Pak. J. Statist.
2016 Vol. 32(5), 329-348

SKEW-GENERALIZED INVERSE WEIBULL DISTRIBUTION AND ITS PROPERTIES

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ABSTRACT

The skew-generalized inverse weibull distribution (SGIW) has four parameters of lifetime distribution. It could have different hazard rates: increasing, decreasing and unimodal. In this paper, the method of Azzalini's (1985) is used to provide a shape of parameter to generalize inverse weibull, which creates a new class of skew-generalized inverse weibull distributions. Different statistical properties of this new distribution are discussed whereas expressions for density, minimum and maximum order statistic and i^{th} moment of the order statistics and the inference of the old parameters and the skewness parameter are studied. In addition, Mont Carlo simulation method was carried out to investigate the properties of the estimations of the unknown parameters of SGIW. Furthermore, the flexibility of SGIW model is illustrated by means of two real data sets applications.

KEYWORDS

Inverse weibull distribution; Selection models; Maximum likelihood estimation; Probabilistic representation.

1. INTRODUCTION

There are various methods that can be used to add a new shape parameter for any distribution. There is no specific distribution for such addition. The shape of the distribution, its result, the used method are identical. In this paper, the method of Azzalini (1985) is used to provide a shape of parameter to generalize inverse Weibull. Azzalini (1985) introduced a new method through which weighted distributions which can be obtained from independently identically distributed (i.i.d) random variables. He depends on a shape of parameter called the class of skew-normal distribution with the density function as follows:

$$g_{SN}\left(y\right)=2\Phi(\lambda y)\Theta(y); \qquad -\infty\le y\le\infty,$$

where $-\infty\le\lambda\le\infty$ is the shape/skewness parameter, $\Theta(.)$ and $\Phi(\lambda y)$ are the PDF and CDF respectively. He argues that the proposed family of distributions uses density function of one random variable and distribution of other random variable. Moreover, Many researches use Azzalini's method for symmetric and non- symmetric distributions such as Gupta and Kundu (2007, 2009), Arnold and Beaver (2000a, b), Genton (2004, 2007), Mahdy (2011, 2013) and Hussian (2013).

The main objective of this paper is to introduce skew-generalized inverse weibull $\left(SGIW\left(\alpha,\beta,\gamma,\lambda\right)\right)$ distribution and to investigate its properties. It has several other special distributions.

The rest of the paper is organized as follows. The definition of $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ distribution is provided in Section 2. The statistical properties of $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ are provide in section 3. In section 4, the order statistics models are provided. The parameter estimation and some asymptotic confidence intervals for estimators are considered in section 5. Finally, Applications upon two real data sets and simulation study are performed in Section 6.

2. DEFINITION OF SGIW DISTRIBUTION

Let Y be a random variable distributed according to a generalized inverse weibull distribution with parameters α, β and γ density function (PDF) as follows:

$$g_{Y|\{\alpha,\beta,\gamma\}}\left(y\right)=\beta\gamma\alpha^{\beta}y^{-(\beta+1)}\exp\left[-\gamma\left(\alpha/y\right)^{\beta}\right],\; y\in\Re^{+} \tag{2.1}$$

where α is a scale parameter, β and γ are shape parameters. The distribution function of $Y\left|\{\alpha,\beta,\gamma\}\right.$ is

$$G_{Y|\{\alpha,\beta,\gamma\}}\left(t\right)=\exp\left[-\gamma\left(\alpha/y\right)^{\beta}\right], y>0\,.$$

Let $N\in\Re^{+}$ and $M\in\Re^{+}$ be two random vectors and Ξ to be a measurable subset of $\Re^{+}$. The weighted distribution is represented by the conditional distribution of $\left(\mathrm{M}\left|\mathrm{N}\;\hat{I}\;\Xi\right.\right)$ in Arellano-Valle and Azzalini's (2006). Clearly, a *p*-dimensional random vector X^{w} is said to have a multivariate weighted density function with parameters depending on the characteristics of M,N and Ξ, if $X^{w}\xrightarrow{D}\left(M\left|N\in\Xi\right.\right)$. If M has a PDF, f_{M}, therefore X^{w} has a pdf as follows:

$$f_{X^{w}}\left(x\right)=f_{M}\left(x\right)=\frac{\Pr\left(N\in\Xi\left|M=x\right.\right)}{\Pr\left(N\in\Xi\right)} \tag{2.2}$$

Therefore, X^{w} is called the weighted random variable with weight function $w(x)$. Some special weighted functions are mentioned in Patil (2002) and Azzalini (1985).

Let Y, denotes the random variable with density function (2.1) and let we selected weighted function as mentioned in Azzalini's (1985) as

$$w(y)=G_{Y}(\lambda y)$$

Then, by (2.2) we have

$$g_{y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=G_{y|\{\alpha,\beta,\gamma\}}(\lambda y)\,g_{y|\{\alpha,\beta,\gamma\}}(y)/E_g\left[G_{Y|\{\alpha,\beta,\gamma\}}(\lambda y)\right],\ \text{for } y\in\Re^+,\quad (2.3)$$

According to (2.3), we have a new class of skew-generalized inverse weibull distribution

$$G_{Y|\{\alpha,\beta,\gamma\}}(\lambda y)=\exp\left[-\gamma\left(\alpha/\lambda y\right)^{\beta}\right],\ y>0;\ \alpha,\beta,\gamma,\lambda>0,$$

and $$E\left[G_{Y|\{\alpha,\beta,\gamma\}}(\lambda y)\right]=\int_y G_{Y|\{\alpha,\beta,\gamma\}}(\lambda t)\,g_{Y|\{\alpha,\beta,\gamma\}}(t)\ dt,=\frac{1}{\left[1+\lambda^{-\beta}\right]}$$

Inserting into (2.3) yields

$$g_{y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=\beta\gamma\alpha^{\beta}\left(1+\lambda^{-\beta}\right)y^{-(\beta+1)}\exp\left(-\gamma\left(\alpha/y\right)^{\beta}\left(1+\lambda^{-\beta}\right)\right);\ \text{for } y>0,\quad (2.4)$$

and $g_{y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=0$ otherwise. In addition, the corresponding the distribution function is:

$$G_{y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=\exp\left(-\gamma\left(\alpha/y\right)^{\beta}\left(1+\lambda^{-\beta}\right)\right);\quad(\alpha,\beta,\gamma,\lambda)\in\Re^+.\qquad (2.5)$$

