Pak. J. Statist.
2016 Vol. 32(3), 207-226

INEQUALITY MEASURES OF WEIGHTED GENERALIZED BETA DISTRIBUTION OF THE FIRST KIND

N. Nakhaei Rad[1] and **G. H. Yari**[2]
[1] Department of Statistics, Islamic Azad University
Tehran, Iran. Email: najme.rad@gmail.com
[2] Iran University of Science and Technology,
Tehran, Iran. Email: yari@iust.ac.ir

ABSTRACT

Generalized Beta distribution of the first kind is an important distribution that is properly flexible as an income distribution in economics. In this paper, a new class of weighted generalized beta distribution of the first kind is introduced. Besides the common statistical properties, we intend to focus on the economic properties of the distribution, including the measures of inequality such as generalized entropy indices (Theil index and the mean log deviation), Gini index, Pietra index and Atkinson index. The maximum likelihood estimation of the parameters of the size-biased generalized Beta distribution of the first kind in both grouped and individual data are also included. Finally, the weighted generalized Beta distribution of the first kind is fitted to the real gross domestic product per capita of 172 countries in 2012 and some its inequality indices are estimated.

KEY WORDS

Generalized Beta distribution of the first kind, Sized-biased distribution, Generalized entropy, Theil index, Gini index, MLD index, Pietra index, Atkinson index.

1. INTRODUCTION

Generalized beta distribution of the first kind (GB_1) is defined as a four-parameter income distribution which is included many important income distributions as special cases. Furthermore, it has several application in fitting the data in economics. (see, McDonald, 1984). The density of the GB_1 is given by:

$$f(x;a,b,p,q) = \frac{ax^{ap-1}}{b^{ap}B(p,q)}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1}, \qquad 0<x<b, \tag{1}$$

where $a,b,p,q>0$, $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$. Note that a,p,q are the shape parameters and b is the scale parameter in (1.1). The r^{th} moments of GB_1 is given by:

$$E\left(X^r\right)=\frac{b^r B\left(p+\frac{r}{a},q\right)}{B(p,q)}.$$

At first, Fisher (1934) clarified and gave an explanation of the notion of weighted distribution to investigate the impacts of ascertainment methods on the shape of distribution of observations. Afterwards, Rao (1965) unified the weighted distributions in general terms, and demonstrated various situations in which weighted distributions may be used to model specifications. The weighted distributions were utilized in some fields such as demography (Menken and Sheps , 1972), economics (Ord, 1975), analysis of data in human population and ecology (Patil and Rao, 1978), biomedicine (Zelen, 1974), forestry (Warren, 1975) and reliability (Cox, 1962 and Scheaffer, 1972).

Consider X as a non-negative random variable with density $f(x;\theta)$ and θ as a parameter. The weighted random variable X_w has density

$$f_w(x;\theta)=\frac{w(x)f(x;\theta)}{E\left(w(X)\right)}, \tag{2}$$

where $w(x)$ is a non-negative weight function with finite expectation. In general, $w(x)$ is defined as follows:

$$w(x)=x^k e^{lx}F^i(x)\bar{F}^j(x), \tag{3}$$

where $F(x)$ is the cumulative distribution function (cdf) of X and $\bar{F}(x)=1-F(x)$. It is clear that in (3), $(k=l=i=0)$, $(k=l=j=0)$, $(k=l=0, j=n-i+1)$, $(l=i=j=0)$, $(k=i=j=0)$, $(k=l=0)$ and $(k=0)$ imply proportional hazard, reversed proportional hazard, i^{th} order statistics, sized-biased, moment generating, Jones model (2004) and probability weighted moment, respectively. The choice of proper weight function is very important in the use of weighted distributions. The weighted distribution with $w(x)=x$, $(k=1,\ l=i=j=0)$, is named the length-biased which was introduced by Cox (1962) in renewal theory. When $w(x)=x^k$, $(l=i=j=0)$, the obtained weighted distribution is named size-biased which is a generalization of the length-biased. Ahmed et al. (2013) introduced the sized-biased generalized beta of the first kind only for $k=1$ ($w(x)=x$) which is in fact length-biased generalized beta of the first kind ($LBGB_1$). The size-biased distributions could be found in the studies of Scheaffer (1972), Gupta (1984), Simon (1980) and Patil and Rao (1978). Other important weight function can be found in works of several authors like Rao (1965), Patil and Ord (1976) and Gupta (1975).

Derivation of the inequality indices in terms of the distributional parameters can be used to obtain their parametric estimations. Derivation of Gini index (1912) in terms of the distributional parameters are expressed in Dagum (1977) and McDonald (1984). Dastrup et al. (2007) summarized some McDonald’s (1981) results about Pietra (1932) and Theil (1967) indices of the Pareto, log-normal, gamma, beta and Singh-Maddala distributions. Kleiber and Kotz (2003) and McDonald and Ransom (2008) obtained these indices for GB_1 and the generalized beta distribution of the second kind (GB_2) and their related distributions. Ye et al. (2012) introduced size-biased generalized beta distribution of second kind ($SBGB_2$) and derived its generalized entropy indices (Shorrocks, 1980).

The second section of this paper focuses on the sized-biased weight as a new class of the weighted generalized Beta distribution of the first kind (WGB_1) which includes the length-biased generalized beta distribution of first kind as a special case. Moreover, the common statistical properties of this weighted distribution, including moments, variance, *coefficient of variation* (CV), skewness, kurtosis and mode are derived.Section 3 deals with obtaining some of the inequality measures of WGB_1 such as generalized entropy indices (Theil index and mean log diviation (MLD)), Gini index, Pietra index and Atkinson index (1970) . In section 4, the estimation of parameters of WGB_1 is presented for data in the grouped format as well as individual data. Finally WGB_1 is fitted to the data consisting the real gross domestic product (GDP) per capita of 172 countries in 2012 and its inequality indices are estimated.

2. WEIGHTED GENERALIZED BETA DISTRIBUTION OF THE FIRST KIND

The five-parameter $WGB_1(a,b,p,q,k)$ with weight function $w(x)=x^k$ is obtained by substituting $w(x)=x^k$ and (1) into (2) as follows:

$$f_{WGB_1}(x;a,b,p,q,k)=\frac{ax^{ap+k-1}}{b^{ap+k}B\left(p+\frac{k}{a},q\right)}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}, \qquad 0<x<b, \qquad (4)$$

where $a,b,p,q,k>0$. b is the scale parameter and a,p,q and also the new parameter k are shape parameters (see, Figure 1).

