$$

$$- \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right) \sum_{i=1}^{n} \frac{\left(1+x_i\right)^{-\hat{\alpha}} \left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right) \left(\log\left(1+x_i\right)\right)^3}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^3} \right) \left(k\hat{\alpha}^{-(k+1)}\right) \tau_{11}{}^2$$

$$\left. \left. - \left(\frac{2n}{\hat{\theta}^3}\right) \left(k\hat{\alpha}^{-(k+1)}\right) \tau_{21}\tau_{22} + 3\tau_{11}\tau_{12} \left(k\hat{\alpha}^{-(k+1)}\right) \sum_{i=1}^{n} \frac{\left(1+x_i\right)^{-\alpha} \left(\log\left(1+x_i\right)\right)^2}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^2} \right\} \right]^{-\frac{1}{k}}, \tag{20}$$

$$\hat{\theta}_{BGEUP} = \left[\hat{\theta}^{-K} + \frac{1}{2} \left\{ \left(k\left(k+1\right)\hat{\theta}^{-(k+2)} \right) \tau_{22} \right. \right.$$

$$- \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right) \sum_{i=1}^{n} \frac{\left(1+x_i\right)^{-\hat{\alpha}} \left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right) \left(\log\left(1+x_i\right)\right)^3}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^3} \right) \left(k\hat{\theta}^{-(k+1)}\right) \tau_{11}\tau_{12}$$

$$-\left(\frac{2n}{\hat{\theta}^3}\right)\left(k\hat{\theta}^{-(k+1)}\right)\tau_{22}{}^2+\left(\tau_{11}\tau_{12}+2\tau_{12}{}^2\right)\left(k\hat{\theta}^{-(k+1)}\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^2}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^2}\Bigg\}\Bigg]^{-\frac{1}{k}}. \quad (21)$$

$$\hat{\alpha}_{BSEGP}=\Bigg[\hat{\alpha}+\frac{1}{2}\Bigg\{\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right)\left(\log\left(1+x_i\right)\right)^3}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^3}\right)\tau_{11}{}^2$$

$$+\left(\frac{2n}{\hat{\theta}^3}\right)\tau_{21}\tau_{22}-3\tau_{11}\tau_{12}\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^2}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^2}\Bigg\}$$

$$+\left(\frac{a_1-1}{\hat{\alpha}}-b_1\right)\tau_{11}+\left(\frac{a_2-1}{\hat{\theta}}-b_2\right)\tau_{12}\Bigg], \quad (22)$$

$$\hat{\theta}_{BSEGP}=\Bigg[\hat{\theta}+\frac{1}{2}\Bigg\{\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right)\left(\log\left(1+x_i\right)\right)^3}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^3}\right)\tau_{11}\tau_{12}$$

$$+\left(\frac{2n}{\hat{\theta}^3}\right)\tau_{22}{}^2-\left(\tau_{11}\tau_{12}+2\tau_{12}{}^2\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^2}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^2}\Bigg\}$$

$$+\left(\frac{a_1-1}{\hat{\alpha}}-b_1\right)\tau_{21}+\left(\frac{a_2-1}{\hat{\theta}}-b_2\right)\tau_{22}\Bigg]. \quad (23)$$

$$\hat{\alpha}_{BSELP}=\Bigg[\hat{\alpha}+\frac{1}{2}\Bigg\{\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right)\left(\log\left(1+x_i\right)\right)^3}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^3}\right)\tau_{11}{}^2$$

$$+\left(\frac{2n}{\hat{\theta}^3}\right)\tau_{21}\tau_{22}-\left(3\tau_{11}\tau_{12}\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^2}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^2}\Bigg\}$$

$$+\frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\hat{\theta}}\right)\tau_{11}+\frac{1}{2}\left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^{2}}\right)\tau_{12}\Bigg]. \tag{24}$$

$$\hat{\theta}_{BSELP}=\Bigg[\hat{\theta}+\frac{1}{2}\Bigg\{\left(\frac{2n}{\hat{\alpha}^{3}}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right)\left(\log\left(1+x_i\right)\right)^{3}}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^{3}}\right)\tau_{11}\tau_{12}$$

$$+\left(\frac{2n}{\hat{\theta}^{3}}\right){\tau_{22}}^{2}-\left(\tau_{11}\tau_{22}+2{\tau_{12}}^{2}\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^{2}}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^{2}}\Bigg\}$$

$$+\frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\hat{\theta}}\right)\tau_{21}+\frac{1}{2}\left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^{2}}\right)\tau_{22}\Bigg]. \tag{25}$$

