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ON DOUBLE k-CLASS ESTIMATORS WITH MULTIVARIATE t-ERRORS

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ABSTRACT

This paper aims at developing a framework for an analysis of exact and asymptotic bias, moment matrix and risk function for double k-class estimators in general linear regression model with multivariate t-errors. It should also contribute to wards a better understanding of the performance of double k-class estimators and the comparison between double k-class estimators and ordinary least squares estimators in the same linear regression model.The study will utilize Monte-Carlo approach to investigate the asymptotic properties of several estimators arising from the family of double k-class estimators.

KEYWORDS

Bias estimators; Double k-class estimators; Multivariate t-errors; and Risk function.

1. INTRODUCTION

Non-linear and biased estimators of regression coefficients in a linear regression model is known to have smaller risk than linear and unbiased estimators under mild constraints. One such family is described by the double k-class estimators proposed by Ullah and Ullah (1978). The family of double k-class estimators is characterized by two characterizing scalars and includes many estimators as its particular cases, including the Stein rule family of estimators lamented by James and Stein (1961). The double k-class estimators was proposed under the assumption of spherical or homoskedastic disturbances in the context of linear regression models. Carter et al. (1993) discovered estimators having better performance than the OLS estimators to have the determinant of two characterizing scalars for several families of estimators with conditions of their dominance over ordinary least squares.

Wan and Chaturvedi (2001)further determined an analysis to the case when disturbances are non-spherical andtheir variance covariance matrix is unknown. They suggested the family offeasiblegeneralized double k-class estimators and analyzed the quadratic risk showing several estimators arising from particular cases under the largesample asymptotic theory. The performance under balanced loss function and general Pitmancloseness criterion was studied by Chaturvedi and Shalabh (2004) while Shalabh et al. (2012) examined the cases using LINEX loss function for non-spherical disturbances when the variance covariance matrix are unknown. They examined the risk associated with the estimators of the feasible generalized double k-class estimators under the LINEX loss function derived in a linear regression model.The review of literature for this paper therefore focused on performance of the double k-class estimators under multivariate normal-errors.

By first using the multivariate t-distributionwhen the original data have longer tails than the normal distribution, the multiple imputed data allow more valid statistical inferences than those using the normal distribution whose "influential" observations are deleted considering the facts that the t-distribution is widely used in applied statistics for robust statistical inferences Kotz and Nadarajah (2004).

Following this introduction, the paper is structured as follows: Section 1 provides introduction of double k-class estimators. The model and estimators are described in Section 2. The properties of the double k-class estimators for the coefficients in a linear regression model under spherical disturbances are derived and analyze withmultivariate t-error in Section 3. A simulation study was conducted using the R program in section 4. The conclusion in section (5). And special function and some of the expectations required for the proofs in section (3) are presented in the Appendix.

2. THE MODEL AND ESTIMATORS

Consider the regression model

$$y = x\beta + u \tag{1}$$

where $y_{T\times 1}$ is a vector of observations on the dependent variable, $x_{T\times p}$ is a matrix of observations on explanatory variables, $\beta_{p\times 1}$ is a parameter vector and $u_{T\times 1}$ is a disturbance vector.

The regression model is valid withthe following assumptions;

Assumption 1: $x_{T\times p}$ is nonstochastic matrix and of rank p

Assumption 2: $u_{T\times 1} \sim T_{\nu}\left[\mu = 0, \sigma^2 I\right]$

Assumption 3: $T > p$

Double k-class estimators are

$$\tilde{b}_{k_1,k_2} = \left[1 - \frac{k_1 \hat{u}'\hat{u}}{y'y - k_2 \hat{u}'\hat{u}}\right] b \tag{2}$$

where k_1, k_2 are arbitrary scalars which may be stochastic or nonstochastic; $\tilde{b}_{k_1,k_2}$, represents a family of Double k-class estimators; $\hat{u} = y - Xb$ and $b = \left(X'X\right)^{-1} X'y$ is the ordinary least squares estimator of β .

The stein-rule estimator is given by;

$$\mathrm{b}_{k_1,1} = \left[1 - \frac{k_1 \hat{u}'\hat{u}}{y'y - \hat{u}'\hat{u}}\right] b \tag{3}$$

3. THE MAIN RESULTS

This section illustrates the main results as follows:

3.1 The Exact Results

The formulas for bias, moment matrix and the risk function of the double k-class estimatorsunder multivariate t-errors for $0 \le k_2 \le 1$ are derived.

1) The sampling error of the estimator in equation (2) is given by;

$$\left(\tilde{b}_{k_1,k_2} - \beta\right) = \left(b - \beta\right) - k_1 cb \qquad (4)$$

where $c = \dfrac{y'My}{y'Ny}, M = I - x\left(x'x\right)^{-1} x', \; N = I - k_2 M$

M is an idempotent matrix with rank n while N is a nonnegative definite matrix provided

$0 \le k_2 \le 1$

where $n = T - p$

2) The assumption two we observe that

$$y \sim T_{\nu}\left[\mu = x\beta, \sigma^2 \mathrm{I}\right] \qquad (5)$$

From equation (4) and (5) ,we can therefore derive the bias and moment matrix, the risk function of $\tilde{b}_{\mathrm{k}_1,\mathrm{k}_2}$ with the needed expectations of cb, cbb' and c^2bb'. This is obtained by first deriving $E\left(c\right)$ and $E\left(c^2\right)$ [Appendix B] using characteristic function of the multivariate t-distributions of Sutradhar (1986), Kotz and Nadarajah (2004) and Ullah and Nagar (1974) expressed in terms of $R(.)$ functions [Appendix A].

The following notations and functions are introduced for simplicity:

$$A_{g,e,o} = R_{\left(k_2^*, r^g, \; \theta, \; -2Tr+e, \; 2Tr-o\right)} \qquad (6)$$

where $g, \; o = 0,1,2, \; \ldots.. \; ; \; e = \ldots, -1, 0, 1, 2, \ldots..., \; \theta = \dfrac{\beta' x' x\beta}{\sigma^2}$ is a noncentrality parameter, $0 \le k_2^* \le 1$, and the function $R(\; . \;)$ is as defined in Appendix A.