From (4), the cdf is readily obtained as:

$$F_{WGB_1}(x;a,b,p,q,k)=I_{\left(\frac{x}{b}\right)^a}\left(p+\frac{k}{a},q\right), \qquad 0<x<b, \qquad (5)$$

where $I_x(p,q)=\frac{B_x(p,q)}{B(p,q)}$, $B_x(p,q)=\int_0^x t^{p-1}(1-t)^{q-1}dt$. The plots of the cdf of WBG_1 for $k=0,1,2,3,4$ are shown in Figure 2. In Table 1 and 2, some percentiles of WGB_1 are presented. The percentiles increase as a,b,p,k increase and decrease as q increases.

The hazared function of WGB_1 is:

$$h_{WGB_1}(x;a,b,p,q,k)=\frac{ax^{ap+k-1}}{b^{ap+k}\left(B\left(p+\frac{k}{a},q\right)-B_{\left(\frac{x}{b}\right)^a}\left(p+\frac{k}{a},q\right)\right)}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1},$$

$$0<x<b. \qquad (6)$$

Plots of hazard function of WGB_1 for $k=0,1,2,3,4$ are shown in Figure 3.

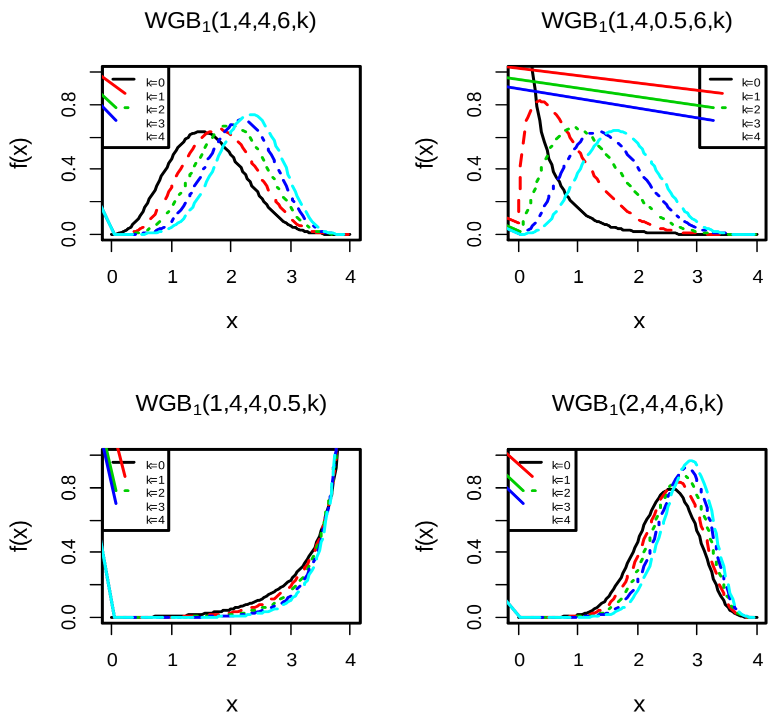

Figure 1: The Plots of Density (4) for $k=0,1,2,3,4$.

Table 1
Percentiles of WGB_1 with $k = 1$

a	b	p	q	25th	50th	75th	95th	99th
2	2	2	2	1.2613	1.5026	1.6888	1.8832	1.9504
4				1.5533	1.7100	1.8290	1.9352	1.9726
6				1.6804	1.7954	1.8807	1.9552	1.9812
8				1.7512	1.8420	1.9083	1.9657	1.9856
2	2	2	2	1.2613	1.5026	1.6888	1.8832	1.9504
	4			2.5227	3.0052	3.3975	3.7664	3.9007
	6			3.7840	4.5078	5.0964	5.6496	5.8511
	8			5.0453	6.0104	6.7952	7.5328	7.8015
2	2	2	2	1.2613	1.5026	1.6888	1.8832	1.9504
		4		1.5239	1.6887	1.8155	1.9298	1.9703
		6		1.6485	1.7732	1.8668	1.9497	1.9788
		8		1.7214	1.8215	1.8958	1.9608	1.9835
2	2	2	2	1.2613	1.5026	1.6888	1.8832	1.9504
			4	0.9965	1.2200	1.4266	1.6702	1.7951
			6	0.8504	1.0533	1.2501	1.5021	1.6474
			8	0.7543	0.9403	1.1254	1.3730	1.5245