$$\hat{\alpha}_{BSEUP}=\Bigg[\hat{\alpha}+\frac{1}{2}\Bigg\{\left(\frac{2n}{\hat{\alpha}^{3}}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right)\left(\log\left(1+x_i\right)\right)^{3}}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^{3}}\right){\tau_{11}}^{2}$$

$$+\left(\frac{2n}{\hat{\theta}^{3}}\right)\tau_{21}\tau_{22}-\left(3\tau_{11}\tau_{12}\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^{2}}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^{2}}\Bigg\}\Bigg], \tag{26}$$

$$\hat{\theta}_{BSEUP}=\Bigg[\hat{\theta}+\frac{1}{2}\Bigg\{\left(\frac{2n}{\hat{\alpha}^{3}}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(1+\left(1+x_i\right)^{-\hat{\alpha}}\right)\left(\log\left(1+x_i\right)\right)^{3}}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^{3}}\right)\tau_{11}\tau_{12}$$

$$+\left(\frac{2n}{\hat{\theta}^{3}}\right){\tau_{22}}^{2}-\left(\tau_{11}\tau_{22}+2{\tau_{12}}^{2}\right)\sum_{i=1}^{n}\frac{\left(1+x_i\right)^{-\hat{\alpha}}\left(\log\left(1+x_i\right)\right)^{2}}{\left(1-\left(1+x_i\right)^{-\hat{\alpha}}\right)^{2}}\Bigg\}\Bigg]. \tag{27}$$

$$\hat{\alpha}_{BPGP}=\left[\hat{\alpha}^2+\frac{1}{2}\left\{2\tau_{11}\right.\right.$$

$$+\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(2\hat{\alpha}\right)\tau_{11}{}^2$$

$$\left.+\left(\frac{4n\hat{\alpha}}{\hat{\theta}^3}\right)\tau_{21}\tau_{11}-\left(6\hat{\alpha}\tau_{11}\tau_{12}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\right\}$$

$$\left.+\left(\frac{a_1-1}{\hat{\alpha}}-b_1\right)\left(2\hat{\alpha}\right)\tau_{11}+\left(\frac{a_2-1}{\hat{\theta}}-b_2\right)\left(2\hat{\alpha}\right)\tau_{12}\right], \tag{28}$$

$$\hat{\theta}_{BPGP}=\left[\hat{\theta}^2+\frac{1}{2}\left\{2\tau_{22}+\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(2\hat{\theta}\right)\tau_{12}\tau_{11}\right.\right.$$

$$\left.+\left(\frac{4n}{\hat{\theta}^2}\right)\tau_{22}{}^2-\left(2\hat{\theta}\left(\tau_{11}\tau_{12}+2\tau_{12}{}^2\right)\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\right\}$$

$$\left.+\left(\frac{a_1-1}{\hat{\alpha}}-b_1\right)\left(2\hat{\theta}\right)\tau_{21}+\left(\frac{a_2-1}{\hat{\theta}}-b_2\right)\left(2\hat{\theta}\right)\tau_{21}\right]. \tag{29}$$

$$\hat{\alpha}_{BPLP}=\left[\hat{\alpha}^2+\frac{1}{2}\left\{2\tau_{11}+\left(\frac{2n}{\hat{\alpha}^3}+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3}\right)\left(2\hat{\alpha}\right)\tau_{11}{}^2\right.\right.$$

$$\left.+\left(\frac{4n\hat{\alpha}}{\hat{\theta}^3}\right)\tau_{21}\tau_{11}-\left(6\hat{\alpha}\tau_{11}\tau_{12}\right)\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2}\right\}$$

$$\left.+\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\theta}}\right)\tau_{11}+\left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}^2}\right)\hat{\alpha}\tau_{21}\right]^{\frac{1}{2}}, \tag{30}$$

$$\hat{\theta}_{BPLP} = \left[\hat{\theta}^2 + \frac{1}{2} \left\{ 2\tau_{22} + \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right) \sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}} \left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right) \left(2\hat{\theta}\right) \tau_{12}\tau_{11} \right.\right.$$

$$\left.\left. + \left(\frac{4n}{\hat{\theta}^2}\right)\tau_{22}{}^2 + \left(2\hat{\theta}\left(\tau_{11}\tau_{12} + 2\tau_{12}{}^2\right)\right) \sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}} \left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \right\} + \left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\theta}}\right)\tau_{11} + \left(\frac{\hat{\alpha}-3\hat{\theta}}{\hat{\theta}}\right)\tau_{22} \right]^{\frac{1}{2}} . \quad (31)$$

$$\hat{\alpha}_{BPUP} = \left[\hat{\alpha}^2 + \frac{1}{2} \left\{ 2\tau_{11} + \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right) \sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}} \left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right) \left(2\hat{\alpha}\right) \tau_{11}{}^2 \right.\right.$$