Theorem 1:

The exact bias of the double k-class estimators of β for $0 \le k_2 \le 1$ is given by

$$E\left(\tilde{b}_{k_1,k_2} - \beta\right) = -2nk_1\sqrt{\pi}.\beta\left[\frac{\nu}{\left(\nu-2\right)}\frac{4T}{k_2^*}A_{2,-1,0} + \frac{1}{k_2^*}A_{1,0,0}\right] \qquad (7)$$

where $A_{2,-1,0}$ is as given in equation (6) for $g=2,\ e=-1$ and $o=0$; $\mathrm{A}_{1,0,0}$ for $g=2,\ e=-1$ and $o=0$

Proof:

$$E\left(\tilde{b}_{k_1,k_2}-\beta\right)=-k_1E\left(cb\right) \tag{8}$$

where $c=\dfrac{y'\,My}{y'\,Ny}$, M is idempotent matrix of rank n ,we can always obtain an orthogonal matrix $\omega_{T\times T}$ such that

$$\omega'M\omega=\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}=D_1,\ \omega'N\omega=\begin{bmatrix} \left(1-k_2\right)I_n & 0 \\ 0 & I_{T-n} \end{bmatrix}=I-k_2D_1=D_2 \tag{9}$$

where I_n represents the identity matrix of order $n\times n$ and 0 is the zero matrix of proper order. We can further, consider D_1 a $T\times T$ diagonal matrix with first n unity elements, $T-n$ zero elements, and D_2 a $T\times T$ diagonal matrix with first n elements $\left(1-k_2\right)$ and $T-n$ unity elements. D_2 will be nonnegative definite for $0\le k_2\le 1$.

The expectation of cb on the right of equation (8) can be obtained as

$$E\left(cb\right)=\left(x'x\right)^{-1}x'\sigma\omega E\left(zc\right) \tag{10}$$

where,

$$z=\frac{\omega'y}{\sigma}\sim T_\nu\left[x\beta\ ,\mathrm{I}\right],\ z'=\frac{\omega'x\beta}{\sigma},\ c=\frac{y'My}{y'Ny}=\frac{z'D_1z}{z'D_2z} \tag{11}$$

Note that,

$$E\left(zc\right)=E\left[\left(z-\overline{z}\right)c\right]+\overline{z}E\left(c\right)$$

$$=\frac{-\nu}{2\left(\dfrac{\nu}{2}-1\right)}\frac{\partial}{\partial\,\overline{z}}E\left(c\right)+\overline{z}E\left(c\right) \tag{12}$$

where,

$$E\left(c\right)=\frac{2n\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,\ r,\ \theta\ ,\ -2Tr,\ 2Tr\right)} \tag{13}$$

Next, the fact that

$$\frac{\partial}{\partial\,\overline{z}}E\left(c\right)=\frac{-8nT\sqrt{\pi}}{k_2^*\,\overline{z}}R_{\left(k_2^*,\ r^2,\ \theta\ ,\ -2Tr,\ 2Tr\right)} \tag{14}$$

And using equation (A.4) in appendix A, equation (12) can be obtained as;

$$E(zc) = \frac{-\nu}{(\nu-2)} \frac{-8nT\sqrt{\pi}}{\bar{z}k_2^*} R_{\left(k_2^*,\ r^2,\ \theta,\ -2Tr,\ 2Tr\right)} + \bar{z}\,\frac{2n\sqrt{\pi}}{k_2^*} R_{\left(k_2^*,\ r,\ \theta,\ -2Tr,\ 2Tr\right)} \tag{15}$$

Equation (15) is substituted in equation (10) to get,

$$E(cb) = 2n\sqrt{\pi}\beta\left[\frac{\nu}{\nu-2}\frac{4T}{k_2^*} R_{\left(k_2^*,r^2,\theta,-2Tr-1,2Tr\right)} + \frac{1}{k_2^*} R_{\left(k_2^*,r,\theta,-2Tr,2Tr\right)}\right] \tag{16}$$

Finally, substitute equation (6) and (16) in equation (8), we get the result declared in the theorem 1.

Theorem 2:

The exact moment matrix of the double k-class estimators of β for $0 \le k_2 < 1$ is given by;

$$\begin{aligned} E\left[\left(\tilde{b}_{k_1,k_2}-\beta\right)\left(\tilde{b}_{k_1,k_2}-\beta\right)'\right] &= \frac{\nu}{\nu-2}\sigma^2 (x'x)^{-1} - 2k_1 \frac{\nu}{(\nu-2)}\sigma^2 (x'x)^{-1}\left[\frac{\nu}{(\nu-4)}.\frac{8nT\sqrt{\pi}}{k_2^*}\right. \\ &\left[-4TA_{(3,-1,0)} - A_{(2,-1,0)}\right] + \frac{16nT\sqrt{\pi}}{k_2^*} A_{(2,0,0)} + \left.\frac{(\nu+T)}{((\nu+T)-2)}\frac{2n\sqrt{\pi}}{k_2^*} A_{(1,0,0)}\right] \\ &-k_1^2 \frac{\nu}{(\nu-2)}\sigma^2 (x'x)^{-1}\left[\frac{\nu}{(\nu-4)}\frac{4n\sqrt{\pi}}{k_2^{*2}}\left[-16nT^2 A_{(4,0,1)} + 4T(3n+2T)A_{(3,0,1)}\right.\right. \\ &\left.-2(2n+3T)A_{(2,0,1)} - A_{(1,0,1)}\right] + \frac{8n\sqrt{\pi}}{k_2^{*2}}\left[4nTA_{(3,1,1)} - (2n+2T)A_{(2,1,1)} + A_{(1,1,1)}\right] \\ &\left.+\left(\frac{(\nu+T)}{((\nu+T)-2)}\right)\frac{2n\sqrt{\pi}}{k_2^{*2}}\left[2nA_{(2,1,1)} - A_{(1,1,1)}\right]\right] - \frac{2nk_1\beta\beta'\sqrt{\pi}}{k_2^*}\left\{\frac{k_1}{k_2^*}\left[-2nA_{(2,1,1)}\right.\right. \\ &\left.\left.+A_{(1,1,1)}\right] - \frac{\nu 8T}{\nu-2}A_{(2,1,0)}\right\} \end{aligned} \tag{17}$$

