Pak. J. Statist.
2016 Vol. 32(2), 97-108

COMPARISON OF DIFFERENT ENTROPY MEASURES

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ABSTRACT

In this article we consider seven entropy measures to calculate the loss of entropy when the underlying life time distribution is truncated Rayleigh on [0, t) instead of Rayleigh on [0, ∞). Numerical comparisons are performed to see which entropy measure has advantages over the other measures in terms of relative loss in entropy.

KEY WORDS AND PHRASES

Shannon entropy, Réyni entropy, Tsallis entropy, truncated distribution.

AMS Subject Classifications: 62F15, 65C05

1. INTRODUCTION

The idea of entropy of random variables is developed by Shannon (1948), for the first time in information theory and now it is described differently by different disciplines. For example, in Physics, the entropy measures the number of ways in which a thermodynamic system is arranged; in the Information Theory, it measures the average amount of information in each message received whereas in Statistics, it measures the uncertainty and dispersion. There are many different entropy measures available in the literature. During the last sixty years or so, a number of research papers and monographs discussing and extending Shannon's original work have appeared. In this paper we will discuss some of them. In the area of information theory as well as engineering sciences, the Shannon entropy is a very important and well known concept which resulted in a new branch of mathematics with far reaching applications. To name a few: Financial Analysis (see Sharpe (1985)), Data Compression (see Salomon (1998)), Statistics (see Kullback (1959)), and Information Theory (see Cover and Thomas (1991)). It is a known fact that in any stochastic process, the probability distribution changes with time and consequently, it becomes obvious that the entropy or uncertainty of a probability distribution also changes with time. It becomes therefore interesting to know how the uncertainty changes with time.

Let X be an absolutely continuous non negative random variable having probability density function $f(x)$. Shannon (1948) defined a formal measure of entropy as

$$H(X) = \int_{R_x} f(x) ln f(x) dx, \text{ where } R_x = \{x: f(x) \neq 0\}, \quad (1.1)$$

called Shannon Entropy.

One of the main drawbacks of $H(X)$ is that for some probability distribution it may be negative and then it is no longer an uncertainty measure.

Alfred Réyni (see, Réyni (1961)) generalized (1.1) and defined the measure as

$$H_\alpha(X) = \frac{1}{1-\alpha} ln \int_{R_x} [f(x)]^\alpha dx; \ \alpha > 0, \alpha \neq 1. \quad (1.2)$$

It has similar properties as the Shannon entropy, but it contains additional parameter α which can be used to make it more or less sensitive to the shape of probability distributions.

Tsallis (1988) generalized (1.1) and defined the measure as

$$T_\alpha(X) = \frac{1}{\alpha-1} ln \int_{R_x} [f(x)]^\alpha dx; \ \alpha > 0, \alpha \neq 1. \quad (1.3)$$

In the same direction, Havrda and Chavrat (1967) suggested another extension of (1.1). This extension is called entropy of degree α and is defined as

$$H^\alpha(X) = \frac{1}{2^{1-\alpha}-1} \left(\int_{R_x} [f(x)]^\alpha \ln \frac{f(x)}{\delta} \ dx \right), \alpha > 0, \alpha \neq 1 \quad (1.4)$$

Awad (1987) extended (1.1) and defined the measure as

$$A(X) = - \int_{R_x} f(x) \ln \frac{f(x)}{\delta} \ dx \ , \quad (1.5)$$

where $\delta = sup_{x \epsilon R_x} f(x)$

Awad et al. (1987) again generalized Réyni entropy (1.2) as

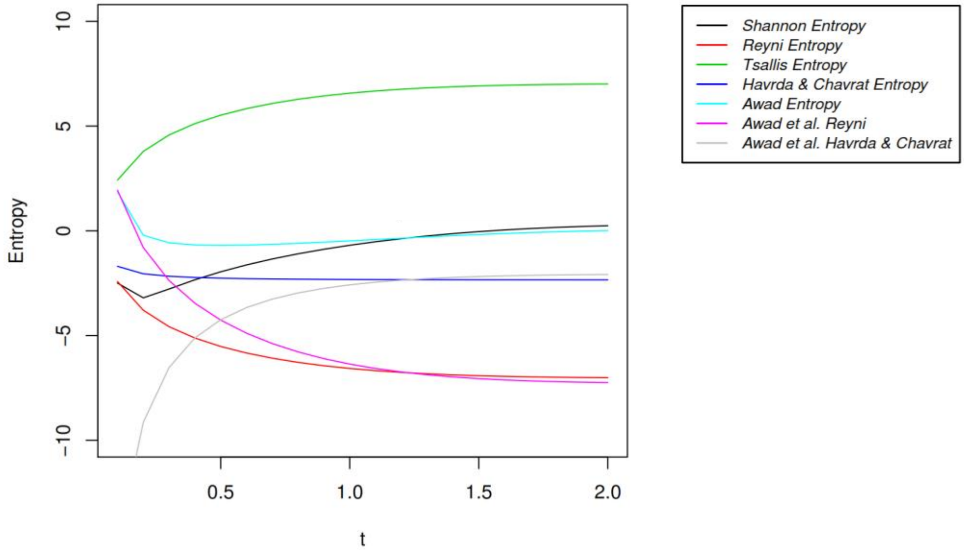
$$A(X)_\alpha = \frac{1}{1-\alpha} \ln \int_{R_x} [\frac{f(x)}{\delta}]^{\alpha-1} \text{ f(x) dx} \quad (1.6)$$