$$\left.\left. + \left(\frac{4n\hat{\alpha}}{\hat{\theta}^3}\right)\tau_{21}\tau_{11} - \left(6\hat{\alpha}\tau_{11}\tau_{12}\right) \sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}} \left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \right\} \right]^{\frac{1}{2}} , \quad (32)$$

$$\hat{\theta}_{BPUP} = \left[\hat{\theta}^2 + \frac{1}{2} \left\{ 2\tau_{22} + \left(\frac{2n}{\hat{\alpha}^3} + \left(\hat{\theta}-1\right) \sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}} \left(1+(1+x_i)^{-\hat{\alpha}}\right)\left(\log(1+x_i)\right)^3}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^3} \right) \left(2\hat{\theta}\right) \tau_{12}\tau_{11} \right.\right.$$

$$\left.\left. + \left(\frac{4n}{\hat{\theta}^2}\right)\tau_{22}{}^2 - \left(2\hat{\theta}\left(\tau_{11}\tau_{12} + 2\tau_{12}{}^2\right)\right) \sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}} \left(\log(1+x_i)\right)^2}{\left(1-(1+x_i)^{-\hat{\alpha}}\right)^2} \right\} \right]^{\frac{1}{2}} . \quad (33)$$

where $\hat{\alpha}$ and $\hat{\theta}$ are the MLEs of α and θ. Moreover, abbreviations of Bayesian estimators are as under:

BLGP	Bayesian LINEX Gamma Prior
BLLP	Bayesian LINEX Levy Prior
BLUP	Bayesian LINEX Uniform Prior
BGEGP	Bayesian General Entropy Gamma Prior
BGELP	Bayesian General Entropy Levy Prior
BGEUP	Bayesian General Entropy Uniform Prior
BSEGP	Bayesian Squared Error Gamma Prior
BSELP	Bayesian Squared Error Levy Prior
BSEUP	Bayesian Squared Error Uniform Prior
BPGF	Bayesian Precautionary Gamma Prior
BPLP	Bayesian Precautionary Levy Prior
BPUP	Bayesian Precautionary Uniform Prior

4. SIMULATION STUDY

In this Section, numerical examples are provided to assess the functioning of proposed Bayesian estimators, samples are generated from the EPD using inverse transformation and we replicated the process 1000 times. Moreover, MLEs and Bayesian estimators are compared in terms of biases and MSEs (within parenthesis) regarding various sample sizes and various parametric values using informative and non-informative priors of both α and θ, and results are listed in Tables 1-4. From the results of simulation study, conclusions are drawn regarding the behavior of the estimators, which are summarized below:

1. In general, the Bayesian estimators under different loss functions are closed to the true values. In terms of MSEs, the Bayesian estimators give the smallest MSEs.
2. For estimating θ Bayesian estimators perform better than MLEs. That is, the approximate Bayesian estimators have smallest MSE. However, BSELP, BGELP and BPLGP tend to underestimate the EPD parameters. While for estimating of α, the staff is different. From Tables 1-4, we can see that the Bayesian estimators i.e., BSELP, BGELP and BPLGP tend to overestimate the parameters.
3. It is also observed that the performances of both Bayesian and MLEs become better when the sample size increases. Further Figure 1-2 depict that the MSEs for all methods of estimation studied, decreases with increasing the sample size.

Table 1
Average Estimates for α and MSEs (Within Parenthesis)