Proof:

Using equation (4), we the moment matrix of the double k-class estimators can be written as:

$$E\left[\left(\tilde{b}_{k_1,k_2}-\beta\right)\left(\tilde{b}_{k_1,k_2}-\beta\right)'\right] = E(b-\beta)(b-\beta)' + 2k_1.E[cb]\beta' - 2k_1.E[cbb'] + k_1^2.E\left[c^2bb'\right] \tag{18}$$

The first term of equation(18) is $\frac{\nu}{\nu-2}\sigma^2 (x'x)^{-1}$

Considering the second from equation (16), its noted that,

$$E\left(cb\right)\beta' = 2n\sqrt{\pi}\beta\beta'\left[\frac{\nu}{\nu-2}\frac{4T}{k_2^*}R_{\left(k_2^*,r^2,\theta,-2Tr-1,2Tr\right)}+\frac{1}{k_2^*}R_{\left(k_2^*,r,\theta,-2Tr,2Tr\right)}\right] \tag{19}$$

The third term of equation (18) is then written as;

$$E\left(cbb'\right)=\sigma^2\left(x'x\right)^{-1}x'\omega E\left[zz'c\right]\omega'x\left(x'x\right)^{-1} \tag{20}$$

where z and c are defined in equation (11), respectively. The procedure in equation (12) will then produce,

$$E\left[zz'c\right]=E\left[\left[\left(z-\bar{z}\right)\left(z-\bar{z}\right)'+\left(z-\bar{z}\right)\bar{z}'+\bar{z}\left(z-\bar{z}\right)'+\bar{z}\,\bar{z}'\right]c\right]$$

$$=\frac{-\nu^2}{\left(\nu-2\right)\left(\nu-4\right)}\cdot\frac{\partial^2}{\partial\bar{z}\partial\bar{z}'}E\left(c\right)-\frac{2\nu}{\left(\nu-2\right)}\bar{z}\frac{\partial}{\partial\bar{z}'}E\left(c\right)$$

$$+\left(\bar{z}\,\bar{z}'+\frac{\nu\left(\nu+T\right)}{\left(\left(\nu+T\right)-2\right)\left(\nu-2\right)}\right)E\left[c\right] \tag{21}$$

where $E\left(c\right)$ is as given in equation (13). Using appendix A, equation (21) can be rewritten as;

$$E\left[zz'c\right]=\frac{\nu^2}{\left(\nu-2\right)\left(\nu-4\right)}\cdot\frac{8nT\sqrt{\pi}}{k_2^*}\left[-4TR_{\left(k_2^*,r^3,-2Tr-1,2Tr\right)}-R_{\left(k_2^*,r^2,-2Tr-1,2Tr\right)}\right]$$

$$+\frac{2\nu}{\left(\nu-2\right)}\cdot\frac{8nT\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r^2,-2Tr,2Tr\right)}+\bar{z}\,\bar{z}'\frac{2n\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r,-2Tr,2Tr\right)}$$

$$+\frac{\nu\left(\nu+T\right)}{\left(\left(\nu+T\right)-2\right)\left(\nu-2\right)}\frac{2n\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r,-2Tr,2Tr\right)} \tag{22}$$

and equation (20) is written as;

$$E\left(cbb'\right)=\sigma^2\left(x'x\right)^{-1}\left[\frac{\nu^2}{\left(\nu-2\right)\left(\nu-4\right)}\cdot\frac{8nT\sqrt{\pi}}{k_2^*}\left[-4TR_{\left(k_2^*,r^3,\theta,-2Tr-1,2Tr\right)}\right.\right.$$

$$\left.-R_{\left(k_2^*,r^2,\theta,-2Tr-1,2Tr\right)}\right]+\frac{2\nu}{\left(\nu-2\right)}\frac{8nT\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r^2,\theta,-2Tr,2Tr\right)}+\frac{\nu}{\left(\nu-2\right)}$$

$$\left.\frac{\left(\nu+T\right)}{\left(\left(\nu+T\right)-2\right)}\frac{2n\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r,\theta,-2Tr,2Tr\right)}\right]+\beta\beta'\frac{2n\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r,\theta,-2Tr,2Tr\right)} \tag{23}$$

Similarly, considering the fourth term of equation (18) we have,

$$E\left[zz'c^2\right] = \frac{-\nu^2}{(\nu-2)(\nu-4)} \cdot \frac{\partial^2}{\partial \bar{z} \partial \bar{z}'} E\left(c^2\right) - \frac{2\nu}{(\nu-2)} \bar{z} \frac{\partial}{\partial \bar{z}'} E\left(c^2\right) + \left(\bar{z}\,\bar{z}' + \frac{\nu(\nu+T)}{((\nu+T)-2)(\nu-2)}\right) E\left[c^2\right] \quad (24)$$

From Appendix B,

$$E\left(c^2\right) = \frac{2n\sqrt{\pi}\bar{z}'\bar{z}}{k_2^{*2}} \left[-2n R_{\left(k_2^*, r^2, -2Tr, 2Tr-1\right)} + R_{\left(k_2^*, r, -2Tr, 2Tr-1\right)}\right] \quad (25)$$