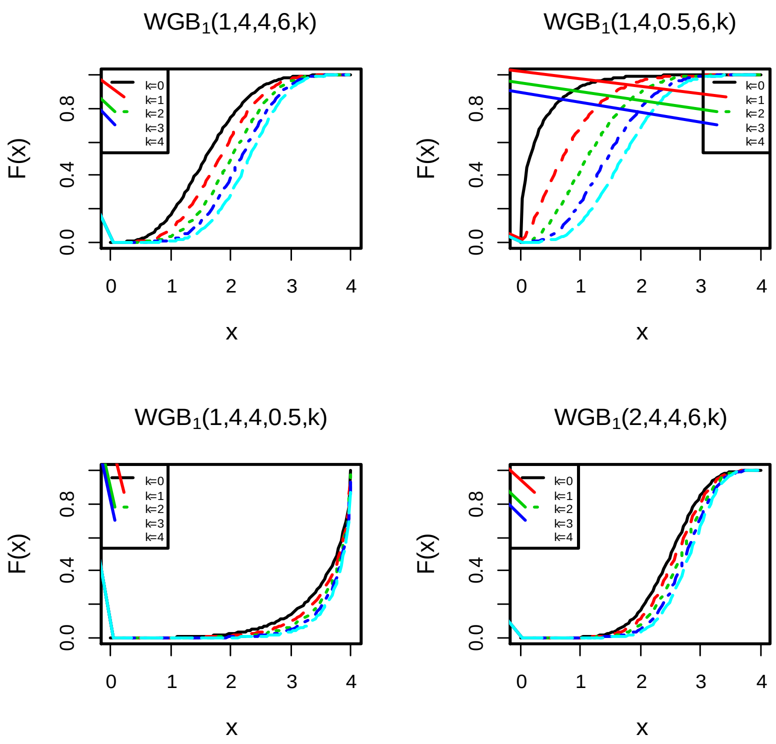

Figure 2: The Plots of cdf in (5) for $k = 0,1,2,3,4$

Table 2
Percentiles of WGB_1 with $k = 2$

a	b	p	q	25th	50th	75th	95th	99th
2	2	2	2	1.3510	1.5675	1.7401	1.8999	1.9575
4				1.5883	1.7335	1.8432	1.9407	1.9750
6				1.6988	1.8075	1.8878	1.9578	1.9823
8				1.7626	1.8493	1.9126	1.9673	1.9863
2	2	2	2	1.3510	1.5675	1.7401	1.8999	1.9575
	4			2.7021	3.1350	3.4802	3.7998	3.9151
	6			4.0531	4.7025	5.2203	5.6996	5.8726
	8			5.4041	6.2700	6.9603	7.5995	7.8302
2	2	2	2	1.3510	1.5675	1.7401	1.8999	1.9575
		4		1.5627	1.7153	1.8318	1.9361	1.9730
		6		1.6701	1.7876	1.8755	1.9531	1.9802
		8		1.7351	1.8306	1.9011	1.9629	1.9844
2	2	2	2	1.3510	1.5675	1.7401	1.8999	1.9575
			4	1.0898	1.2983	1.4875	1.7072	1.8187
			6	0.9398	1.1322	1.3136	1.5488	1.6814
			8	0.8380	1.0170	1.1924	1.4239	1.5643

The r^{th} moments of WGB_1 is

$$E(X^r) = \frac{b^r B\left(p + \frac{k}{a} + \frac{r}{a}, q\right)}{B\left(p + \frac{k}{a}, q\right)}. \tag{7}$$

The mean, the variance and CV are respectively obtained as below

$$\mu = E(X) = \frac{bB\left(p + \frac{k}{a} + \frac{r}{a}, q\right)}{B\left(p + \frac{k}{a}, q\right)}, \tag{8}$$

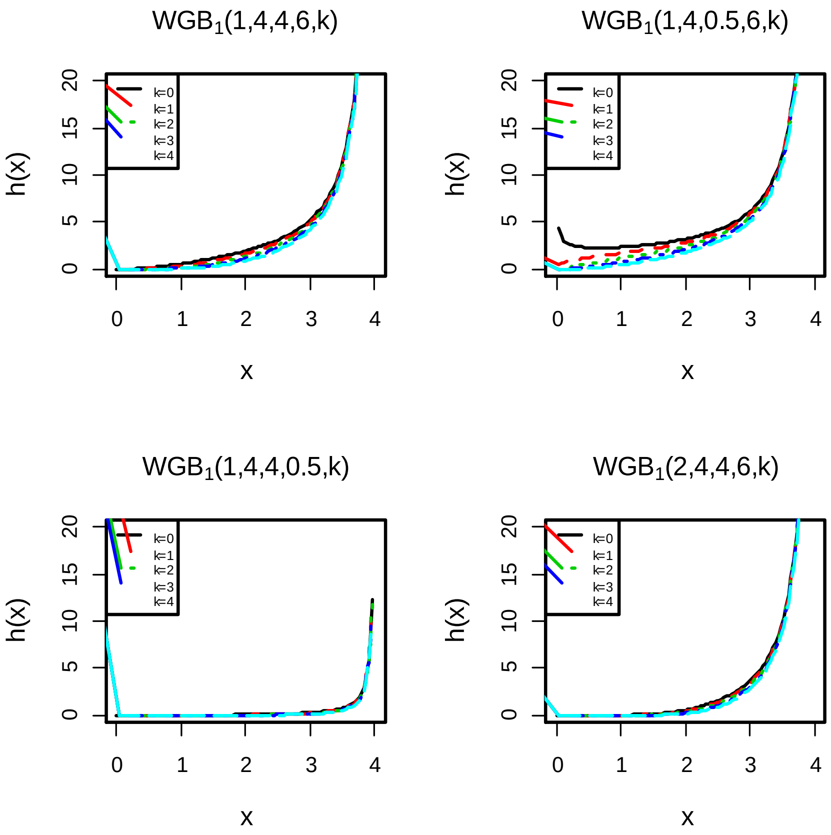

Figure 3: The Plots of Hazard Function in (6) for $k = 0,1,2,3,4$

$$Var(X) = b^2 \left[\frac{B\left(p + \frac{k}{a} + \frac{2}{a}, q\right)}{B\left(p + \frac{k}{a}, q\right)} - \left[\frac{B\left(p + \frac{k}{a} + \frac{1}{a}, q\right)}{B\left(p + \frac{k}{a}, q\right)} \right]^2 \right], \tag{9}$$

$$CV = \sqrt{\frac{B\left(p + \frac{k}{a} + \frac{2}{a}, q\right) B\left(p + \frac{k}{a}, q\right)}{B^2\left(p + \frac{k}{a} + \frac{1}{a}, q\right)} - 1}.$$

Furthermore, the coefficient of skewness (CS) and kurtosis (CK) of WGB_1 can be found from (7), (8), (9) and the following formulas:

$$CS = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{(Var(X))^{3/2}},$$

$$CK = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{(Var(X))^2}.$$

The mode of WGB_1 is:

$$Mode = b\left(\frac{ap+k-1}{aq+ap+k-a-1}\right)^{\frac{1}{a}}.$$

When $ap+k>1, q>1$, the WGB_1 is unimodal; when $ap+k<1, q<1$, the WGB_1 is uniantimodal; when $ap+k>1, q<1$, the WGB_1 is increasing and finally when $ap+k<1, q>1$, the WGB_1 is decreasing (see Figure1).

For different values of parameters a, b, p, q and k the mode, mean, variance, CV, CS and CK of WGB_1 are calculated and the results are summarized in Table 3 and Table 4. From Table 3 and 4, it can be seen that as k increases, mode, mean and CK increase but variance, CV and CS decrease.

Table 3
The Mode, Mean, Variance, CV, CS and CK of WGB_1 with $k=1$

a	b	p	q	mean	mode	var	CV	CS	CK
2	2	2	2	1.4583	1.9330	1.0955	0.2119	-0.6521	3.0009
4				1.6714	1.8072	0.0427	0.1236	-0.9621	3.9176
6				1.7643	1.8693	0.0238	0.0874	-1.1037	4.4638
8				1.8162	1.9012	0.0151	0.0676	-1.1854	4.8188
2	2	2	2	1.4583	1.9330	0.0955	0.2119	-0.6521	3.0009
	4			2.9167	3.2660	0.3819	0.2119	-0.6521	3.0009
	6			4.3750	4.8990	0.8594	0.2119	-0.6521	3.0009
	8			5.8333	6.5320	1.5278	0.2119	-0.6521	3.0009
2	2	2	2	1.4583	1.9330	0.0955	0.2119	-0.6521	3.0009
		4		1.6500	1.7888	0.0467	0.1310	-0.9064	3.7353
		6		1.7410	1.8516	0.0275	0.0952	-1.0311	4.1868
		8		1.7944	1.8856	0.0181	0.0749	-1.1062	4.4894
2	2	2	2	1.4583	1.9330	0.0955	0.2119	-0.6521	3.0009
			4	1.2031	1.2649	0.0909	0.2507	-0.2666	2.6339
			6	0.0473	1.0690	0.0795	0.2692	-0.0999	2.6151
			8	0.9397	0.9428	0.0693	0.2802	-0.0049	2.6443

The WGB_1 distribution includes several distributions as special or limiting cases (see Figure 4). As it can be seen in Figure 4, when $a = 2, b = \beta(p+q)^{\frac{1}{a}}, q \to \infty$, then the WGB_1 tends to the truncated weighted generalized normal distribution. In addition to above assumptions, If we take $p = \frac{1}{2}$ and $k = 0$ then the WGB_1 tends to the truncated normal distribution.