n↓	α→	1	1.5	2
30	ML	1.0859(0.0859)	1.6031(0.1468)	2.1485(0.2339)
	BLGP	1.0609(0.0745)	1.6021(0.1297)	2.1299(0.2335)
	BLLP	1.0700(0.0797)	1.6029(0.1395)	2.1470(0.2337)
	BSEGP	1.0850(0.0803)	1.6030(0.1397)	2.1481(0.2338)
	BSELP	1.2559(0.1697)	1.8708(0.3331)	2.5398(0.6249)
	BGEGP	1.0854(0.0840)	1.6027(0.1393)	2.1374(0.2308)
	BGELP	1.2452(0.1518)	1.8067(0.2587)	2.4143(0.4414)
	BPLGP	1.2180(0.1501)	1.8591(0.3404)	2.5640(0.7002)
	BPLLP	1.0698(0.0774)	1.6026(0.1391)	2.1477(0.2333)
	BLUP	1.0761(0.0605)	1.6028(0.1297)	2.1459(0.2329)
	BSEUP	1.0670(0.0840)	1.6020(0.1358)	2.1471(0.2336)
	BGEUP	1.0759(0.0804)	1.6025(0.1355)	2.1379(0.2330)
	BPLUP	1.0847(0.0861)	1.6023(0.1359)	2.1396(0.2334)
50	ML	1.0495(0.0437)	1.5656(0.0751)	2.0631(0.1117)
	BLGP	1.0384(0.0430)	1.5619(0.0695)	2.0623(0.1073)
	BLLP	1.0396(0.0435)	1.5655(0.0748)	2.0630(0.1110)
	BSEGP	1.0490(0.0406)	1.5654(0.0711)	2.0624(0.1090)
	BSELP	1.2473(0.0706)	1.7210(0.1373)	2.2833(0.2208)
	BGEGP	1.0489(0.0436)	1.5642(0.0747)	2.0629(0.1115)
	BGELP	1.1433(0.0659)	1.6856(0.1138)	2.2152(0.1699)
	BPLGP	1.1261(0.0640)	1.7167(0.1408)	2.3018(0.2459)
	BPLLP	1.0480(0.0433)	1.5647(0.0739)	2.0494(0.1023)
	BLUP	1.0393(0.0389)	1.5652(0.0698)	2.0611(0.1036)
	BSEUP	1.0412(0.0432)	1.5637(0.0745)	2.0625(0.1115)
	BGEUP	1.0470(0.0429)	1.5483(0.0736)	2.0627(0.1028)
	BPLUP	1.0471(0.0434)	1.5618(0.0740)	2.0628(0.1111)
80	ML	1.0371(0.0262)	1.5331(0.0430)	2.0526(0.0672)
	BLGP	1.0362(0.0255)	1.5176(0.0424)	2.0499(0.0588)
	BLLP	1.0365(0.0260)	1.5317(0.0427)	2.0521(0.0627)
	BSEGP	1.0359(0.0230)	1.5312(0.0420)	2.0516(0.0590)
	BSELP	1.0975(0.0371)	1.6275(0.0644)	2.1520(0.1659)
	BGEGP	1.0368(0.0254)	1.5330(0.0426)	2.0488(0.0621)
	BGELP	1.0953(0.0354)	1.6066(0.0563)	2.1464(0.0916)
	BPLGP	1.0854(0.0347)	1.6262(0.0660)	2.2008(0.1213)
	BPLLP	1.0369(0.0258)	1.5327(0.0679)	2.0309(0.0559)
	BLUP	1.0348(0.0256)	1.5209(0.0416)	2.0523(0.0574)
	BSEUP	1.0334(0.0241)	1.5245(0.0423)	2.0514(0.0590)
	BGEUP	1.0313(0.0225)	1.5321(0.0415)	2.0141(0.0506)
	BPLUP	1.0323(0.0219)	1.5297(0.0410)	2.0202(0.0497)

Table 2
Average Estimates for α and MSEs (Within Parenthesis)

n↓	α→	1	1.5	2
100	ML	1.0309(0.0208)	1.5270(0.0347)	2.0387(0.0495)
	BLGP	1.0300(0.0201)	1.5245(0.0344)	2.0362(0.0448)
	BLLP	1.0304(0.0200)	1.5264(0.0346)	2.0376(0.0481)
	BSEGP	1.0307(0.0203)	1.5206(0.0338)	2.0357(0.0483)
	BSELP	1.0789(0.0272)	1.6017(0.0481)	2.1451(0.0757)
	BGEGP	1.0302(0.0201)	1.5244(0.0345)	2.0378(0.0482)
	BGELP	1.0774(0.0262)	1.5855(0.0431)	2.1329(0.0640)
	BPLGP	1.0692(0.0256)	1.6004(0.0490)	2.1355(0.0548)
	BPLLP	1.0305(0.0205)	1.5207(0.0339)	2.0384(0.0477)
	BLUP	1.0304(0.0206)	1.5268(0.0340)	2.0379(0.0479)
	BSEUP	1.0306(0.0205)	1.5237(0.0335)	2.0197(0.0274)
	BGEUP	1.0303(0.0204)	1.5269(0.0343)	2.0275(0.0374)
	BPLUP	1.0308(0.0207)	1.5234(0.0342)	2.0355(0.0390)
150	ML	1.0185(0.0127)	1.5302(0.0241)	2.0224(0.0381)
	BLGP	1.0143(0.0124)	1.5258(0.0209)	2.0202(0.0295)
	BLLP	1.0154(0.0125)	1.5292(0.0233)	2.0215(0.0368)
	BSEGP	1.0183(0.0126)	1.5259(0.0239)	2.0217(0.0363)
	BSELP	1.0502(0.0156)	1.5801(0.0313)	2.0923(0.0495)
	BGEGP	1.0179(0.0121)	1.5301(0.0236)	2.0222(0.0319)
	BGELP	1.0493(0.0152)	1.5692(0.0289)	2.0713(0.0444)
	BPLGP	1.0440(0.0149)	1.5796(0.0317)	2.0995(0.0524)
	BPLLP	1.0152(0.0116)	1.5233(0.0240)	2.0215(0.0363)
	BLUP	1.0182(0.0125)	1.5278(0.0211)	2.0301(0.0291)
	BSEUP	1.0127(0.0124)	1.5280(0.0232)	2.0356(0.0359)
	BGEUP	1.0119(0.0114)	1.5272(0.0249)	2.0220(0.0315)
	BPLUP	1.0179(0.0123)	1.5216(0.0235)	2.0294(0.0319)
200	ML	1.0106(0.0089)	1.5153(0.0155)	2.0190(0.0269)
	BLGP	1.0100(0.0082)	1.5092(0.0146)	2.0102(0.0236)
	BLLP	1.0103(0.0087)	1.5093(0.0154)	2.0119(0.0255)
	BSEGP	1.0105(0.0088)	1.5018(0.0153)	2.0189(0.0267)
	BSELP	1.0136(0.0104)	1.5523(0.0188)	2.0714(0.0335)
	BGEGP	1.0104(0.0088)	1.5138(0.0152)	2.0132(0.0159)
	BGELP	1.0130(0.0090)	1.5440(0.0176)	2.0557(0.0306)
	BPLGP	1.0192(0.0101)	1.5520(0.0190)	2.0773(0.0352)
	BPLLP	1.0106(0.0088)	1.5124(0.0150)	2.0189(0.0265)
	BLUP	1.0102(0.0088)	1.5107(0.0148)	2.0100(0.0134)
	BSEUP	1.0106(0.0089)	1.5133(0.0145)	2.0188(0.0275)
	BGEUP	1.0104(0.0086)	1.5152(0.0155)	2.0131(0.0269)
	BPLUP	1.0101(0.0083)	1.5136(0.0151)	2.0172(0.0249)