By taking the partial derivatives of $R(\ .\ )$ given in the Appendix A, we have equation (24) and hence $E\left(c^2 bb'\right)$ as;

$$\begin{aligned} E\left(c^2 bb'\right) = \sigma^2 \left(x'x\right)^{-1} \Bigg\{ & \frac{\nu^2}{(\nu-2)(\nu-4)} \frac{4n\sqrt{\pi}}{k_2^{*2}} \Big[16nT^2 R_{\left(k_2^*, r^4, \theta, -2Tr, 2Tr-1\right)} \\ & -4T(3n+2T) R_{\left(k_2^*, r^3, \theta, -2Tr, 2Tr-1\right)} + 2(2n+3T) R_{\left(k_2^*, r^2, \theta, -2Tr, 2Tr-1\right)} \\ & + R_{\left(k_2^*, r, \theta, -2Tr, 2Tr-1\right)}\Big] + \frac{2\nu}{(\nu-2)} \frac{4n\sqrt{\pi}}{k_2^{*2}} \Big[-4nT R_{\left(k_2^*, r^3, \theta, -2Tr+1, 2Tr-1\right)} \\ & + (2n+2T) R_{\left(k_2^*, r^2, \theta, -2Tr+1, 2Tr-1\right)} - R_{\left(k_2^*, r, \theta, -2Tr+1, 2Tr-1\right)}\Big] - \frac{2n\sqrt{\pi}}{k_2^{*2}} \\ & \left(\frac{\nu(\nu+T)}{((\nu+T)-2)(\nu-2)}\right) \Big[2n R_{\left(k_2^*, r^2, \theta, -2Tr+1, 2Tr-1\right)} - R_{\left(k_2^*, r, \theta, -2Tr+1, 2Tr-1\right)}\Big] \Bigg\} \\ & -\beta\beta' \frac{2n\sqrt{\pi}}{k_2^{*2}} \Big[2n R_{\left(k_2^*, r^2, \theta, -2Tr+1, 2Tr-1\right)} - R_{\left(k_2^*, r, \theta, -2Tr+1, 2Tr-1\right)}\Big] \end{aligned} \quad (26)$$

Finally, substituting equations (6), (19), (23) and (26) in equation (18), the result stated in theorem 2 follows.

Corollary 1:

The exact risk function of the double k-class estimators of β for $0 \le k_2 < 1$ is given by;

$$E\left[\left(\tilde{b}_{k_1,k_2}-\beta\right)'\left(\tilde{b}_{k_1,k_2}-\beta\right)\right]=\frac{\nu}{\nu-2}\sigma^2 tr\left(x'x\right)^{-1}-2k_1\frac{\nu}{\left(\nu-2\right)}\sigma^2 tr\left(x'x\right)^{-1}\left[\frac{\nu}{\left(\nu-4\right)}.\frac{8nT\sqrt{\pi}}{k_2^*}\right.$$

$$\left[-4TA_{(3,-1,0)}-A_{(2,-1,0)}\right]+\frac{16nT\sqrt{\pi}}{k_2^*}A_{(2,0,0)}+\frac{\left(\nu+T\right)}{\left(\left(\nu+T\right)-2\right)}\frac{2n\sqrt{\pi}}{k_2^*}A_{(1,0,0)}\Bigg]$$

$$-k_1^2\frac{\nu}{\left(\nu-2\right)}\sigma^2 tr\left(x'x\right)^{-1}\left[\frac{\nu}{\left(\nu-4\right)}\frac{4n\sqrt{\pi}}{k_2^{*2}}\left[-16nT^2A_{(4,0,1)}+4T\left(3n+2T\right)A_{(3,0,1)}\right.\right.$$

$$-2\left(2n+3T\right)A_{(2,0,1)}-A_{(1,0,1)}\Big]+\frac{8n\sqrt{\pi}}{k_2^{*2}}\left[4nTA_{(3,1,1)}-\left(2n+2T\right)A_{(2,1,1)}+A_{(1,1,1)}\right]$$

$$+\left(\frac{\left(\nu+T\right)}{\left(\left(\nu+T\right)-2\right)}\right)\frac{2n\sqrt{\pi}}{k_2^{*2}}\left[2nA_{(2,1,1)}-A_{(1,1,1)}\right]\Bigg]-\frac{2nk_1\beta\beta'\sqrt{\pi}}{k_2^*}\left\{\frac{k_1}{k_2^*}\left[-2nA_{(2,1,1)}\right.\right.$$

$$\left.\left.+A_{(1,1,1)}\right]-\frac{\nu 8T}{\nu-2}A_{(2,1,0)}\right\} \qquad (27)$$

where $tr\left(x'x\right)^{-1}=\sum_{j=1}^{p}\lambda_j$ and λ_j is the jth characteristic root of $\left(x'x\right)^{-1}$

Corollary 1 follows by applying the trace on both sides of equation (17).

3.2 Large-θ Asymptotic Expansion

This section examines the asymptotic expansion of the bias, moment matrix and the risk function of the double k-class estimators for $0\le k_2<1$ under multivariate t-errors in terms of the inverse of θ. The results help in analyzing the complicated expressions of the exact bias, exact moment matrix and exact risk function given in equation (7), equation (17) and (27), respectively. Sufficiently large θ which according to $\theta=\frac{\beta'x'x\beta}{\sigma^2}$ means relatively small σ is noted and the terms of order $\frac{1}{\theta^{2T-e}}$ as the terms of order $\sigma^{2(2T-e)}$ is considered in the asymptotic expansion.

Theorem 3:

The asymptotic of the bias of double k-class estimators of β under multivariate t-errors in equation (7) up to order $\frac{1}{\theta^{4T+1}}$ is given by;

$$E\left(\tilde{b}_{k_1,k_2}-\beta\right)=-2nk_1\sqrt{\pi}\left[\frac{1}{2T}+\left(\frac{4T\theta\nu}{\left(\nu-2\right)}+1\right)\frac{\left(2T-1\right)k_2^{*\,2n-1}\zeta}{\theta^{2T}}\right]\beta \qquad (28)$$

where,

$$\zeta=\left(-1\right)^{T+1}\frac{\nu^2}{8}.\frac{3\Gamma\left(2T-1\right)}{\left(\frac{\nu-4}{2}\right)_2}$$

Proof:

Using equation (6) and substituting (A.7), (A.8), (A.9), (A.10) and (A.11) in equation (7), the results in equation (28) can easily be established.