Suppose Y_1 and Y_2 are two i.i.d. random variables with probability density function $g_{Y|\{\alpha,\beta,\gamma\}}(.)$ and distribution function $G_{Y|\{\alpha,\beta,\gamma\}}(.)$. If $Y=Y_1$ given that $\lambda Y_1>Y_2$, for $\lambda\in\Re^+$ and $\beta=2$, therefore, PDF of Y is given as:

$$g_{Y^w|\{\alpha,2,\gamma,\lambda\}}(y)=G_{Y|\{\alpha,\beta,\gamma\}}(\lambda y)\,g_{Y|\{\alpha,\beta,\gamma\}}(y)/P_{Y_1,Y_2}\left(\left\{(y_1,y_2):\alpha y_1>y_2\right\}\right).\quad (2.6)$$

Now, (2.3) can be obtained explicitly from (2.6) by inserting

$$g_{y^w|\{\alpha,2,\gamma,\lambda\}}(y)=\beta\gamma\alpha^2y^{-3}\left[1+\lambda^{-\beta}\right]\exp\left[-\gamma\left(\alpha/y\right)^{\beta}\left(1+\lambda^{-\beta}\right)\right],\quad \text{for } y\in\Re^+.$$

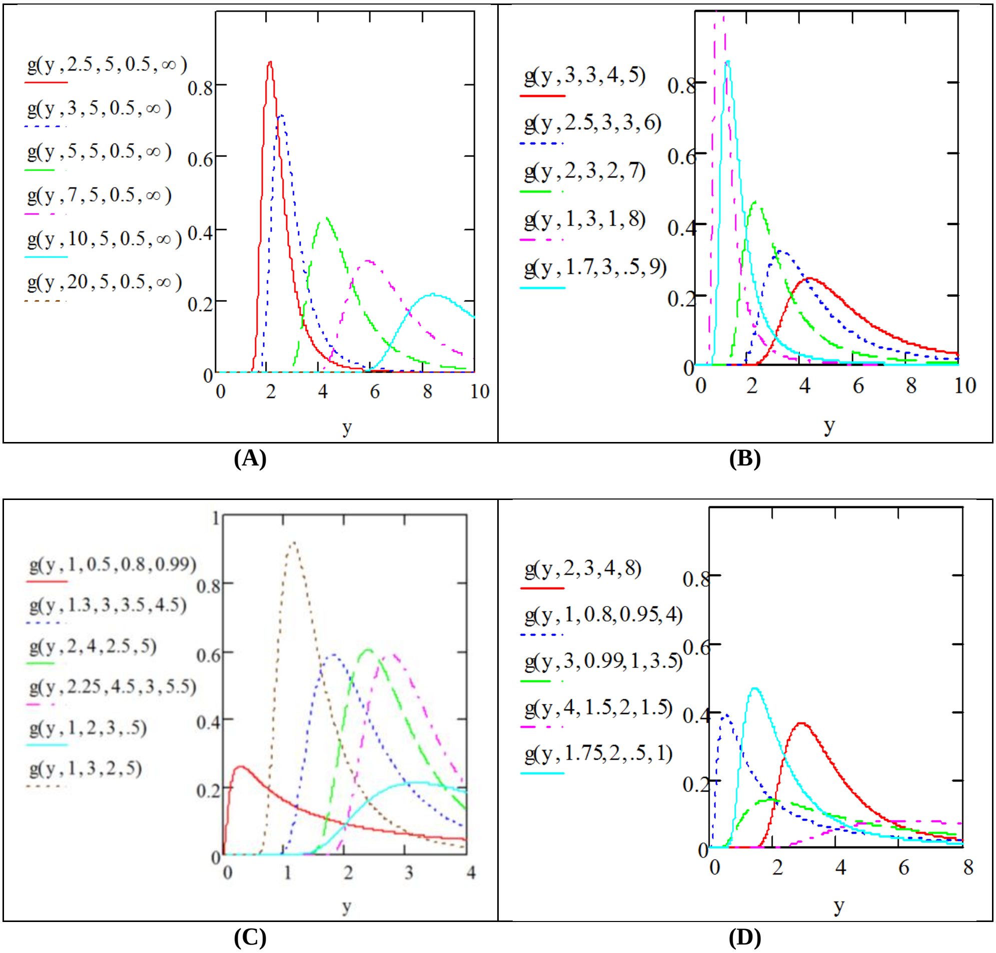
In addition, the distribution function is given by

$$G_{Y^w|\{\alpha,2,\gamma,\lambda\}}(y)=\exp\left[-\gamma\left(\alpha/y\right)^2\left(1+\lambda^{-2}\right)\right]$$

where

$$P_{Y_1,Y_2}\left(\left\{(y_1,y_2):\lambda y_1>y_2\right\}\right)=\int_0^{\infty}\int_0^{\lambda y_1}g_{Y_1,Y_2}(y_1,y_2)\,dy_2dy_1\ =1\Big/\left(1+\lambda^{-2}\right).$$

The graphs of $SGIW(\alpha,\beta,\gamma,\lambda)$ for different values of are illustrated in Figures 2.1. It is easy to see that as increases, the skewness of the distribution increases.



Figures 2.1: PDF of $SGIW(\alpha, \beta, \gamma, \lambda)$ Density for Some Values of the Parameters

Furthermore, we have derivative special models by using our model such as a weighted inverse weibull distribution $(WIW(\alpha, \beta, \lambda))$, weighted inverse exponential distribution $(WIE(\alpha, \lambda))$, a weighted inverse rayleigh distribution $(WIR(\alpha, \lambda))$, a generalized inverse weibull distribution $(GIW(\alpha, \beta, \lambda))$, an inverse weibull distribution $(IW(\alpha, \beta))$, an inverse exponential distribution $(IE(\alpha))$, and an inverse rayleigh distribution $(IR(\alpha))$. The probability density functions of the considered distributions are given in Table 2.1.