Table 3
Average Estimates for θ and MSEs (Within Parenthesis)

n↓	θ→	1	1.5	2
30	ML	1.1082(0.1009)	1.6770(0.2803)	2.2753(0.5799)
	BLGP	1.0554(0.0964)	1.5970(0.2661)	2.1634(0.5632)
	BLLP	1.0127(0.0890)	1.5315(0.2632)	2.0737(0.5927)
	BSEGP	1.0242(0.0647)	1.4887(0.1430)	1.9250(0.2273)
	BSELP	0.9810(0.0552)	1.4161(0.1227)	1.8176(0.2022)
	BGEGP	1.0368(0.0786)	1.5592(0.2022)	2.0867(0.3731)
	BGELP	0.9936(0.0670)	1.4866(0.1659)	1.9793(0.2945)
	BPLGP	0.9547(0.0715)	1.3857(0.1554)	1.7650(0.2560)
	BPLLP	1.0012(0.0717)	1.4598(0.1513)	1.8749(0.2301)
	BLUP	1.0816(0.1075)	1.6511(0.3082)	2.2555(0.6700)
	BSEUP	1.0495(0.0729)	1.5405(0.1665)	2.0130(0.2704)
	BGEUP	1.0620(0.0880)	1.6110(0.2371)	2.1747(0.4594)
	BPLUP	1.0740(0.0967)	1.5967(0.2256)	2.1016(0.3750)
50	ML	1.0539(0.0507)	1.6006(0.1329)	2.1451(0.2644)
	BLGP	1.0259(0.0496)	1.5553(0.1270)	2.0830(0.2555)
	BLLP	1.0022(0.0474)	1.5175(0.1232)	2.0327(0.2575)
	BSEGP	1.0092(0.0395)	1.5003(0.0926)	1.9660(0.1663)
	BSELP	0.9854(0.0362)	1.4604(0.0849)	1.9082(0.1542)
	BGEGP	1.0136(0.0440)	1.5352(0.1102)	2.0453(0.2095)
	BGELP	0.9897(0.0403)	1.4954(0.0989)	1.9875(0.1857)
	BPLGP	0.9738(0.0415)	1.4486(0.0969)	1.8956(0.1765)
	BPLLP	0.9988(0.0418)	1.4889(0.0973)	1.9521(0.1728)
	BLUP	1.0398(0.0528)	1.5842(0.1390)	2.1311(0.2849)
	BSEUP	1.0229(0.0421)	1.5284(0.1003)	2.0126(0.1813)
	BGEUP	1.0273(0.0468)	1.5633(0.1204)	2.0919(0.2338)
	BPLUP	1.0376(0.0496)	1.5603(0.1196)	2.0638(0.2174)
80	ML	1.0378(0.0269)	1.5657(0.0736)	2.0936(0.1416)
	BLGP	1.0218(0.0262)	1.5397(0.0712)	2.0551(0.1372)
	BLLP	1.0072(0.0251)	1.5168(0.0690)	2.0233(0.1360)
	BSEGP	1.0112(0.0231)	1.5070(0.0594)	1.9894(0.1072)
	BSELP	0.9967(0.0219)	1.4830(0.0564)	1.9555(0.1020)
	BGEGP	1.0132(0.0245)	1.5266(0.0655)	2.0346(0.1225)
	BGELP	0.9987(0.0232)	1.5026(0.0614)	2.0006(0.1138)
	BPLGP	0.9912(0.0234)	1.4785(0.0606)	1.9494(0.1109)
	BPLLP	1.0062(0.0238)	1.5022(0.0613)	1.9832(0.1105)
	BLUP	1.0301(0.0273)	1.5567(0.0754)	2.0833(0.1473)
	BSEUP	1.0194(0.0240)	1.5237(0.0624)	2.0170(0.1130)
	BGEUP	1.0214(0.0255)	1.5433(0.0692)	2.0622(0.1311)
	BPLUP	1.0293(0.0266)	1.5440(0.0695)	2.0471(0.1266)