Theorem 4:

The asymptotic expansion of the moment matrix of the double k-class estimators of β under multivariate t-errors in equation (17) up to order $\frac{1}{\theta^{4T}}$ is given by;

$$E\left[\left(\tilde{b}_{k_1,k_2}-\beta\right)\left(\tilde{b}_{k_1,k_2}-\beta\right)'\right]=\frac{\nu}{\nu-2}\sigma^2\left(x'x\right)^{-1}-\frac{2\nu n\sqrt{\pi}\left(x'x\right)^{-1}}{\left(\nu-2\right)}\psi \tag{29}$$

where,

$$\psi=\eta_1 k_1 k_2^{*-1}+\eta_2 k_1^2 k_2^{*-2}+\eta_3\zeta k_1 k_2^{*2n-1}+\eta_4\zeta k_1^2 k_2^{*2n-2}$$

$$\eta_1=\frac{\left(\nu+T\right)}{T\left(\nu+T-2\right)}$$

$$\eta_2=\left\{\frac{\nu}{T\left(\nu-4\right)}-\frac{2\theta}{T}+\frac{\theta}{2T}\left[\frac{-\left(\nu-2\right)\theta}{\nu}+\left(\frac{\nu+T}{\nu+T-2}\right)\right]\right\}$$

$$\eta_3=\frac{2}{\theta^{2T-1}}\left[\frac{-4\nu T}{\left(\nu-4\right)}\left(4T+1\right)+8T-4T\theta+\frac{\left(\nu+T\right)}{\left(\nu+T-2\right)}\right]\left(2T-1\right)$$

$$\eta_4=\frac{1}{\theta^{2T-1}}\left[\frac{2\nu}{\left(\nu-4\right)}\left(-16nT^2+4T\left(3n+2T\right)-2\left(2n+3T\right)-1\right)+4\left(4nT-2T\right)\right.$$
$$\left.+\left(2n-1\right)\left(4-\frac{\left(\nu-2\right)\theta}{\nu}+\frac{\left(\nu+T\right)}{\nu+T-2}\right)\right]$$

Proof:

Using equation (6) and substituting (A.7), (A.8), (A.9), (A.10) and (A.11) in equation (17), the results in equation (29) can easily be established.

Note: All the theorems are based on assumption 1-3 and k1 is considered as an arbitrary constant.

Corollary 2:

The asymptotic expansion of risk function of the double k-class estimators of β under multivariate t-errors in equation (27) up to order $\frac{1}{\theta^{4\mathrm{T}}}$ is given by;

$$E\left[\left(\tilde{b}_{k_1,k_2}-\beta\right)'\left(\tilde{b}_{k_1,k_2}-\beta\right)\right]=\frac{\nu}{\nu-2}\sigma^2\, tr\left(x'x\right)^{-1}-\frac{2\nu n\sqrt{\pi}\, tr\left(x'x\right)^{-1}}{\left(\nu-2\right)}\psi \tag{30}$$

Corollary 3:

The double k-class estimator of β in equation (2) dominates the least squares estimator of β in large θ asymptotic expansion up to the order $\frac{1}{\left(\theta\right)^{(2T)-2}}$ when number of observations is even, in the sense,

$$\lim_{\theta\to\infty}\left(\theta\right)^{(2T)-2}\left\{E\left[\left(\tilde{b}_{k_1,k_2}-\beta\right)'\left(\tilde{b}_{k_1,k_2}-\beta\right)\right]-E\left[\left(b-\beta\right)'\left(b-\beta\right)\right]\right\}<0 \tag{31}$$

For $0\le k_1\le\frac{2}{n+2}\left(d-2\right)$, $d=\sum_{j=1}^{p}\frac{\lambda_j}{\lambda_L}>2;\lambda_j$ eigen values and λ_L largest of eigen values and for any k_2 in $0\le k_2<1$.

The result in equation (31) is exactly the same as what is obtained under the multivariate normal. So the result developed under the normality in Ullah and ullah (1978) is robust against t-errors.

4. SIMULATION STUDY

In this section, comparisons of the performances of the double k-class estimators of β with multivariate t-errors under risk function established. A simulation study was conducted using the R program. The linear regression model in equation (1) is considered and observations on x are nonstochastic. The error term is generated from multivariate t-distribution with parameters 0, Iσ2 and degree of freedom ν.The number of variables (p), the number of observation is (T): $\left(T,\ p\right)$ = {(10, 4), (20, 4), (28, 4), (40, 20)} and u is iid random sample from multivariate t-distribution with ν=T-1 and σ=1. The risk performance of the estimators under varying degrees of multicollinearity are then considered. The values of correlation coefficient(ρ) are between 0 and 1.

In the simulations, the following versions of the double k-class estimators are considered.

a) Least squares(LS) estimator; k1=0.
b) Minimum mean squared error estimator (MMSE), proposed by Ohtani(1996);

$$k_1=\frac{1}{1-p} \text{ and } k_2=1-k_1.$$

c) Adjusted minimum mean squared error estimator (AMMSE), proposed by Ohtani (1996a);

$$k_1=\frac{p}{T-p} \text{ and } k_2=1-k_1.$$

d) Double k-class estimator (KK-C), proposed by Carter et al. (1993);

$$k_1 = \frac{p-2}{T-\text{p+2}} \text{ and } k_2 = 1 - k_1 .$$

These estimators are exposed to criteria of bias and risk with quadratic loss function under multivariate t-error. The bias and the risk ratio function for the double k-class estimators of β under multivariate t-errors are defined as,

$$\sqrt{\left(\tilde{b}_{k_1,k_2} - \beta\right)'\left(\tilde{b}_{k_1,k_2} - \beta\right)} \text{ and } \frac{E\left[\left(\tilde{b}_{k_1,k_2} - \beta\right)'\left(\tilde{b}_{k_1,k_2} - \beta\right)\right]}{E\left[\left(b - \beta\right)'\left(b - \beta\right)\right]} \text{ respectively.}$$

These results in the simulation study of the experiment are based on 1000 replications. The bias and the risk ratio function compare MMSE,AMMSE and KKC with LS respectively.