Let Y is a random variable with $SGIW(\alpha,\beta,\gamma,\lambda)$, then survival function can illustrate as

$$S_{y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=1-\exp\left[-\gamma(\alpha/y)^{\beta}\left(1+\lambda^{-\beta}\right)\right],(\alpha,\beta,\gamma,\lambda)\in R^{+}.$$

In addition to, if $\Omega(x)=\gamma(\alpha/x)^{\beta}\left(1+\lambda^{-\beta}\right)$, we have the hazard rate as follows:

$$h_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=\left(\beta\Omega(y)/y\right)\exp\left[-\Omega(y)\right]\Big/\left(1-\exp\left[-\Omega(y)\right]\right).$$

Furthermore, the reverse hazard function of the $SGIW(\alpha,\beta,\gamma,\lambda)$ is given as

$$r_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)=\beta\gamma\alpha^{\beta}y^{-(\beta+1)}\left(1+\lambda^{-\beta}\right)=\beta\Omega(y)/y.$$

Table 2.1
Sub-Models of the $SGIW(\alpha,\beta,\gamma,\lambda)$

β	γ	λ	**Distribution**	**Probability Density Functions**
-	1	-	$WIW(\alpha,\beta,\lambda)$	$g(x)=\beta\alpha^{\beta}x^{-(\beta+1)}\left(1+\lambda^{-\beta}\right)\exp\left[-(\alpha/x)^{\beta}\left(1+\lambda^{-\beta}\right)\right]$
1	1	-	$WIE(\alpha,\lambda)$	$g(x)=\alpha x^{-2}\left(1+\lambda^{-1}\right)\exp\left[-(\alpha/x)\left(1+\lambda^{-1}\right)\right]$
2	1	-	$WIR(\alpha,\lambda)$	$g(x)=2\alpha^{2}x^{-3}\left(1+\lambda^{-2}\right)\exp\left[-(\alpha/x)^{2}\left(1+\lambda^{-2}\right)\right]$
-	-	$\lambda\to\infty$	$GIW(\alpha,\beta,\lambda)$	$g(x)=\gamma\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-\gamma(\alpha/x)^{\beta}\right]$
-	1	$\lambda\to\infty$	$IW(\alpha,\beta)$	$g(x)=\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-(\alpha/x)^{\beta}\right]$
1	1	$\lambda\to\infty$	$IE(\alpha)$	$g(x)=\alpha x^{-2}\exp\left[-(\alpha/x)\right]$
2	1	$\lambda\to\infty$	$IR(\alpha)$	$g(x)=2\alpha^{2}x^{-3}\exp\left[-(\alpha/x)^{2}\right]$

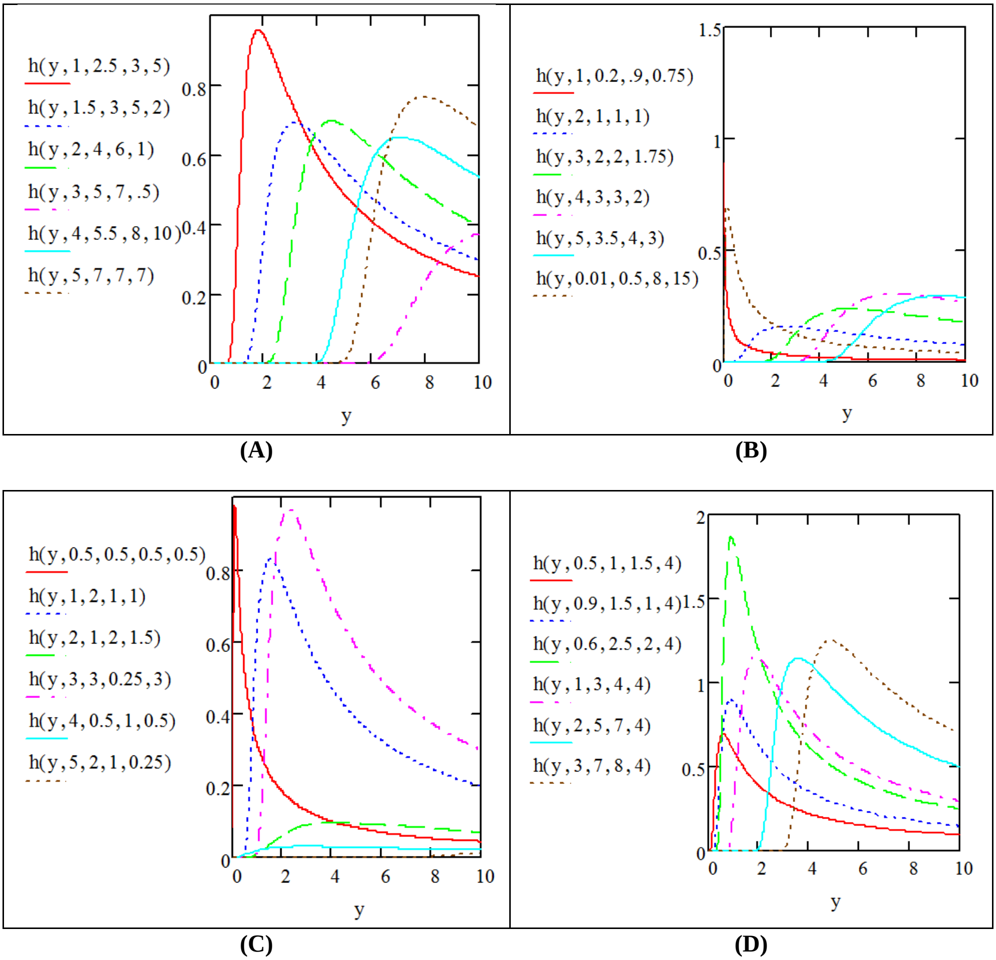


Figure 2.2: Hazard Rate Function of for Selected Values of the Parameters

Figure 2.2 shows some of the possible shapes of the hazard function of the $SGIW(\alpha, \beta, \gamma, \lambda)$ for selected values of the parameters by keeping and different values of the parameters are respectively.

3. STATISTICAL PROPERTIES

On investigating the different properties of the $SGIW(\alpha,\beta,\gamma,\lambda)$ distribution as follows:

i) The *P-quantile* of the $SGIW(\alpha,\beta,\gamma,\lambda)$ distribution, denoted by Q_P is defined as value of Y which satisfies

$$Q_P = F^{-1}_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(P)$$

Then, the quantile Q_P of $SGIW(\alpha,\beta,\gamma,\lambda)$ is given by the following

$$Q_p = \left(-\ln P / \left(\gamma\alpha^{\beta}\left(1+\lambda^{-\beta}\right)\right)\right)^{-\frac{1}{\beta}} \tag{3.1}$$

If $p = 0.5$ in (3.1), therefore, we have the median of $SGIW(\alpha,\beta,\gamma,\lambda)$.

ii) The generating random variable of the $SGIW(\alpha,\beta,\gamma,\lambda)$ is defined as

$$y = \left[\frac{-\ln(v)}{\gamma\alpha^{\beta}\left(1+\lambda^{-\beta}\right)}\right]^{-\frac{1}{\beta}}, \quad v \in U(0,1).$$

iii) If $Y \in SGIW(\alpha,\beta,\gamma,\lambda)$, then the k^{th} moment of $Y^w\left|\{\alpha,\beta,\gamma,\lambda\}\right.$ is given by

$$E_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y^k) = \gamma^{\frac{k}{\beta}}\alpha^k\left(1+\lambda^{-\beta}\right)^{\frac{k}{\beta}}\Gamma\left(1-\frac{k}{\beta}\right); \quad \beta > k \tag{3.2}$$