Table 4
Average Estimates for θ and MSEs (Within Parenthesis)

n↓	θ→	1	1.5	2
100	ML	1.0270(0.0205)	1.5419(0.0557)	2.0615(0.1014)
	BLGP	1.0144(0.0201)	1.5209(0.0547)	2.0310(0.0995)
	BLLP	1.0029(0.0195)	1.5027(0.0537)	2.0056(0.0991)
	BSEGP	1.0062(0.0183)	1.4966(0.0479)	1.9815(0.0834)
	BSELP	0.9948(0.0176)	1.4781(0.0462)	1.9553(0.0808)
	BGEGP	1.0076(0.0191)	1.5114(0.0514)	2.0158(0.0914)
	BGELP	0.9962(0.0184)	1.4928(0.0491)	1.9895(0.0869)
	BPLGP	0.9962(0.0186)	1.4739(0.0491)	1.9505(0.0865)
	BPLLP	1.0024(0.0188)	1.4926(0.0492)	1.9769(0.0857)
	BLUP	1.0209(0.0208)	1.5341(0.0570)	2.0528(0.1049)
	BSEUP	1.0127(0.0188)	1.5096(0.0495)	2.0029(0.0863)
	BGEUP	1.0140(0.0197)	1.5244(0.0534)	2.0371(0.0960)
	BPLUP	1.0204(0.0203)	1.5248(0.0537)	2.0260(0.0939)
150	ML	1.0176(0.0121)	1.5353(0.0347)	2.0327(0.0687)
	BLGP	1.0096(0.0120)	1.5213(0.0340)	2.0127(0.0680)
	BLLP	1.0021(0.0117)	1.5091(0.0333)	1.9960(0.0677)
	BSEGP	1.0040(0.0113)	1.5055(0.0310)	1.9812(0.0613)
	BSELP	0.9965(0.0110)	1.4933(0.0301)	1.9642(0.0603)
	BGEGP	1.0047(0.0116)	1.5151(0.0326)	2.0030(0.0647)
	BGELP	0.9972(0.0113)	1.5029(0.0315)	1.9860(0.0629)
	BPLGP	0.9942(0.0114)	1.4906(0.0313)	1.9614(0.0628)
	BPLLP	1.0018(0.0114)	1.5030(0.0315)	1.9786(0.0624)
	BLUP	1.0139(0.0122)	1.5299(0.0351)	2.0268(0.0701)
	BSEUP	1.0082(0.0115)	1.5140(0.0318)	1.9950(0.0625)
	BGEUP	1.0089(0.0118)	1.5236(0.0335)	2.0168(0.0666)
	BPLUP	1.0137(0.0121)	1.5240(0.0337)	2.0101(0.0657)
200	200ML	1.0143(0.0091)	1.5247(0.0245)	2.0393(0.0528)
	BLGP	1.0086(0.0090)	1.5145(0.0242)	2.0247(0.0520)
	BLLP	1.0030(0.0088)	1.5055(0.0239)	2.0122(0.0515)
	BSEGP	1.0042(0.0085)	1.5027(0.0226)	2.0007(0.0474)
	BSELP	0.9986(0.0084)	1.4936(0.0221)	1.9878(0.0464)
	BGEGP	1.0047(0.0087)	1.5097(0.0234)	2.0170(0.0498)
	BGELP	0.9990(0.0085)	1.5006(0.0228)	2.0042(0.0484)
	BPLGP	0.9971(0.0086)	1.4920(0.0228)	1.9863(0.0478)
	BPLLP	1.0028(0.0087)	1.5011(0.0229)	1.9991(0.0480)
	BLUP	1.0117(0.0091)	1.5209(0.0248)	2.0352(0.0535)
	BSEUP	1.0073(0.0087)	1.5091(0.0230)	2.0111(0.0484)
	BGEUP	1.0078(0.0089)	1.5160(0.0239)	2.0275(0.0512)
	BPLUP	1.0116(0.0090)	1.5167(0.0240)	2.0227(0.0506)