Table(1)
Values of Bias

(T,p)	ρ	LS	MMSE	AMMSE	KK-C
(10,4)	0.6	0.02200888	0.12182388	0.37592032	0.17102804
	0.7	0.02462165	0.10873118	0.34373379	0.15360883
	0.8	0.02905164	0.09954747	0.31540843	0.13988310
	0.9	0.03914522	0.09467979	0.29112591	0.13017799
(20,4)	0.6	0.06790635	0.13519485	0.36105021	0.22078438
	0.7	0.07606411	0.13529054	0.35540836	0.19273922
	0.8	0.08917872	0.13624491	0.32133740	0.18702174
	0.9	0.1176784	0.1419392	0.2992607	0.1822328
(28,4)	0.6	0.06032913	0.11083958	0.32912038	0.17278303
	0.7	0.06854307	0.10710046	0.28461987	0.15924841
	0.8	0.08284285	0.10456381	0.25317143	0.14409645
	0.9	0.1158854	0.1271818	2.2724728	0.1522278
(40,20)	0.6	0.05087376	0.05055462	0.05102050	0.04998829
	0.7	0.02560520	0.02565317	0.03298890	0.03083079
	0.8	0.03020576	0.03024275	0.03461879	0.03330382
	0.9	0.04029763	0.04033729	0.04296840	0.04220550

Table(2)
Values of Ratio Risk Function

(T,p)	ρ	lS	MMSE	AMMSE	KK-C
(10,4)	0.6	1.0000000	1.0000000	0.2041423	0.9999938
	0.7	1.0000000	1.0000000	0.9933664	0.9999999
	0.8	1.0000000	1.0000000	0.9998497	1.0000000
	0.9	1.0000000	1.0000000	0.9999712	1.0000000
(20,4)	0.6	1.0000000	1.0000000	0.3108969	1.0000000
	0.7	1.0000000	1.0000000	0.9971598	1.0000000
	0.8	1.0000000	1.0000000	0.9999943	1.0000000
	0.9	1.0000000	1.0000000	1.0000000	1.0000000
(28,4)	0.6	1.0000000	1.0000000	0.4519305	1.0000000
	0.7	1.0000000	1.0000000	1.0000000	1.0000000
	0.8	1.0000000	1.0000000	1.0000000	1.0000000
	0.9	1.0000000	1.0000000	1.0000000	1.0000000
(40,20)	0.6	1.0000000	1.0000000	1.0000000	1.0000000
	0.7	1.0000000	1.0000000	1.0000000	1.0000000
	0.8	1.0000000	1.0000000	1.0000000	1.0000000
	0.9	1.0000000	1.0000000	1.0000000	1.0000000

The analysis of the performance of these estimators from the simulation results are based on Table (1) which shows that, the biased LS estimator is smaller than MMSE, AMMSE and KKC estimators. The LS is best from MMSE, KKC and AMMSE estimators respectively. Table (2) shows that the performance of AMMSE dominates KKC, MMSE and LS respectively. As the values of correlation coefficient (ρ) and sample size get smaller, the performance of double k-class estimators dominates LS estimator. The performance of double k-class estimators approximately equal LS estimator when the values of correlation coefficient (ρ) and sample size get larger.

5. CONCLUSION

The analysis on exact and asymptotic biasedness, moment matrix and risk function properties was based on LS and the double k-class estimators in regression models with spherical disturbances in a multivariate t-error. In particular, we have derived the large noncentrality parameter (θ) of the double k-class estimators and investigated the conditions under which the double k-class estimators dominates the least squares estimator under a quadratic loss function. The exact and asymptotic biasedness, moment matrix and risk function properties of the double k-class estimators are true when number of observations is even.

The double k-class estimators dominates the LS estimator in equation (31) and explains the difference between risk under quadratic loss function of the multivariate t-errors in large θ asymptotic up to the order $\frac{1}{(\theta)^{(2T)-2}}$.The result in equation (31) is exactly the same as the one obtained under the multivariate normal. Further, we have considered the performance of the estimators in large noncentrality parameter (θ) and examined the effects of varying degrees of multicollinearity on the properties of the estimators. The performance of AMMSE dominates KKC, MMSE and LS respectively. As the values of correlation coefficient(ρ) and sample size get smaller, the performance of double k-class estimators dominates LS estimator.

Finally, the result for the simulation study of double k-class estimator estimators dominates ordinary least square in linear regression under multivariate t-errors.

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APPENDIX

A. Special Function

1) The Gamma functions Γ have the following power series representation (see Andrews)

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt;\ x > 0$$

2) The Beta function,

$$B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt$$

3) The pochhammer symbol,

$$(r)_0 = 1$$

$$(r)_n = r(r+1)\ldots\ldots(r+n-1);\ n = 1, 2, \ldots..$$

$$(r)_n = \frac{\Gamma(r+n)}{\Gamma(r)};\ n = 0, 1, 2, \ldots\ldots.$$

4) Similarly, the power series presentation of the function R,

$$R_{\left(k_2^*, r^g, \theta, -2Tr+e, 2Tr-o\right)} = \sum_{r=0}^{\infty} (-1)^{Tr+r} \frac{\nu^{2r} r^g k_2^{*2nr} (\theta)^{-(2Tr)+e}}{2^{2r}\, r!} \frac{\left(r+\frac{1}{2}\right)_r \Gamma(2Tr-o)}{\left(\frac{\nu}{2}-2r\right)_r \left(\frac{\nu}{2}-r\right)_r} \qquad \text{(A.1)}$$

where $\theta = \bar{z}'\bar{z} > 0;\ 0 \le k_2^* \le 1;\ g,\ o\ =\ 0,\ 1,\ 2,\ \ldots..\ ;\ e\ = \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots..$