Table 4

The Mode, Mean, Variance, CV, CS and CK of WGB_1 with $k = 2$

a	b	p	q	mean	mode	var	CV	CS	CK
2	2	2	2	1.5238	1.6903	0.0780	0.1832	-0.7382	3.2229
4				1.6970	1.8243	0.0370	0.1133	-0.9940	4.0336
6				1.7778	1.8774	0.0213	0.0821	-1.1192	4.5318
8				1.8246	1.9059	0.0138	0.0644	-1.1940	4.8602
2	2	2	2	1.5238	1.6903	0.0780	0.1832	-0.7382	3.2229
	4			3.0476	3.3806	0.3120	0.1832	-0.7382	3.2229
	6			4.5714	5.0701	0.7020	0.1832	-0.7382	3.2229
	8			6.0952	6.7612	1.2481	0.1832	-0.7382	3.2229
2	2	2	2	1.5238	1.6903	0.0780	0.1832	-0.7382	3.2229
		4		1.6783	1.8091	0.0404	0.1197	-0.9448	3.8677
		6		1.7569	1.8619	0.0245	0.0891	-1.0532	4.2732
		8		1.8045	1.8918	0.0164	0.0711	-1.1205	4.5500
2	2	2	2	1.5238	1.6903	0.0780	0.1832	-0.7382	3.2229
			4	1.2787	1.3484	0.0791	0.2200	-0.3407	2.7360
			6	1.1233	1.1547	0.0716	0.2382	-0.1617	2.6790
			8	1.0134	1.0260	0.0638	0.2492	-0.0673	2.6799

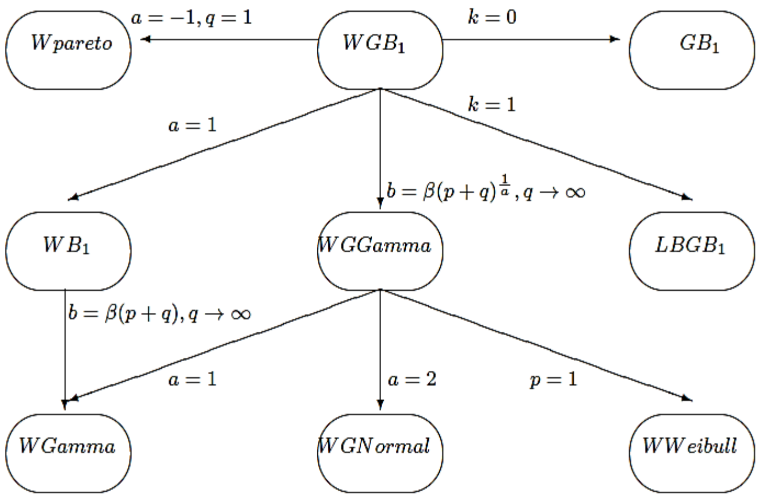

Figure 4: Some Related Distributions of WGB_1.

3. INEQUALITY MEASURES

In this section, some of the popular inequality measures such as Theil index, MLD index, Gini index, Pietra index and Atkinson index are derived for WGB_1. Derivation of these indices in terms of the distributional parameters can be used to obtain their parametric estimations based on a income data set. According to Figure 3, these indices can be found for the distributions which are related to WGB_1.

3.1 Generalized Entropy Indices

Generalized entropy (GE) is used to measure income inequality and is defined as:

$$I(\alpha)=\frac{\nu(\alpha)\mu^{-\alpha}-1}{\alpha(\alpha-1)},\quad \alpha\neq 0,1, \tag{10}$$

where $\nu(\alpha)=E(X^{\alpha})$ and $\mu=E(X)$. The MLD index is given by:

$$I(0)=\lim_{\alpha\to 0} I(\alpha)=\log\mu-\nu'(0),$$

where $\nu'(\alpha)=\frac{d}{d\alpha}\nu(\alpha)$. Also, Theil index is derived by:

$$I(1)=\lim_{\alpha\to 1} I(\alpha)=\frac{1}{\mu}\nu'(1)-\log\mu,$$

The GE of WGB_1 is obtained by replacing (7) and (8) into (10) as follows:

$$I_{WGB_1}(\alpha)=\frac{B\left(p+\frac{k}{a}+\frac{\alpha}{a},q\right)B^{-\alpha}\left(p+\frac{k}{a}+\frac{1}{a},q\right)-B^{1-\alpha}\left(p+\frac{k}{a},q\right)}{\alpha(\alpha-1)B^{1-\alpha}\left(p+\frac{k}{a},q\right)}.$$

The MLD and Theil indices of WGB_1 are:

$$I_{WGB_1}(0)=\log\frac{B\left(p+\frac{k}{a}+\frac{1}{a},q\right)}{B\left(p+\frac{k}{a},q\right)}-\frac{\varphi\left(p+\frac{k}{a}\right)}{a}+\frac{\varphi\left(p+q+\frac{k}{a}\right)}{a}$$

and

$$I_{WGB_1}(1)=\frac{\varphi\left(p+\frac{k}{a}+\frac{1}{a}\right)}{a}-\frac{\varphi\left(p+q+\frac{k}{a}+\frac{1}{a}\right)}{a}-\log\frac{B\left(p+\frac{k}{a}+\frac{1}{a},q\right)}{B\left(p+\frac{k}{a},q\right)},$$

respectively, where $\phi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ and $\Gamma'(z)=\frac{d}{dz}\Gamma(z)$. Theil index of GB_1, ($k=0$), was derived by McDonald and Ransom (2008).