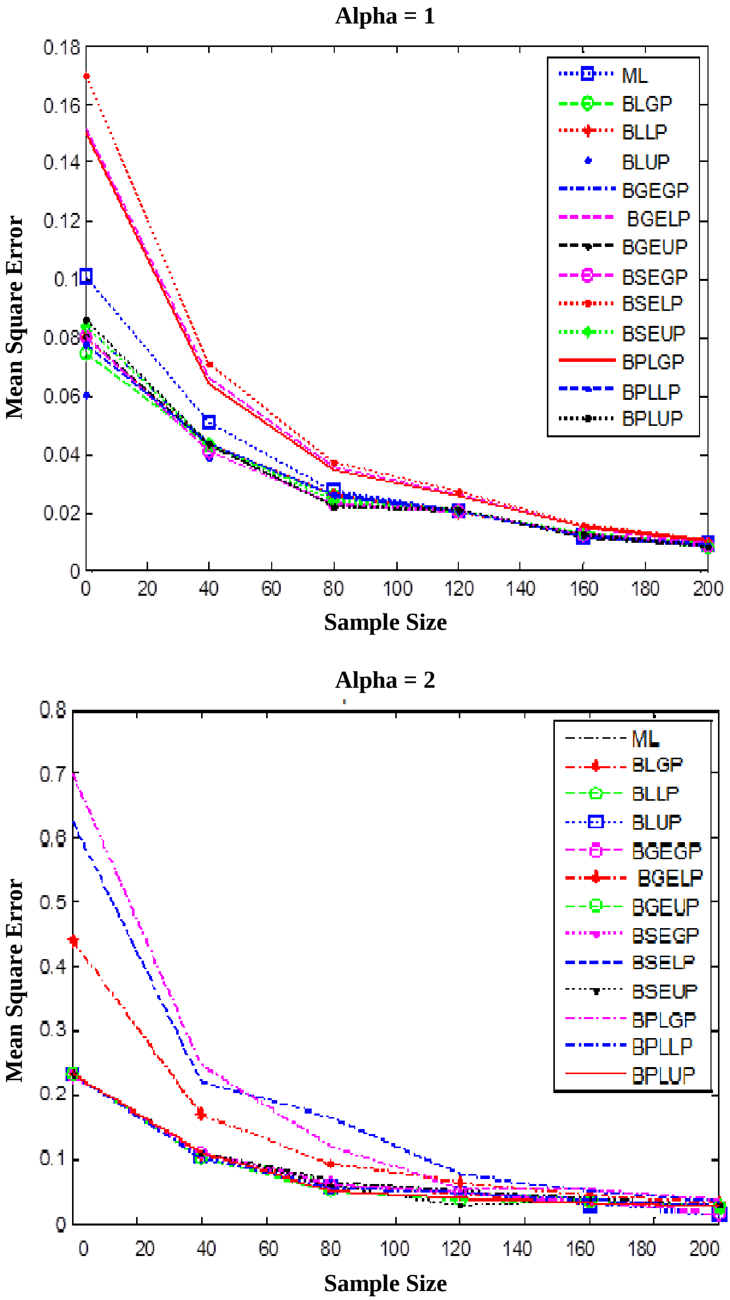

Figure 1: Graph of Mean squared error for different values of α

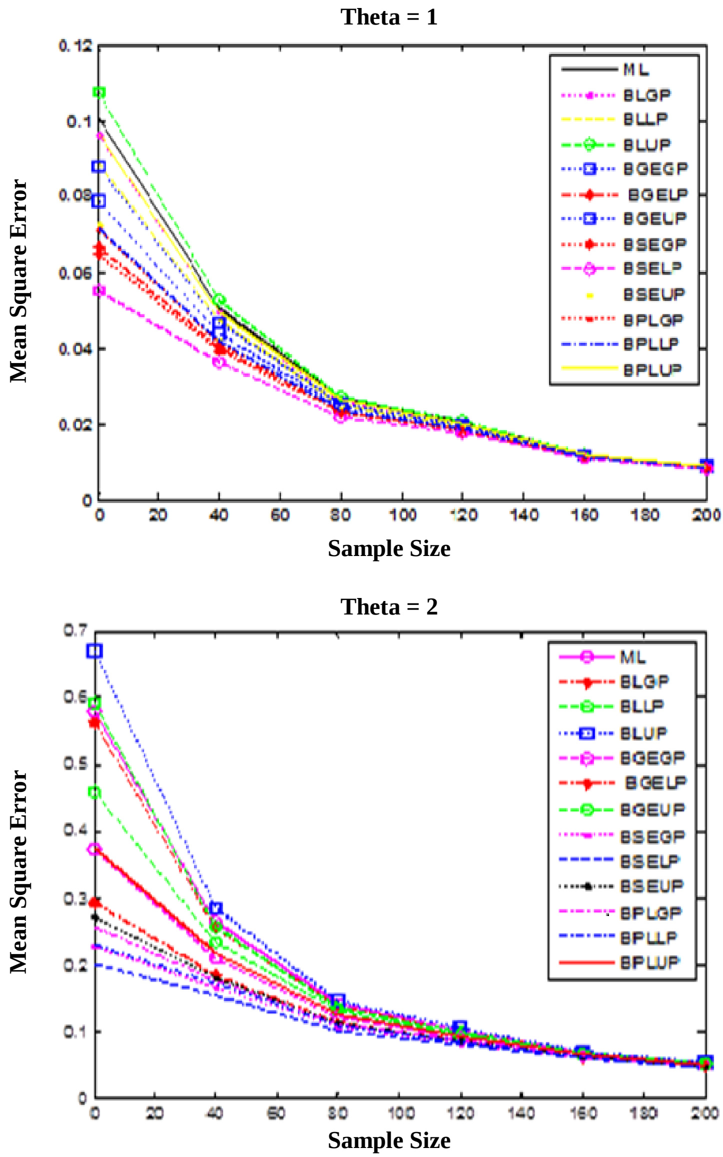

Figure 2: Graph of Mean squared error for different values of θ.

4. DATA ANALYSIS

In this section we consider the real data set which is taken from Nichols and Padgett (2006) and is presented in Table 5. The data gives 100 observations on breaking stress of carbon fibers in Gross building area (Gba).

Table 5
Breaking Stress of Carbon Fibers (in Gba)

3.7	2.74	2.73	2.5	3.6	3.11	3.27	2.87	1.47	3.11	4.42	2.41
3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	3.75	2.43	2.95	2.97
2.96	2.53	2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31
2.85	2.56	3.56	3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83
1.92	1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59	3.19
1.57	0.81	5.56	1.73	1.59	2.0	1.22	1.12	1.71	2.17	1.17	5.08
2.48	1.18	3.51	2.17	1.69	1.25	4.38	1.84	0.39	3.68	2.48	0.85
1.61	2.79	4.70	2.03	1.80	1.57	1.08	2.03	1.61	2.12	1.89	2.88
2.82	2.05	3.65	3.39								

Table 6
Average estimates and standard deviation (SD) of α and θ.

Estimator	α	SD	θ	SD
ML	3.0698	0.0507	27.6963	43.1718
BLGP	3.4019	0.0495	24.2969	39.1880
BLLP	3.4673	0.0504	25.8123	43.1609
BSEGP	3.3778	0.0483	20.1001	40.4333
BSELP	3.4232	0.0435	18.7756	36.1996
BGEGP	3.1566	0.0502	23.4999	42.8693
BGELP	3.2585	0.0507	22.5835	43.1718
BPLGP	3.6108	0.0523	18.8168	42.1609
BPLLP	4.0944	0.0602	18.1421	41.1287
BLUP	3.3584	0.0501	24.3350	40.4812