For $k_2^* = 1$, (A.1) reduce to

$$R_{\left(1, r^g, \theta, -2Tr+e, 2Tr-o\right)} = \sum_{r=0}^{\infty} (-1)^{Tr+r} \frac{\nu^{2r} r^g (\theta)^{-(2Tr)+e}}{2^{2r}\, r!} \frac{\left(r+\frac{1}{2}\right)_r \Gamma(2Tr-o)}{\left(\frac{\nu}{2}-2r\right)_r \left(\frac{\nu}{2}-r\right)_r} \qquad \text{(A.2)}$$

The partial derivatives of R underrespect to z write

$$\frac{\partial}{\partial \bar{z}} R_{\left(k_2^*, r^g,\ \theta,\ -2Tr+e,\ 2Tr-o\right)} = 2\bar{z}\Big[-2T\, R_{\left(k_2^*, r^{g+1},\ \theta,\ -2Tr+e-1,\ 2Tr-o\right)} + eR_{\left(k_2^*, r^g,\ \theta,\ -2Tr+e-1,\ 2Tr-o\right)}\Big] \qquad \text{(A.3)}$$

$$\frac{\partial^2}{\partial \bar{z}\partial \bar{z}'} R_{\left(k_2^*, r^g, \theta, -2Tr+e, 2Tr-o\right)} = 16T^2 \bar{z}' \bar{z} R_{\left(k_2^*, r^{g+2}, \theta, -2Tr+e-2, 2Tr-o\right)} - \left(16Te - 4T\right) R_{\left(k_2^*, r^{g+1}, \theta, -2Tr+e-1, 2Tr-o\right)} + e\left(4e-2\right) R_{\left(k_2^*, r^g, \theta, -2Tr+e-1, 2Tr-o\right)} \tag{A.4}$$

and partial derivatives of R under respect to k_2^* write

$$\frac{\partial}{\partial k_2^*} R_{\left(k_2^*, r^g, \theta, -2Tr+e, 2Tr-o\right)} = \frac{2n}{k_2^*} R_{\left(k_2^*, r^{g+1}, \theta, -2Tr+e, 2Tr-o\right)} \tag{A.5}$$

$$\frac{\partial^2}{\partial k_2^{*2}} R_{\left(k_2^*, r^g, \theta, -2Tr+e, 2Tr-o\right)} = \frac{4n^2}{k_2^{*2}} R_{\left(k_2^*, r^{g+2}, \theta, -2Tr+e, 2Tr-o\right)} - \frac{2n}{k_2^{*2}} R_{\left(k_2^*, r^{g+1}, \theta, -2Tr+e, 2Tr-o\right)} \tag{A.6}$$

Finally, for large θ, the asymptotic expansion of the function R, in (A.1) up to orde $\frac{1}{\theta^{8T-e}}$ is given by

When: $g = 1$

$$R_{\left(k_2^*, r, \theta, -2Tr, 2Tr\right)} = \frac{1}{2T}\left(-1\right)^{T+1} \frac{\nu^2 k_2^{*2n}}{4\left(\theta\right)^{(2T)}} \frac{\frac{3}{2}\Gamma\left(2T\right)}{\left(\frac{\nu-4}{2}\right)_2} + \frac{\nu^4 k_2^{*4n}}{8\left(\theta\right)^{(4T)}} \frac{\left(\frac{5}{2}\right)_2 \Gamma\left(4T\right)}{\left(\frac{\nu-8}{2}\right)_4} + \left(-1\right)^{T+1} \frac{\nu^6 3 k_2^{*6n}}{128\left(\theta\right)^{(6T)}} \frac{\left(\frac{7}{2}\right)_3 \Gamma\left(6T\right)}{\left(\frac{\nu-12}{2}\right)_6} + o\left(\theta^{-8T}\right) \tag{A.7}$$

$$R_{\left(k_2^*, r, \theta, -2Tr, 2Tr-1\right)} = \frac{-1}{2T} + \left(-1\right)^{T+1} \frac{\nu^2 k_2^{*2n}}{4\left(\theta\right)^{(2T)}} \frac{\frac{3}{2}\Gamma\left(2T-1\right)}{\left(\frac{\nu-4}{2}\right)_2} + \frac{\nu^4 k_2^{*4n}}{8\left(\theta\right)^{(4T)}} \frac{\left(\frac{5}{2}\right)_2 \Gamma\left(4T-1\right)}{\left(\frac{\nu-8}{2}\right)_4} + \left(-1\right)^{T+1} \frac{\nu^6 3 k_2^{*6n}}{128\left(\theta\right)^{(6T)}} \frac{\left(\frac{7}{2}\right)_3 \Gamma\left(6T-1\right)}{\left(\frac{\nu-12}{2}\right)_6} + o\left(\theta^{-8T}\right) \tag{A.8}$$

$$R_{\left(k_2^*,r,\theta,-2Tr-1,2Tr\right)}=\frac{1}{2T\theta}+(-1)^{T+1}\frac{\nu^2 k_2^{*2n}}{4(\theta)^{(2T+1)}}\frac{\frac{3}{2}\Gamma(2T)}{\left(\frac{\nu-4}{2}\right)_2}+\frac{\nu^4 k_2^{*4n}}{8(\theta)^{(4T+1)}}\frac{\left(\frac{5}{2}\right)_2\Gamma(4T)}{\left(\frac{\nu-8}{2}\right)_4}$$

$$+(-1)^{T+1}\frac{\nu^6 3k_2^{*6n}}{128(\theta)^{(6T+1)}}\frac{\left(\frac{7}{2}\right)_3\Gamma(6T)}{\left(\frac{\nu-12}{2}\right)_6}+o\left(\theta^{-8T-1}\right) \qquad (A.9)$$

$$R_{\left(k_2^*,r,\theta,-2Tr+1,2Tr-1\right)}=\frac{-\theta}{2T}+(-1)^{T+1}\frac{\nu^2 k_2^{*2n}}{4(\theta)^{(2T-1)}}\frac{\frac{3}{2}\Gamma(2T-1)}{\left(\frac{\nu-4}{2}\right)_2}+\frac{\nu^4 k_2^{*4n}}{8(\theta)^{(4T-1)}}\frac{\left(\frac{5}{2}\right)_2}{\left(\frac{\nu-8}{2}\right)_4}$$

$$\Gamma(4T-1)+(-1)^{T+1}\frac{\nu^6 3k_2^{*6n}}{128(\theta)^{(6T-1)}}\frac{\left(\frac{7}{2}\right)_3\Gamma(6T-1)}{\left(\frac{\nu-12}{2}\right)_6}+o\left(\theta^{-8T+1}\right) \qquad (A.10)$$