3.2 Gini Index and Pietra Index

The most common inequality index in economics is the Gini index which can be derived as below:

$$G=\left(J^*(1,0)-J^*(0,1)\right)/\mu,$$

where

$$J^*(i,j)=\int_0^\infty x^i f(x)\int_0^x y^j f(y)dydx.$$

Another popular inequality index is Pietra index, which can obtained as following:

$$P=J(\mu,0)-J(\mu,1)/\mu,$$

where

$$J(x,i)=\int_0^x y^i f(y)dy.$$

For WGB_1 Gini and Pietra indices are obtained by derivation J and J^* as follows:

$$G_{WGB_1}=\frac{B\left(2p+\frac{2k}{a}+\frac{1}{a},q\right)}{B\left(p+\frac{k}{a},q\right)B\left(p+\frac{k}{a}+\frac{1}{a},q\right)}\left\{\frac{1}{\left(p+\frac{k}{a}\right)}\right.$$
$$\times{}_3F_2\left(2p+\frac{2k}{a}+\frac{1}{a},p+\frac{k}{a},1-q;1;2p+q+\frac{2k}{a}+\frac{1}{a},p+\frac{k}{a}+1\right)$$
$$-\frac{1}{\left(p+\frac{k}{a}+\frac{1}{a}\right)}$$
$$\left.\times{}_3F_2\left(2p+\frac{2k}{a}+\frac{1}{a},p+\frac{k}{a}+\frac{1}{a},1-q;1;2p+q+\frac{2k}{a}+\frac{1}{a},p+\frac{k}{a}+\frac{1}{a}+1\right)\right\}.$$

and

$$P_{WGB_1}=\frac{\left(\frac{\mu}{b}\right)^{ap+k}}{B\left(p+\frac{k}{a},q\right)}\left\{\frac{1}{p+\frac{k}{a}}\times{}_2F_1\left(p+\frac{k}{a},1-q;\left(\frac{\mu}{b}\right)^a;p+\frac{k}{a}+1\right)\right.$$
$$\left.-\frac{1}{p+\frac{k}{a}+\frac{1}{a}}\times{}_2F_1\left(p+\frac{k}{a}+\frac{1}{a},1-q;\left(\frac{\mu}{b}\right)^a;p+\frac{k}{a}+\frac{1}{a}+1\right)\right\}.$$

where ${}_mF_n(a_1,...,a_m;z;b_1,...,b_n) = \sum_{i=0}^{\infty} \frac{(a_1)_i...(a_m)_i}{(b_1)_i...(b_n)_i} \frac{z^i}{i!}$ and $(a)_i = a(a+1)\ldots(a+i-1)$.

For more details on the obtaining G and P, see Mcdonald (1984) and Mcdonald and Ransom (2008).

3.3 Atkinson Index

Another inequality measure which is more sensitive to what happens to the poor is Atkinson index and is defined as:

$$A(\varepsilon) = 1 - \left[\int_0^{\infty} \left(\frac{x}{\mu} \right)^{1-\varepsilon} f(x)dx \right]^{\frac{1}{(1-\varepsilon)}}, \quad \varepsilon > 0,$$

where ε controls the inequality aversion. When $\varepsilon \to 1$, Atkinson index is defined as:

$$A(1) = 1 - \frac{1}{\mu} \exp \int_0^{\infty} \log x f(x)dx.$$

Atkinson index of WGB_1 is obtained as follows:

$$A_{WGB_1}(\varepsilon) = 1 - \frac{B^{1-\frac{1}{(1-\varepsilon)}}\left(p + \frac{k}{a}, q \right) B^{\frac{1}{1-\varepsilon}}\left(p + \frac{k}{a} + \frac{1-\varepsilon}{a}, q \right)}{B\left(p + \frac{k}{a} + \frac{1}{a}, q \right)}$$

and

$$A_{WGB_1}(1) = 1 - \frac{B\left(p + \frac{k}{a}, q \right)}{bB\left(p + \frac{k}{a} + \frac{1}{a}, q \right)} \exp[\frac{\varphi\left(p + \frac{k}{a} \right)}{a} - \frac{\varphi\left(p + q + \frac{k}{a} \right)}{a} + \log b].$$

For different values of parameters a, p, q and k Theil index, MLD index, Gini index, Pietra index and Atkinson index of WGB_1 are calculated and the results are summarized in Table 5 and Table 6.

Table 5
The Theil, MLD, Gini, Pietra, A(0.5) and A(1) of WGB_1 with $k=1$

a	p	q	Theil	MLD	Gini	Pietra	A(0.5)	A(1)
2	2	2	0.0241	0.0270	0.1417	0.0860	0.0127	0.0266
4			0.0080	0.0085	0.0749	0.0490	0.0041	0.0085
6			0.0040	0.0041	0.0508	0.0343	0.0020	0.0041
8			0.0024	0.0024	0.0385	0.0264	0.0012	0.0024
2	2	2	0.0241	0.0270	0.1417	0.0860	0.0127	0.0266
	4		0.0090	0.0096	0.0801	0.0522	0.0046	0.0096
	6		0.0047	0.0049	0.0560	0.0376	0.0024	0.0049
	8		0.0029	0.0030	0.0430	0.0294	0.0015	0.0030
2	2	2	0.0241	0.0270	0.1417	0.0860	0.0127	0.0266
		4	0.0332	0.0366	0.1933	0.0102	0.0172	0.0360
		6	0.0379	0.0416	0.2303	0.1094	0.0196	0.0407
		8	0.0408	0.0446	0.2608	0.1136	0.0211	0.0436

Table 6
The Theil, MLD, Gini, Pietra, A(0.5) and A(1) of WGB_1 with $k=2$

a	p	q	Theil	MLD	Gini	Pietra	A(0.5)	A(1)
2	2	2	0.0180	0.0197	0.1394	0.0739	0.0093	0.0195
4			0.0067	0.0071	0.0745	0.0448	0.0034	0.0071
6			0.0035	0.0036	0.0508	0.0322	0.0018	0.0036
8			0.0021	0.0022	0.0385	0.0251	0.0011	0.0022
2	2	2	0.0180	0.0197	0.1394	0.0739	0.0093	0.0195
	4		0.0075	0.0080	0.0797	0.0475	0.0039	0.0079
	6		0.0041	0.0043	0.0558	0.0351	0.0021	0.0043
	8		0.0026	0.0027	0.0430	0.0278	0.0013	0.0027
2	2	2	0.0180	0.0197	0.1394	0.0739	0.0093	0.0195
		4	0.0255	0.0277	0.2154	0.0893	0.0131	0.0273
		6	0.0296	0.0320	0.2830	0.0966	0.0152	0.0315
		8	0.0322	0.0347	0.3474	0.1009	0.0165	0.0341

4. APPLICATION

The common way to derive the estimation of parameters of a distribution is maximizing the log-likelihood function. The estimation of parameters of WGB_1 based on maximizing the multinomial likelihood function is presented in both grouped and individual data in the following subsections.