When $\beta \le k$, the k^{th} moment of $Y^w\left|\{\alpha,\beta,\gamma,\lambda\}\right.$ does not exist. When $k=1$, the Eq. (3.2) becomes the expected value. From (3.2), the first four moments function of $SGIW(\alpha,\beta,\gamma,\lambda)$ can be found by using $k = 1,2,3,4$ as follows:

a. $l_1 = E_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y) = \gamma^{\frac{1}{\beta}}\alpha\left(1+\lambda^{-\beta}\right)^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right),$

b. $l_2 = E_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y^2) = \gamma^{\frac{2}{\beta}}\alpha^2\left(1+\lambda^{-\beta}\right)^{\frac{2}{\beta}}\Gamma\left(1-\frac{2}{\beta}\right),$

c. $l_3 = E_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y^3) = \gamma^{\frac{3}{\beta}}\alpha^3\left(1+\lambda^{-\beta}\right)^{\frac{3}{\beta}}\Gamma\left(1-\frac{3}{\beta}\right),$

d. $l_4 = E_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y^4) = \gamma^{\frac{4}{\beta}}\alpha^4\left(1+\lambda^{-\beta}\right)^{\frac{4}{\beta}}\Gamma\left(1-\frac{4}{\beta}\right).$

In addition to, the variance $\left(V_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)$, the coefficient of variation $\left(CV_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)$, the coefficient skewness $\left(CS_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)$ and coefficient of kurtosis $\left(CK_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)$ of $SGIW(\alpha,\beta,\gamma,\lambda)$ respectively can acquire as:

1. $V_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y) = l_2 - (l_1)^2$;
2. $CV_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y) = \left(l_2 - (l_1)^2\right)^{1/2} / l_1$;
3. $CS_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y) = \left(l_3 - 3(l_2 l_1) + 2(l_1)^3\right) / \left(\left(l_2 - (l_1)^2\right)^{3/2}\right)$,
4. $CK_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y) = \left(l_4 - 4(l_3 l_1) + 6\left(l_2 (l_1)^2\right) - 3(l_1)^4\right) / \left(\left(l_2 - (l_1)^2\right)^2\right)$.

iv) The general form of moments of residual life can consider as follows:

$$\mu_r(y) = E\left\{(X-y)^r \,|X > y\right\} = \frac{1}{S_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}} \int_y^\infty (u-y)^r g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(u)du\,.$$

Therefore,

$$\mu_r(y) = \frac{\left(\sum_{i=0}^{r} (-y)^i \binom{r}{i} \gamma_{\left(\frac{i-r}{\beta}+1,t\right)} \gamma^{\frac{r-i}{\beta}} \alpha^{r-i} \left(1+\left(\frac{1}{\lambda}\right)^{\beta}\right)^{\frac{r-i}{\beta}}\right)}{\left(1-\exp\left(-\gamma(\alpha/y)^{\beta}\left(1+\lambda^{-\beta}\right)\right)\right)},$$

where

$$\gamma_{\left(\frac{i-r}{\beta}+1,t\right)} = \int_0^t t^{\frac{i-r}{\beta}} \exp(-t)\,dt,$$

is the lower incomplete gamma function

v) The moment generating function $\left(\mathrm{M}_Y(x)\right)$ of the $SGIW(\alpha,\beta,\gamma,\lambda)$ distribution for $|\lambda|<1$, can be obtained as

$$\mathrm{M}_Y(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} \int_0^\infty g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(t)dt = \sum_{k=0}^{\infty} \frac{y^k}{k!} \mu_r' \qquad (3.3)$$

By using Eq. (3.2) in result (3.3), we obtain

$$\mathrm{M}_Y(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} \gamma^{\frac{k}{\beta}} \alpha^k \left(1+\lambda^{-\beta}\right)^{\frac{k}{\beta}} \Gamma\left(1-\frac{k}{\beta}\right).$$

4. DISTRIBUTION OF ORDER STATISTICS

Let $Y_1, Y_2, ..., Y_n$ is a random sample drawn from $SGIW(\alpha,\beta,\gamma,\lambda)$. Since $Y_1, Y_2, ..., Y_n$ are i.i.d. continuous random variables, then the probability that any two (or more) observation in random sample take the same magnitude (the same value is equal to zero). Therefore, there exists a unique ordered arrangement of the sample observation according to magnitude.

Let $Y_{(1:n)}, Y_{(2:n)}, ..., Y_{(n:n)}$ be the order statistics. Then, the PDF of $Y_{(r:n)}$, $1 \le r \le n$, denoted by $g_{r:n}(y)$, is given by

$$g_{r:n}(t) = C_{r:n}\left[G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right]^{r-1}\left[S_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right]^{n-r} g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y), \qquad (4.1)$$

where $C_{r:n} = n!/\left((r-1)!(n-r)!\right)$.

4.1 Distribution of Minimum and Maximum

The smallest observation in the sample as $Y_{(1:n)} = \min\left(Y_1,...,Y_n\right)$, the largest observation in the sample $Y_{(n)} = \max\left(Y_1,...,Y_n\right)$ and the median order as $Y_{(m+1:n)}$, if $n = 2m+1$ thus $Y_{(1)} < Y_{(2)} < ... < Y_{(n)} < \infty$ are given by**:**

i) $$g_{1:n}(y) = n\left[1 - G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right]^{n-1} g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)$$
$$= n\left[1-\left(\exp(-\Omega)\right)\right]^{n-1}\left(\Omega\beta / y\right)\left(\exp(-\Omega)\right)$$

ii) $$g_{n:n}(y) = n\left[G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right]^{n-1} . g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y) = n\left(\exp(-\Omega\left(n+1\right)+1)\right)\left(\Omega\beta / y\right)$$

iii) $$g_{m+1:n}(y) = (2m+1)!/\left(m!\right)^2 G^m{}_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\left(1 - G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^m g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)$$
$$= \left((2m+1)!\Omega\beta / y\left(m!\right)^2\right)\left[1-\left(\exp(-\Omega)\right)\right]^m\left[\exp(-\Omega(m+1))\right];$$
$$\Omega = \left[\gamma\left(\alpha/y\right)^\beta\left(1+\lambda^{-\beta}\right)\right]$$

4.2 Joint Distribution of the r^{th} and j^{th} Order Statistics

The bivariate probability density function of $Y_{(r:n)}$ and $Y_{(j:n)}$, $1 \le r \le j \le n$, from the $SGIW(\alpha,\beta,\gamma,\lambda)$ distribution is given by:

$$g_{r,j:n}\left(y_r, y_j\right) = C_{r:j:n}\left(G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}\left(y_r\right)\right)^{r-1}\left(G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}\left(y_j\right) - G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}\left(y_r\right)\right)^{j-r-1}$$
$$\times\left(1 - G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}\left(y_j\right)\right)^{n-j} g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}\left(y_r\right) g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}\left(y_j\right) \quad (4.2)$$