When: $g \ge 2$

$$R_{\left(k_2^*,r^g,\theta,-2Tr+e,2Tr-o\right)}=(-1)^{T+1}\frac{\nu^2 k_2^{*2n}}{4(\theta)^{(2T)-e}}\frac{\frac{3}{2}\Gamma(2T-o)}{\left(\frac{\nu-4}{2}\right)_2}+\frac{\nu^4 2^g k_2^{*4n}}{16(\theta)^{(4T)-e}}\frac{\left(\frac{5}{2}\right)_2}{\left(\frac{\nu-8}{2}\right)_4}$$

$$\Gamma(4T-o)+(-1)^{T+1}\frac{\nu^6 3^g k_2^{*6n}}{384(\theta)^{(6T)-e}}\frac{\left(\frac{7}{2}\right)_3\Gamma(6T-o)}{\left(\frac{\nu-12}{2}\right)_6}+o\left(\theta^{-8T+e}\right) \qquad (A.11)$$

where : $2T \ \ge \ e$, T is number of observations, ν is degrees of freedom,

$$k_2^*=1-k_2, 0\le k_2\le 1, \left(g,o=0,1,2,\ldots ;e=\ldots,-1,0,1,2,\ldots\right),\ \theta=\frac{\beta' x' x\beta}{\sigma^2}.$$

B. Evaluation of Expectations required in Subsection (3.1)

Let w be a $T \times 1$ Multivariate t distribution random vector such that

$$E(w)=0, E(w-\bar{w})(w-\bar{w})' = \frac{\nu}{\nu-2}\mathrm{I} \tag{B.1}$$

$$c=\frac{y'My}{y'Ny}=\frac{z'D_1z}{z'D_2z}=\frac{w'D_1w}{w'D_2w} \tag{B.2}$$

the joint characteristic function of the quadratic forms $w'D_2w$ and $w'D_1w$ can be written as

$$\Phi_{c_w}(t_1,t_2)=E\left[e^{it_1w'D_2w+it_2w'D_1w}\right] \tag{B.3}$$

Note that

$$Q=t_1-(k_2t_1-t_2)D_1, Q\bar{z}=t_1\bar{z} \tag{B.4}$$

We can obtain equation (B.3) as

$$\Phi_{c_w}(t_1,t_2)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\sum_{r=0}^{\infty}(-1)^r\,\frac{\nu^{2r}\left(k_2^*t_1+t_2\right)^{2nr}t_1^{(T-n)2r}}{2r!}$$

$$B\left(2r+\frac{1}{2},\frac{\nu+1}{2}-2r-\frac{1}{2}\right) \tag{B.5}$$

where $k_2^*=1-k_2$

The characteristic function of the quadratic forms $z'D_2z$ and $z'D_1z$ can be written as

$$\Phi_{c_z}(t_1,t_2)=e^{iQ\bar{z}'\bar{z}}\Phi_{c_w}(t_1,t_2)=e^{it_1\bar{z}'\bar{z}}\Phi_{c_w}(t_1,t_2) \tag{B.6}$$

The the following derivatives of (B.5) can then be obtained as follows.

$$\frac{-i\partial\Phi_{c_z}(t_1,t_2)}{\partial t_2}\Big|_{t_2=0}=\frac{-i\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\partial}{\partial k_2^*}\sum_{r=1}^{\infty}(-1)^r\,\frac{\nu^{2r}k_2^{*2nr}e^{it_1\bar{z}'\bar{z}}t_1^{2Tr-1}}{2r!}$$

$$B\left(2r+\frac{1}{2},\frac{\nu+1}{2}-2r-\frac{1}{2}\right) \tag{B.7}$$

and

$$\frac{-\partial^2\Phi_{c_z}\left(t_1,t_2\right)}{\partial {t_2}^2}\Big|_{t_2=0}=\frac{-\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\partial^2}{\partial k_2^{*2}}\sum_{r=1}^{\infty}\left(-1\right)^r\frac{\nu^{2r}k_2^{*2nr}\,e^{it_1\bar{z}'\bar{z}}\,t_1^{2Tr-2}}{2r!}$$

$$B\left(2r+\frac{1}{2},\frac{\nu+1}{2}-2r-\frac{1}{2}\right) \tag{B.8}$$

Finally using (B.7), (B.8), and (A.1) we obtain the following expectations for $0\le k_2^*\le 1$.

$$E\left(\frac{z'D_1z}{z'D_2z}\right)=i\int_0^{\infty}\frac{-i\partial\Phi_{c_z}\left(t_1,t_2\right)}{\partial t_2}\Big|_{t_2=0}\,dt_1$$

$$=\frac{2n\sqrt{\pi}}{k_2^*}R_{\left(k_2^*,r\ ,\theta,-2Tr,2Tr\right)} \tag{B.9}$$

and

$$E\left(\frac{z'D_1z}{z'D_2z}\right)^2=i\int_0^{\infty}\frac{\partial^2\Phi_{c_z}\left(t_1,t_2\right)}{\partial t_2^2}\Big|_{t_2=0}\,dt_1$$

$$=\frac{2n\sqrt{\pi}\bar{z}'\bar{z}}{k_2^{*2}}\left[-2nR_{\left(k_2^*,r^2\ ,\theta,-2Tr,2Tr-1\right)}+R_{\left(k_2^*,r\ ,\theta,-2Tr,2Tr-1\right)}\right] \tag{B.10}$$