4.1 Estimation of Parameters in Grouped Data

Jones and Faddy (2003) introduced the class of generalized beta-F distributions with the following density:

$$\frac{1}{B(\alpha,\lambda)} f(y)[F(y)]^{\alpha-1}[1-F(y)]^{\lambda 1}, \tag{11}$$

where $\alpha, \lambda > 0$ and $f(y)$ is derivation of the cdf $F(y)$. Therefore, the density of WGB_1 in (4) is obtained by substituting $F(x) = \left(\frac{x}{b}\right)^a, 0 < x < b, a > 0$ in (11); hence, the WGB_1 belongs to the beta-F distributions . The estimation of parameters of WGB_1 in a data set with r groups denoted by the intervals $I_i = [y_{i-1}, y_i]$ is obtained by maximizing the following log-likelihood function:

$$\log(L(\theta)) = \sum_{i=1}^{r} n_i \log P_i(\theta),$$

where

$$P_i(\theta) = \int_{F(y_{i-1})}^{F(y_i)} \frac{1}{B\left(p+\frac{k}{a}, q\right)} t^{p+\frac{k}{a}-1}(1-t)^{q-1} dt,$$

and n_i denotes the estimated proportion and observed frequency in i th interval, respectively. Obtaining $P_i(\theta)$, as it has been mentioned above, is suggested by Sepanski and Kong (2007) to reduce the complexity of calculations. The first derivative of the log-likelihood function with respect to θ is as below:

$$\frac{d\log(L(\theta))}{d\theta} = \sum_{i=1}^{r} \frac{n_i}{P_i(\theta)} \frac{dP_i(\theta)}{d\theta}. \tag{12}$$

The partial derivatives of $P_i(\theta)$ with respect to $\theta = (a, b, p, q)^T$ are as follows:

$$\frac{\partial P_i(\theta)}{\partial a} = \left(\frac{x_i}{b}\right)^a log\left(\frac{x_i}{b}\right) h\left(F(x_i)\right) - \left(\frac{x_{i-1}}{b}\right)^a log\left(\frac{x_{i-1}}{b}\right) h\left(F\left(x_{i-1}\right)\right)$$
$$+ \int_{F(y_{i-1})}^{F(y_i)} \frac{k}{a^2} h(t)\left[\varphi\left(p+q+\frac{k}{a}\right) - \varphi\left(p+\frac{k}{a}\right) - \log t\right] dt, \tag{13}$$

$$\frac{\partial P_i(\theta)}{\partial b} = \frac{-a}{b^2}\left[\left(\frac{x_i}{b}\right)^{a-1} h\left(F(x_i)\right) - \left(\frac{x_{i-1}}{b}\right)^{a-1} h\left(F(x_{i-1})\right)\right], \tag{14}$$

$$\frac{\partial P_i(\theta)}{\partial p} = \int_{F(x_{i-1})}^{F(x_i)} h(t)\left[-\varphi\left(p+q+\frac{k}{a}\right) + \varphi\left(p+\frac{k}{a}\right) + logt\right] dt, \tag{15}$$

$$\frac{\partial P_i(\theta)}{\partial q} = \int_{F(x_{i-1})}^{F(x_i)} h(t)\left[\varphi(q) - \varphi\left(p+q+\frac{k}{a}\right) + log(1-t)\right] dt. \tag{16}$$

where $h(t)=\frac{1}{B\left(p+\frac{k}{a},q\right)}t^{p+\frac{k}{a}-1}(1-t)^{q-1}$. By substituting the equations (13)-(16) in (12), the gradient functions of the log-likelihood function with respect to a,b,p,q can be obtained.

4.2 Estimation of Parameters in Individual Data

In this case, the estimation of vector parameter $\theta=(a,b,p,q)^T$ can be found directly through maximizing the log-likelihood function of WGB_1 distribution. The log-likelihood function of WGB_1 is:

$$\log(L(\theta))=n\log a-n(ap+k)\log b-n\log B\left(p+\frac{k}{a},q\right)$$
$$+(ap+k-1)\sum_{i=1}^{n}\log x_i+(q-1)\sum_{i=1}^{n}\log\left(1-\left(\frac{x_i}{b}\right)^a\right). \tag{17}$$

Finally, the partial derivations of the log-likelihood function (4.7) with respect to $\theta=(a,b,p,q)^T$ are derived as follows:

$$\frac{\partial\log(L(\theta))}{\partial a}=\frac{n}{a}+p\sum_{i=1}^{n}\log x_i-np\log b-\frac{nk}{a^2}\left(\phi\left(p+q+\frac{k}{a}\right)-\phi\left(p+\frac{k}{a}\right)\right)$$
$$-(q-1)\sum_{i=1}^{n}\frac{y_i\log\left(\frac{x_i}{b}\right)}{1-y_i}, \tag{18}$$

$$\frac{\partial\log(L(\theta))}{\partial b}=\frac{-n(ap+k)}{b}+\frac{a(q-1)}{b}\sum_{i=1}^{n}\frac{y_i}{1-y_i}, \tag{19}$$

$$\frac{\partial\log(L(\theta))}{\partial p}=a\sum_{i=1}^{n}\log x_i-na\log b-n\left(\phi\left(p+\frac{k}{a}\right)-\phi\left(p+q+\frac{k}{a}\right)\right), \tag{20}$$

$$\frac{\partial\log(L(\theta))}{\partial q}=-n\left(\phi(q)-\phi\left(p+q+\frac{k}{a}\right)\right)+\sum_{i=1}^{n}\log(1-y_i), \tag{21}$$

where $y_i=\left(\frac{x_i}{b}\right)^a$. Equating (18)-(21) to zero gives the maximum likelihood estimations (MLE) of the vector parameter $\theta=(a,b,p,q)^T$, denoted by $\theta=(\hat{a},\hat{b},\hat{p},\hat{q})^T$, for given k. The MLEs have not close forms, so a numerical method is needed.

4.3 Illustrative Example

To demonstrate the procedure of the previous subsections, a real GDP (million dollars) per capita of 172 countries in 2012 has been considered (The data were taken from http://www.unctadstat.unctad.org). The numerical values of $\hat{a}, \hat{b}, \hat{p}, \hat{q}$ and the corresponding log-likelihood are obtained for $k = 0,1,2,3,4,5$, and the results are summarized in Table 7. Furthermore, the Anderson-Darling (AD) test and also the *average scaled absolute error* (ASAE) are used for goodness of fit.