By using (2.5), we have

$$g_{r:j:n}\left(y_r, y_j\right)=C_{r:j:n}\left(\exp\left(-\Omega_r\right)\right)^{r-1}\left(\exp\left(-\Omega_j\right)-\exp\left(-\Omega_r\right)\right)^{j-r-1}\left(1-\exp\left(-\Omega_j\right)\right)^{n-j}$$
$$\times\,\Omega_r\Omega_j\left(\beta/y\right)^2\exp\left(-\left(\Omega_r+\Omega_j\right)\right),$$
$$=C_{r:j:n}\Omega_r\Omega_j\left(\beta/y\right)^2\sum_{k=0}^{j-r-1}\sum_{d=0}^{n-j}\binom{j-r-1}{k}\binom{n-j}{d}(-1)^{n-r-k-1}$$
$$\exp\left(\Omega_r\left(k-j+1\right)+\Omega_j\left(j+d-k-n-1\right)\right).$$

whereas:

1. $-\infty < y_r < y_j < \infty$;
2. $C_{r:j:n} = n!/\left((r-1)!(j-r-1)!(n-j)!\right)$;
3. $\Omega_k = \gamma\left(\alpha/y_r\right)^{\beta}\left(1+\lambda^{-\beta}\right);\ \ k = r, j.$

Consider (4.2), then we can conclude that the minimum and maximum bivariate probability density of the $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ denoted by $g_{1:n::n}(y_r, y_j)$, that can be investigated from equation (4.2) by substituting $j = n$ and $r = 1$ as follows

$$g_{1:n:n}\left(y_1, y_n\right)=\frac{n!\Omega_r\Omega_j\left(\beta/y\right)^2}{(n-2)!}\left(\exp\left(-\Omega_j\right)-\exp\left(-\Omega_r\right)\right)^{n-2}\exp\left(-\left(\Omega_r+\Omega_j\right)\right).$$

4.3 The i^{th} Moment of Order Statistic

The i^{th} moment of order statistic $Y_{(r:n)}$ is defined as

$$\mu_{r:n}^{(i)}=E\left(Y_{r:n}^{(i)}\right)=\binom{n}{r}\int_0^\infty g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)G^{r-1}{}_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\left(1-G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^{n-r}dy \tag{4.3}$$

An alternative the expression of (4.3) can be derived by consider Barakat and Abdelkader (2004), as follows

$$\mu_{r:n}^{(i)}=E\left(Y_{r:n}^{(i)}\right)=i\sum_{s=n-r+1}^{n}(-1)^{s-n+r-1}\binom{s-1}{n-r}\binom{n}{s}\int_0^\infty y^{i-1}\left(1-G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^{s}dy \tag{4.4}$$

By using the binomial expansion for $\left(1-G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^s$ we get

$$\left(1-G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^{x}=\sum_{d=0}^{x}(-1)^d\binom{x}{d}\left(G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^d.$$

Therefore, the integration in Eq. (4.4) can be written as

$$\int_0^\infty y^{i-1}\left(1-G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^{s}dy=\sum_{k=0}^{s}(-1)^k\binom{s}{k}\int_0^\infty y^{i-1}\left(G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)\right)^k dy \tag{4.5}$$

Substituting (2.5) for $G_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y)$ in (4.5), we get

$$=\frac{1}{\beta}\sum_{k=0}^{s}(-1)^k\binom{s}{k}k^{\frac{i}{\beta}}\gamma^{\frac{i}{\beta}}\alpha^i\left(1+\lambda^{-\beta}\right)\Gamma\left(-\frac{i}{\beta}\right).$$

Finally, the Eq. (4.4) can be written as

$$E\left(Y_{r:n}^{(i)}\right)=\frac{i}{\beta}\gamma^{\frac{i}{\beta}}\alpha^i\left(1+\lambda^{-\beta}\right)^{\frac{i}{\beta}}\Gamma\left(-\frac{i}{\beta}\right)\sum_{s=n-r+1}^{n}\sum_{k=0}^{s}(-1)^{s-n+r-1}\binom{s-1}{n-r}\binom{n}{s}(-1)^k\binom{s}{k}k^{\frac{i}{\beta}}.$$

5. THE PARAMETER ESTIMATION OF SGIW DISTRIBUTION

In this section, we obtain an estimate of parameters of $SGIW(\alpha,\beta,\gamma,\lambda)$ by using moment estimator and maximum likelihood are derived and their asymptotic are given

5.1 The Moment Estimators

Let $Y_1,Y_2,...,Y_n$ be an independent a weighted function, then the method of moments estimator of α,γ and λ when β known. If $E(Y)$ and $E\left(Y^2\right)$ equal to the corresponding sample moments, that is:

$$\frac{1}{n}\sum_{i=1}^{n}Y_i=\gamma^{\frac{1}{\beta}}\alpha\left(1+\lambda^{-\beta}\right)^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right) \text{ and } \frac{1}{n}\sum_{i=1}^{n}Y_i^2=\gamma^{\frac{2}{\beta}}\alpha^2\left(1+\left(\frac{1}{\lambda}\right)^{\beta}\right)^{\frac{2}{\beta}}\Gamma\left(1-\frac{2}{\beta}\right). \tag{5.1}$$

For fixed $\beta>1$, the method of moment estimate (MME) of α,γ and λ. Thus, the moment estimators of α,γ and λ can be obtained as solution equations in (5.1) we get the following:

$$\hat{\alpha}=\frac{RK}{\sqrt[\beta]{\gamma}},\ \hat{\gamma}=\left[\frac{RK}{\alpha}\right]^{\beta} \text{ and } \hat{\lambda}=\left[\left\{\left(\left(\frac{R}{\alpha}\right)^{\beta}\frac{1}{\gamma}\right)-1\right\}\right]^{-\frac{1}{\beta}}$$

where $R=\overline{T}/\Gamma\left(1-\frac{1}{\beta}\right)$ and $K=\left(1+\lambda^{-\beta}\right)^{-\frac{1}{\beta}}$