BSEUP	3.3458	0.0520	21.2217	42.1367
BGEUP	3.1736	0.0505	24.3360	41.9716
BPLUP	3.5835	0.0493	21.5553	43.1217

The point estimates of α and θ obtained by all the methods are presented in Table 6. Since the Bayesian estimators of the model parameters cannot be obtained analytically, approximate Bayesian estimates are obtained numerically using Lindley approximation with shape parameter $k = 1$. We use non-informative priors with the entire hyper parameters equal to zero, i.e., $a_1 = a_2 = b_1 = b_2 = 0$, for Bayes estimates.

5. CONCLUSION

In this paper, the Bayesian estimation of the parameters of EPD based on various loss functions are considered by using informative and non-informative priors. It is observed that Bayesian estimators cannot be obtained in explicit forms. In present study Lindley's approximation is utilized to obtain the posterior estimates numerically and it is notice that approximation works very well. However, Markov Chain Monte Carlo (MCMC) can be used to generate samples from posterior distribution in turn computing Bayesian estimators. Moreover, the results obtained from Lindley's approximation can be compared with MCMC. More work is needed in that direction. Based on simulation study it is concluded that the Bayesian estimators perform well since the MSE is significantly smaller. However, for large sample sizes the Bayesian and MLEs become closer in terms of MSEs.

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