Table 7
The Values of $\hat{a}, \hat{b}, \hat{p}, \hat{q}$ and the Corresponding Log-Likelihood, ASAE and the P-value of the AD Test for WGB_1

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$\hat{a}$	0.0638	0.0234	**0.0107**	0.0001	0.0001	0.0001
$\hat{b}$	95736.1949	85533.60	**98745.46**	100015.08	100015.09	100015.09
$\hat{p}$	23.6446	39.5893	**12.4620**	0.0001	0.0000	0.0000
$\hat{q}$	37.9278	34.6525	**35.1982**	45.6239	60.6657	75.7075
Log-likelihood	237.3814	237.5578	**237.6094**	234.5073	222.1915	204.3412
P-value	0.7705	0.7784	**0.7838**	0.0823	0.0008	0.0000
ASAE	0.0042	0.0037	**0.0036**	0.0105	0.0149	0.0170

According to the P-value of the AD test in Table 7, it can be concluded that the WGB_1 for $k = 0,1,2,3$ can be fitted on the mentioned data. But, for $k = 2$, the log-likelihood is maximum and ASAE is minimum, thus the WGB_1 with $k = 2$ provides the best fit on the income data.

Ye et al. (2012) obtained the maximum likelihood estimation of parameters of the size-biased generalized beta distribution of the second kind from U.S family income data (2001-2009). Based on the sum of square error values, they concluded that the length-biased generalized beta distribution of second kind ($k = 1$) provides a better fit than GB_2 ($k = 0$) but also, provides a better fit than sized-biased generalized beta distribution of second kind ($k = 2,3,4$). Finally to show the fitness of the weigthed generalized beta distribution of secend kind (WGB_2) for $k > 1$ and to compare it with WGB_1, the estimated parameters, the corresponding log-likelihood, the AD test and the ASAE are derived from GDP data (2012) for $k = 0,1,2,3,4,5$ and the results are summarized in Table 8.

According to the P-value of the AD test in Table 8, it is observed that the WGB_2 for $k = 0,1,2,3,4,5$ can be fitted on the mentioned data. But, for $k = 2$, the log-likelihood is maximum , thus the WGB_2 with $k = 2$ provides the best fit on the data . Comparing the results in Table 7 with Table 8, it is shown that WGB_1 provides a better fit than WGB_2 on the mentioned data set.

Table 8
The Values of $\hat{a}, \hat{b}, \hat{p}, \hat{q}$ and the Corresponding Log-Likelihood, ASAE and the P-value of the AD Test for WGB_2

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$\hat{a}$	0.0470	0.0364	**0.0361**	0.0366	0.0386	0.0408
$\hat{b}$	49406.0115	17149.0499	**22350.5959**	28053.9480	7963.4587	46670.5403
$\hat{p}$	107.6201	163.9837	**139.6493**	106.4244	67.2642	24.6870
$\hat{q}$	212.6457	340.0120	**357.4580**	395.8153	382.5507	387.9122
Log-likelihood	237.3572	237.4258	**237.5361**	237.5347	237.5329	237.5243
P-value	0.7639	0.7752	**0.7781**	0.7753	0.7757	0.7757
ASAE	0.0098	0.0095	**0.0095**	0.0095	0.0095	0.0095

Paap and Djik (1998) analyzed the distribution of the GDP per capita of 120 countries in 1960-1989. They suggested a mixture model of a Weibull distribution and a truncated normal distribution as follows:

$$f(x;\alpha,\beta,\lambda,\mu,\sigma)=\lambda\frac{\beta}{\alpha^{\beta}}x^{\beta-1}\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)+(1-\lambda)\frac{\phi(x;\mu,\sigma^2)}{\Phi(\mu/\sigma)}, \qquad x>0,$$

where $0<\lambda<1$, $\alpha,\beta,\mu,\sigma>0$. $\phi(.)$ is the probability density function and $\Phi(.)$ is the cumulative distribution function of the normal distribution. To compare this five-parameter model with WGB_1, the corresponding log-likelihood, the AD test and the ASAE are derived from GDP data (2012) and the results are summarized in Table 9.

Table 9
The Estimated Parameters and the Corresponding Log-Likelihood, ASAE and the P-Value of the AD Test for Mixture Model

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$	**Log-likelihood**	**P-value**	**ASAE**
0.0642	0.4601	0.9648	0.0000	7.2996	231.3475	0.0870	0.0038

According to Table 9, the mentioned mixture model can be fitted on the data, but comparing the results of Table 7 and 8 with Table 9 shows that the WGB_1 and WGB_2 provide a better fit than the mentioned mixture model on the data set. As it can be seen in Figure 5, the WGB_1 and WGB_2 are unimodal while the mixture model is a decreasing function.

Analyzing the GDP per capita of 120 countries in 1960-1989, Paap and Djik (1998) found that the gap between poor and rich countries is increasing. Also, their analysis not only showed a significant increase in the number of poor countries but also showed that the middle groups between poor and rich countries disappeared. This disappearance can be seen as a huge gap in the histogram of data in Figure 5. Therefore, calculating the inequality indices can be useful when indicating the amount of wealth inequality between nations. The higher values of these indices show higher levels of inequality. From the invariance property of the MLE, the MLEs of the inequality indices in the previous section may be derived for the WGB_1. For example, $Theil = 0.3940$, $MLD = 0.5007$, $Gini = 0.7730$, $Pietra = 0.6155$, $A(0.1) = 0.1869$, $A(0.5) = 0.6777$ and $A(1) = 0.9240$. Among the above indices, the GE indice family with higher α values are more sensitive to inequality in upper income groups; so, the Theil index can be used to explains the inequality changes at the upper tail of the income distribution over years. The Gini index is sensitive to inequality changes in the middle part of the income distribution and the Atkinson index with the higher ε values becomes more sensitive to inequality changes at the bottom of the income distribution such as $A(1)$. Regarding different abilities of the mentioned indices when explaining the inequality, the information may be lost when the degree of the inequality is described with just one inequality index. According to the histogram of our data in Figure 5, it seems that following the changes at the bottom and top of the distribution is more important i.e. the Theil index and the Atkinson index are suitable. While for a too uneaven society ($Gini = 0.7730$), Ryu (2008) showed that it is suitble to use the Gini index to study year to year income inequality changes.