5.2 Maximum Likelihood Estimators

The maximum likelihood estimators (MLEs) for the parameters of the *SGIW* distribution are studied in this section. Let $Y_1,Y_2,...,Y_n$ be a random sample of size n from the *SGIW* distribution with (2.4), therefore the likelihood function can be written as follows:

$$L(\alpha,\beta,\gamma,\lambda,y)=\prod_{i=1}^{n}g_{Y^w|\{\alpha,\beta,\gamma,\lambda\}}(y),$$

the likelihood function based on the observed sample $\{y_1, y_2, ..., y_n\}$ is

$$L(y_1, y_2, ..., y_n \mid \alpha, \beta, \gamma, \lambda) = \exp\left[-\gamma\alpha^{\beta}\left(1+\lambda^{-\beta}\right)\sum_{i=1}^{n} y^{-\beta}\right]\left[1+\lambda^{-\beta}\right]^{n}\gamma^{n}\beta^{n}\alpha^{n\beta}\prod_{i=1}^{n} y_i^{-(\beta+1)} . \tag{5.3}$$

By using (5.3), we have

$$\ln L(y_1, y_2, ..., y_n \mid \alpha, \beta, \gamma, \lambda) = -\gamma\alpha^{\beta}\left(1+\lambda^{-\beta}\right)\sum_{i=1}^{n} y^{-\beta} + n\ln\left(1+\lambda^{-\beta}\right) + n\ln\gamma$$
$$+n\beta\ln\alpha + n\ln\beta - (\beta+1)\sum_{i=1}^{n}\ln y_i \tag{5.4}$$

The components of the score vector for the parameters α, β, γ and λ are given by

$$\frac{\partial \ln L}{\partial \alpha} = -\beta\alpha^{\beta-1}\gamma\left(1+\lambda^{-\beta}\right)\sum_{i=1}^{n} y_i^{-\beta} + \frac{n\beta}{\alpha} \tag{5.5}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n\lambda^{-\beta}\ln\left(\lambda^{-1}\right)}{\left[1+\lambda^{-\beta}\right]} + \frac{n}{\beta} + n\ln(\alpha) - \gamma\alpha^{\beta}\sum_{i=1}^{n} y^{-\beta}\ln\left(\alpha y^{-1}\right)\left[\left(1+\lambda^{-\beta}\right)\ln\left(\lambda^{-1}\right)\right] - \sum_{i=1}^{n}\ln y_i \tag{5.6}$$

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} - \alpha^{\beta}\left(1+\lambda^{-\beta}\right)\sum_{k=1}^{n} y_k^{-\beta} \tag{5.7}$$

$$\frac{\partial \ln L}{\partial \lambda} = \gamma\beta\alpha^{\beta}\lambda^{-\beta-1}\sum_{k=1}^{n} y_k^{-\beta} - \frac{n\beta\lambda^{-\beta-1}}{\left[1+\lambda^{-\beta}\right]} \tag{5.8}$$

We should to solve the nonlinear equations (5.5), (5.6), (5.7) and (5.8) simultaneously, because these equations does not have a closed forms solution and must be solved iteratively to obtain the MLEs of the unknown parameters.

For the observed information matrix of the parameters ($\alpha, \beta, \gamma, \lambda$), we calculate the second partial derivatives of (5.4) with respect to $\alpha, \beta, \gamma, \lambda$ respectively, can derivative as follows:

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\beta(\beta-1)\gamma\alpha^{\beta-2}\sum_{i=1}^{n} y_i^{-\beta}\left(1+\lambda^{-\beta}\right) - \frac{n\beta}{\alpha^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\gamma\left(\alpha / \sum_{i=1}^{n} y_i\right)^{\beta}\left(\ln\left(\frac{\alpha}{y_i}\right)\right)^2\left[\left(1+\lambda^{-\beta}\right)\ln\left(\lambda^{-1}\right)\right] + \frac{n\left(\lambda^{-\beta}\right)\left(\ln\left(\lambda^{-1}\right)\right)^2}{\left(\left(1+\lambda^{-\beta}\right)\right)^2} - \frac{n}{\beta^2}$$

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = -\frac{n}{\gamma^2} \text{ and } \frac{\partial^2 \ln L}{\partial \lambda^2} = -\beta(\beta+1)\gamma\alpha^{\beta}\lambda^{-(\beta+2)}\sum_{i=1}^{n} y_i^{-\beta} + \frac{n\beta\lambda^{-(\beta+2)}\left[\beta+1+\lambda^{-\beta}\right]}{\left(1+\lambda^{-\beta}\right)^2}.$$

In addition to, the partial derivatives of (5.5), (5.6), (5.7) and (5.8) with respect to $\alpha,\beta,\gamma,\lambda$ are:

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} = \frac{n}{\alpha} - \gamma\alpha^{\beta-1}\ln(\alpha)\sum_{i=1}^{n} y_i^{-\beta}\ln\left(y^{-1}\right)\left[\left(1+\lambda^{-\beta}\right)\ln\left(\lambda^{-1}\right)\right],$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\gamma} = -\beta\alpha^{\beta-1}\sum_{i=1}^{n} y_i^{-\beta}\left(1+\lambda^{-\beta}\right),\ \frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} = \beta^2\gamma\alpha^{\beta-1}\sum_{i=1}^{n} y_i^{-\beta}\left(\lambda\right)^{-(\beta+1)},$$

$$\frac{\partial^2 \ln L}{\partial\gamma\partial\beta} = -\alpha^{\beta}\sum_{i=1}^{n} y_i^{-\beta}\ln\left(\frac{\alpha}{y}\right)\left(1+\lambda^{-\beta}\ln\left(\lambda^{-1}\right)\right),$$

$$\frac{\partial^2 \ln L}{\partial\gamma\partial\lambda} = +\beta\left(\alpha\Big/\sum_{i=1}^{n} y_i\right)^{\beta}\left(\lambda\right)^{-(\beta+1)},$$

$$\frac{\partial^2 \ln L}{\partial\lambda\partial\beta} = \gamma\left(\alpha\Big/\lambda\sum_{i=1}^{n} y_i\right)^{\beta}\left(\lambda^{-1}\right)\ln\left(\frac{\alpha}{\lambda y_i}\right) - \frac{n\left(\lambda\right)^{-(\beta+1)}\ln\left(\lambda^{-1}\right)\left[1+\lambda^{-\beta}-\beta\lambda^{-\beta}\right]}{\left(1+\lambda^{-\beta}\right)^2}.$$