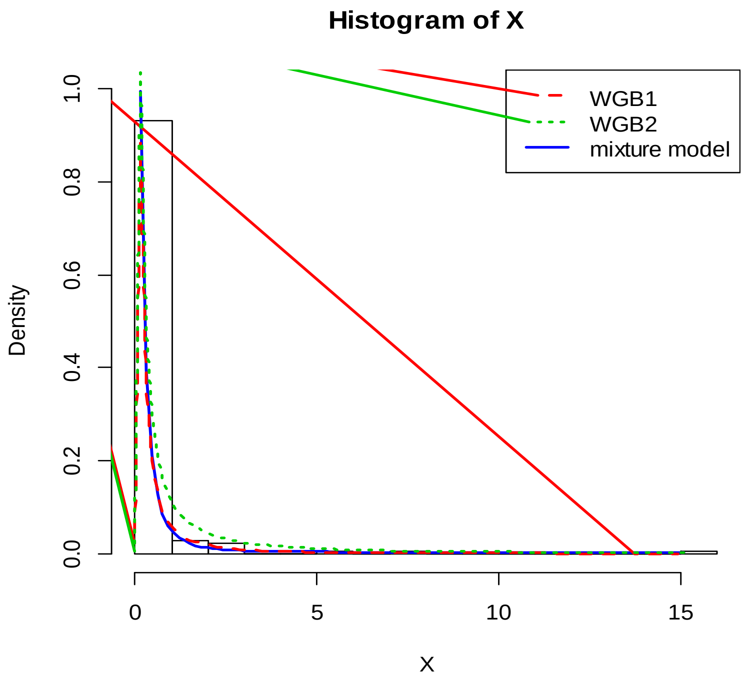

Figure 5: The Histogram of the Data and the Fitted Distributions

ACKNOWLEDGMENTS

The authors are deeply thankful to the technical editor and two reviewers for their valuable suggestions to improve the quality of the manuscript.

REFERENCES

1. Ahmed, A., Mir, K.A. and Reshi, J.A. (2013). On size biased generalized beta distribution of first kind. IOSR Journal of Mathematics, 5, 41-48.
2. Atkinson, A.B. (1970). On the measurement of inequality. *Journal of Economic Theory*, 2, 244-263.
3. Cox, D.R. (1962). Renewal Theory. Barnes & Noble, New York.
4. Dagum, C. (1977). A new model for personal income distribution: specification and estimation. *Economie Applique'e*, 30, 413-437.
5. Dastrup, S.R., Hartshorn, R. and McDonald, J.B. (2007). The impact of taxes and transfer payments on the distribution of income: a parametric comparison. *Journal of Economic Inequality*, 5, 353-369.
6. Fisher, R.A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *Annals of Human Genetics*, 6, 439-444.
7. Gini, C. (1912). Variability and Mutability. C. Cuppini, Bologna.
8. Gupta, R.C. (1975). Some characterizations of discrete distributions by properties of their moment distribution. *Communications in Statistics-Theory and Methods*, 4, 761-765.
9. Gupta, R.C. (1984). Some characterizations of renewal densities with emphasis in reliability. *Math Operations Forshu Statist*, 15, 571-579.
10. Jones, M.C. (2004). Families of distributions arising from distributions of order statistics. *An official Journal of the Spanish Society of Statistics and Operating Research*, 13, 1-43.
11. Jones, M.C. and Faddy, M.J. (2003). A skew extension of the t distribution with applications. *Journal of The Royal Statistical Society, Series B*, 65, 159-174.
12. Kleiber, C. and Kotz, S. (2003). *Statistical Size Distribution Economics and Actuarial Sciences*. Wiley, New York.
13. McDonald, J.B. (1981). Some issues associated with the measurement of income inequality. *Statistical Distribution in Scientific Work*, 6, 161-179.
14. McDonald, J.B. (1984). Some generalized function for size distribution of income. *Econometrica*, 25, 647-663.
15. McDonald, J.B. and Ransom, M. (2008). The generalized beta distribution as a model for the distribution of income: estimation of related measure of inequality. *Economic Studies in Equality, Social Exclusion and Well-Being*, 5, 147-166.
16. Menken, J.A., and Sheps, M.C. (1972). The sampling frame as a determinant of observed distibutions of duration variable. *Population Dynamics*, Academic Press, New York, 57-87.
17. Ord, J.K. (1975). Statistical model for personal income distribution. *Statistical Distributions in Scientific Work*, 2, 151-158.
18. Paap, R. and Dijk, H.K. (1998). Distribution and mobility of wealth of nations. In Chotikapanich, Duang Kamon (Ed.), *Modeling Income Distributions and Lorenz Curves*, Springer, 71-96.

19. Patil, G.P. and Ord, J.K. (1976). On size-biased sampling and related form-invariant weighted distribution. *Sankhya B*, 38, 48-61.
20. Patil, G.P. and Rao, R. (1978). Weighted distributions and size-biased sampling with applications to wildlife and human families. *Biometrics,* 34, 179-189.
21. Pietra, G. (1932). Nuovi contributi alla methodologia degli indici di variabilita e di concentrazione, in: Atti del Reale Instituto Veneto di Scienze. *Lettere ed Arti*, 91, 989-1008.
22. R Development Core Team (2012). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org/.
23. Rao, C.R. (1965). On discrete distributions arising out of methods of ascertainment. *The Indian Journal of Statistics,* 27, 320-332.
24. Ryu, H. (2008). Maximum entropy estimation of income distributions from Bonferroni indices. In Chotikapanich, Duang Kamon (Ed.), *Modeling Income Distributions and Lorenz Curves*, Springer, 193-211.
25. Sepanski, J.H. and Kong, L. (2007). A family of generalized beta distribution for income. *International Business*, 1, 129-145.
26. Scheaffer, R.L. (1972). Size-biased sampling. *Technometrics*, 14, 635-644.
27. Shorrocks, A.F. (1980). The class of additively decomposable inequality measures. *Econometrica*, 48, 613-625.
28. Simon, R. (1980). Length-biased sampling in etiological studies. *American Journal of Epidemoiology*, 27, 444-452.
29. Theil, H. (1967). Economics and Information Theory. Amsterdam: North-Holland.
30. Ye, Y., Oluyede, B.O. and Pararai, M. (2012). Weighted generalized beta distribution of the second kind and related distributions. *Journal of Statistical and Econometric Methods*, 1, 13-31.
31. Warren, W.G. (1975). Statistical distribution in forestry and forest products research. *Statistical Distributions in Scientific Works,* 2, 369-384.
32. Zelen, M. (1974). Problems in cell kinetics and early detection of disease. *Reliability and Biometry*, 56, 701-726.