Therefore, we have

$$\left(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\lambda}\right)^T \in Normal\left[(\alpha,\beta,\gamma,\lambda)^T, V_0^{-1}\right], \tag{5.9}$$

with the information matrix

$$V_0 = -E\begin{pmatrix} V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha\gamma} & V_{\alpha\lambda} \\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta\gamma} & V_{\beta\lambda} \\ V_{\gamma\alpha} & V_{\gamma\beta} & \hat{V}_{\gamma\gamma} & V_{\gamma\lambda} \\ V_{\lambda\alpha} & V_{\lambda\beta} & V_{\lambda\gamma} & V_{\lambda\lambda} \end{pmatrix},$$

where $V\varphi_1\varphi_2 = \frac{\partial^2 \ln L}{\partial\varphi_1\varphi_2}$ denotes the second derivative of $\ln L$ with respect to φ_1 and φ_2. The matrix V_0^{-1} represents the asymptotic variance and covariance matrix of the MLEs. According to (5.9), approximate 100(1-ϕ)% confidence intervals of the parameters α, β, γ and λ are determined respectively as

$$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{Var\left(\hat{\alpha}\right)},\ \hat{\beta} \pm Z_{\alpha/2}\sqrt{Var\left(\hat{\beta}\right)},\ \hat{\gamma} \pm Z_{\alpha/2}\sqrt{Var\left(\hat{\gamma}\right)} \text{ and } \hat{\lambda} \pm Z_{\alpha/2}\sqrt{Var\left(\hat{\lambda}\right)}$$

Here, $Z_{\alpha/2}$ is the upper $\alpha/2$ the percentile of the standard normal distribution.

6. NUMERICAL ILLUSTRATION

In this section, random numbers are generated and two real data sets are considered, therefore, the maximum likelihood estimates for certain values of the parameters α, β and γ and different value of the parameter λ are obtained.

6.1 Generated Random Number

In this section, Mont Carlo simulation method is used to study the properties of the estimations of the unknown parameters of the $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ distribution. The simulation procedures are presented through the following steps:

- **Step 1.** A random sample $Y_1,Y_2,...,Y_n$ of sizes $n=10$, 50, 70, 100 and 150 is generated from $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ distribution. The parameters are fixed at values (1.25, 2, 0.25) respectively and the parameter is studied at different values (0.5, 1, and 3);
- **Step 2.** MLEs of $\hat{\alpha},\hat{\beta},\hat{\gamma}$, and $\hat{\lambda}$ can be investigated by solving the non-liner equations (5.5), (5.6), (5.7) and (5.8);
- **Step 3.** Steps 1 and 2 will be repeated 1000 times for each sample sizes and for the selected value of the parameters;
- **Step 4.** The mean, biases, variance, skewness and kurtosis are obtained for different sample sizes.
- **Step 5.** Computing the mean square error associated with the MLE of each parameter based on the following relation.

$$MSE(\hat{\Psi})=E\left(\hat{\Psi}-\Psi\right)^2=\mathrm{var}(\hat{\Psi})+Bias^2\left(\hat{\Psi}\right),\text{ whereas }\hat{\Psi}=(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\lambda}).$$

These steps were performed using MATHCAD (2001) software. Table (6.1), (6.2) and (6.3) presents the mean, biases, variance, skewness and kurtosis for the MLEs of the parameters α,β,γ and for *1000* random samples of sizes 10, 50, 70, 100 and 150 generated by the proposed random number generator for the parameters α,β and are fixed at value 1.25, 2, 0.25 respectively and different value of the parameter λ. These tables shows that the mean square error of the estimates decreases as the sample sizes increase

Table 6.1
The Parameters Estimation from SGIW Distribution using MLE:
$\alpha = 1.25,\ \beta = 2,\ \gamma = 0.25$ **and** $\lambda = 0.5$

n	Ψ	**Mean**	**Bias**	**Variance**	**MSE**	**Sek**	**Kurt**
10	α	1.535	0.285	0.06	0.142	0.686	2.114
	β	2.335	0.335	0.469	0.581	1.162	2.108
	γ	0.158	-0.092	0.002024	0.01	0.744	0.892
	λ	0.584	0.084	0.007337	0.014	0.147	-0.157
50	α	1.53	0.28	0.01	0.089	-0.094	-0.134
	β	2.045	0.045	0.058	0.06	0.717	1.729
	γ	0.16	-0.09	0.0004612	0.00854	0.307	-0.475
	λ	0.562	0.062	0.001666	0.005537	-0.246	0.165
70	α	1.528	0.278	0.006373	0.085	-0.263	0.192
	β	2.031	0.031	0.038	0.039	0.444	0.522
	γ	0.159	-0.091	0.0003536	0.00853	0.258	-0.863
	λ	0.561	0.061	0.001188	0.00491	-0.427	0.39
100	α	1.532	0.282	0.00506	0.084	-0.103	-0.066
	β	2.026	0.026	0.025	0.026	0.286	-0.051
	γ	0.16	-0.09	0.0003007	0.008456	0.168	-0.953
	λ	0.564	0.064	0.000821	0.004865	-0.437	0.702
150	α	1.535	0.285	0.003108	0.083	-0.04	0.216
	β	2.016	0.016	0.017	0.017	0.271	0.091
	γ	0.16	-0.09	0.0002494	0.008342	0.228	-0.776
	λ	0.564	0.064	0.0004888	0.004593	-0.416	1.14

Table 6.2
The Parameters Estimation from SGIW Distribution using MLE:
$\alpha = 1.25,\ \beta = 2,\ \gamma = 0.25$ **and** $\lambda = 1$

n	Ψ	**Mean**	**Bias**	**Variance**	**MSE**	**Sek**	**Kurt**
10	α	1.223	-0.027	0.097	0.097	3.22	14.636
	β	2.341	0.341	0.484	0.6	1.576	4.581
	γ	0.13	-0.12	0.0007461	0.015	2.635	11.444
	λ	0.744	-0.256	0.072	0.138	3.174	11.101
50	α	1.227	-0.023	0.009197	0.00971	2.056	15.167
	β	2.053	0.053	0.053	0.056	0.399	0.063
	γ	0.13	-0.12	0.0001178	0.014	0.173	4.241
	λ	0.686	-0.314	0.01	0.109	4.588	58.896
70	α	1.231	-0.019	0.005422	0.005794	0.042	-0.353
	β	2.034	0.034	0.04	0.041	0.341	0.041
	γ	0.131	-0.119	0.00007907	0.014	-0.655	1.825
	λ	0.686	-0.314	0.005246	0.104	-1.026	0.461
100	α	1.241	-0.009303	0.003788	0.003875	-0.022	0.079
	β	2.03	0.03	0.028	0.029	0.441	0.198
	γ	0.131	-0.119	0.00005423	0.014	-0.74	1.615
	λ	0.699	-0.301	0.003682	0.095	-1.026	0.266
150	α	1.251	0.0005896	0.002242	0.002242	-0.118	-0.121
	β	2.012	0.012	0.017	0.017	0.171	-0.073
	γ	0.132	-0.118	0.00003458	0.014	-0.255	-0.018
	λ	0.706	-0.294	0.003067	0.089	-1.422	1.528

Table 6.3
The Parameters Estimation from SGIW Distribution using MLE:
$\alpha = 1.25,\ \beta = 2,\ \gamma = 0.25$ **and** $\lambda = 3$

n	Ψ	**Mean**	**Bias**	**Variance**	**MSE**	**Sek**	**Kurt**
10	α	0.933	-0.317	0.058	0.159	20.356	515.105
	β	2.262	0.262	0.532	0.6	0.46	9.008
	γ	0.348	0.098	0.01	0.02	0.707	2.588
	λ	2.599	-0.401	0.796	0.957	5.887	67.26
50	α	0.917	-0.333	0.0007534	0.112	-0.097	1.026
	β	2.045	0.045	0.058	0.06	0.717	1.727
	γ	0.36	0.11	0.004019	0.016	-0.301	-0.607
	λ	2.536	-0.464	0.065	0.281	-0.109	2.685
70	α	0.918	-0.332	0.0005883	0.111	-0.306	1.208
	β	2.042	0.042	0.039	0.041	0.365	0.104
	γ	0.356	0.106	0.003443	0.015	-0.139	-0.67
	λ	2.527	-0.473	0.051	0.275	0.005271	0.484
100	α	0.918	-0.332	0.0004481	0.11	-0.488	0.364
	β	2.03	0.03	0.024	0.025	0.27	-0.002082
	γ	0.357	0.107	0.002806	0.014	-0.115	-0.78
	λ	2.54	-0.46	0.048	0.259	-0.118	-0.815
150	α	0.92	-0.33	0.0003831	0.109	-0.898	0.776
	β	2.018	0.018	0.017	0.017	0.361	0.36
	γ	0.356	0.106	0.002211	0.013	0.098	-0.718
	λ	2.552	-0.448	0.048	0.249	0.0005609	-1.411

6.2 Applications

In this subsection, the two real data sets to illustrate that the $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ distribution might fit better than a model based on the $\left(GIW\left(\alpha,\beta,\gamma,y\right)\right)$ distribution.

Data Set (1): The first data sets is excluded from Lee and Wang (2003) which represent remission times (in months) of a random sample of 128 bladder cancer patients. The data are as follows:

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2	2.23
9.02	13.29	0.4	2.26	3.57	5.06	7.09	9.22	13.8	25.74
5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.7	5.17	7.28
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88
14.83	34.26	0.9	2.69	4.18	5.34	7.59	10.66	15.96	36.66
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63
2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
1.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46	4.4	5.85
1.76	3.25	4.5	6.25	8.37	12.02	2.02	3.31	4.51	6.54
3.52	4.98	6.97	0.5	2.46	3.64	9.74	14.76	26.31	5.32
7.39	10.34	1.05	2.69	4.23	17.12	46.12	1.26	7.87	11.64
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63
17.36	8.26	11.98	19.13	8.53	12.03	20.28	22.69		

In order to compare the proposed distribution $SGIW\left(\alpha,\beta,\gamma,\lambda\right)$ with $GIW\left(\alpha,\beta,\gamma,y\right)$ distribution we consider criteria like the Kolmogorov- Smirnov test statistic, $(-2\ln L)$, Akaike Information Criterion (AIC), Bayesian information criterion (BIC) and Consistent Akaike Information Criterion (CAIC) which are defined, respectively, by, $AIC=-2\ln L+2k$, $BIC=k\ln L-2\ln L$ and $CAIC=AIC+\dfrac{2k(k+1)}{n-k-1}$, where k is the number of parameter in the statistical model, n denotes the sample size and $\left(\ln L\right)$ is the maximized value of the log-likelihood function. The better distribution corresponds to smaller $\left(K-S\right)$, $\left(-\ln L\right)$, AIC, CAIC, BIC values. All the computations were done using the Mathcad software. Summary of all these fitted distributions is introduced in tables (6.4) and (6.5).

Table 6.4
Estimate of Models for the GIW and SGIW Distributions

Model	**Parameter Estimate**				$\ln L(.,y)$
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	
GIW	1.193	0.715	1.105	-	-466.064
SGIW	1.215	0.751	0.981	0.906	-444.057

Table 6.5
Goodness of Fit Criteria

Model	**K-S**	**-2logL(.,t)**	**AIC**	**CAIC**	**BIC**
GIW	0.361	932.127	938.127	938.321	946.683
SGIW	0.147	888.115	896.115	896.44	907.523

Tables (6.4) and (6.5) shows that the $SGIW(\alpha,\beta,\gamma,\lambda)$ distribution gives better fit than the $GIW(\alpha,\beta,\gamma,y)$ distribution.

Data Set (2): The second data set is obtained Lawless (2003) which represent on distance between cracks in a pipe data set and it is provided below:

30.94	18.51	16.62	51.56	22.85	22.38	19.08	49.56
17.12	10.67	25.43	10.24	27.47	14.7	14.1	29.93
27.98	36.02	19.4	14.97	22.57	12.26	18.14	18.84

Table 6.6
Estimate of Models for the GIW and SGIW Distribution

Model	**Parameter Estimate**				$\ln L(.,y)$
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	
GIW	0.176	0.351	2.789	_	-126.479
SGIW	1.127	2.389	0.011	$9.68e^{-03}$	-86.396

Table 6.7
Goodness of Fit Criteria

Model	**K-S**	**-2logL(.,t)**	**AIC**	**CAIC**	**BIC**
GIW	0.512	252.958	258.958	260.158	262.492
SGIW	0.103	172.792	180.792	182.897	185.504

Similarly, the results given in tables (6.3) and (6.4) show that the $SGIW(\alpha,\beta,\gamma,\lambda)$ distribution is the better distribution for fitting these data sets compared to $\left(GIW(\alpha,\beta,\gamma,y)\right)$ distribution.

CONCLUSION

The skew generalized inverse weibull distribution has been studied. At first, the PDF of the $SGIW(\alpha,\beta,\gamma,\lambda)$ have been obtained considering weight as $w(y) = G_y(\lambda y)$ by the idea proposed by Azzalini (1985). Besides, characterizing the distribution of a random variable Y of the $SGIW(\alpha,\beta,\gamma,\lambda)$, many functions have been introduced. Expressions for density, minimum and maximum order statistic and i^{th} moment of the order statistics are derived. The estimation of the parameters of the $SGIW(\alpha,\beta,\gamma,\lambda)$ are introduced by maximum likelihood and moments methods are introduced.

ACKNOWLEDGEMENTS

The authors would like to express their sincere gratitude and thankfulness to the editor, the associate editor, and the examiners for their academic guidance and their valuable contribution.

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