Awad et al. (1987) version of Havrda and Charvat entropy (1.4) is given by

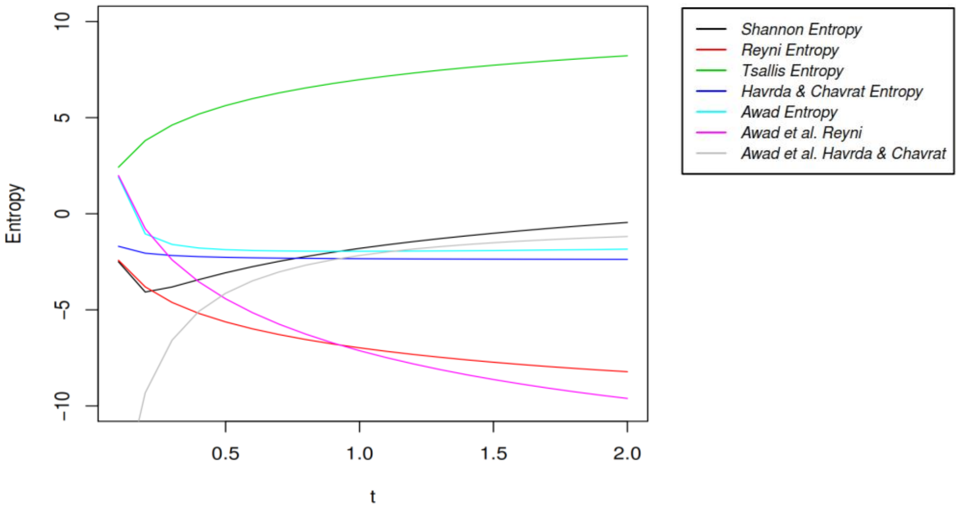
$$A^\alpha(X) = \frac{1}{2^{1-\alpha}-1} \ (\int_{R_x} [\frac{f(x)}{\delta}]^{\alpha-1} f(x) dx - 1) \ . \quad (1.7)$$

Awad and Alawneh (1987) obtained relative loss of entropy based on six entropy measures using truncated exponential distribution. In this paper, we consider seven different entropy measures to calculate the loss of entropy when the life time distribution is assumed to be truncated Rayleigh on [0, t) instead of considering Rayleigh distribution on [0, ∞). Plots of different entropy measures are shown in Figure 1 when the underlying distribution is truncated Rayleigh. Numerical comparisons are performed to see which entropy measure has advantages over the other measures.

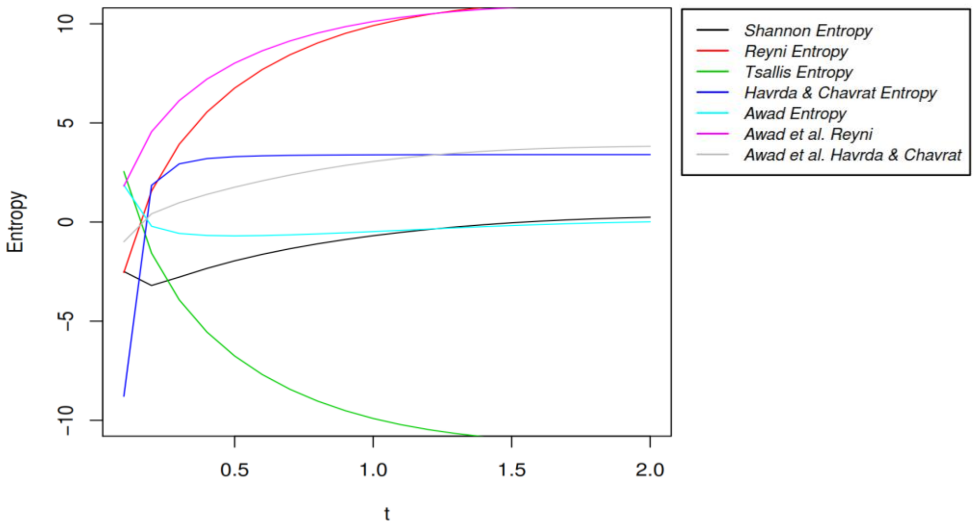
The paper is organized as follows. In Section 2, we discuss the preliminaries of the model and loss of entropy. Different entropy measures based on Rayleigh and truncated Rayleigh distributions are presented in Section 3. Numerical comparisons based on relative loss of entropy using truncated Rayleigh distribution instead of Rayleigh distribution are presented and discussed in Section 4. In the last Section, we draw some conclusions about this article.



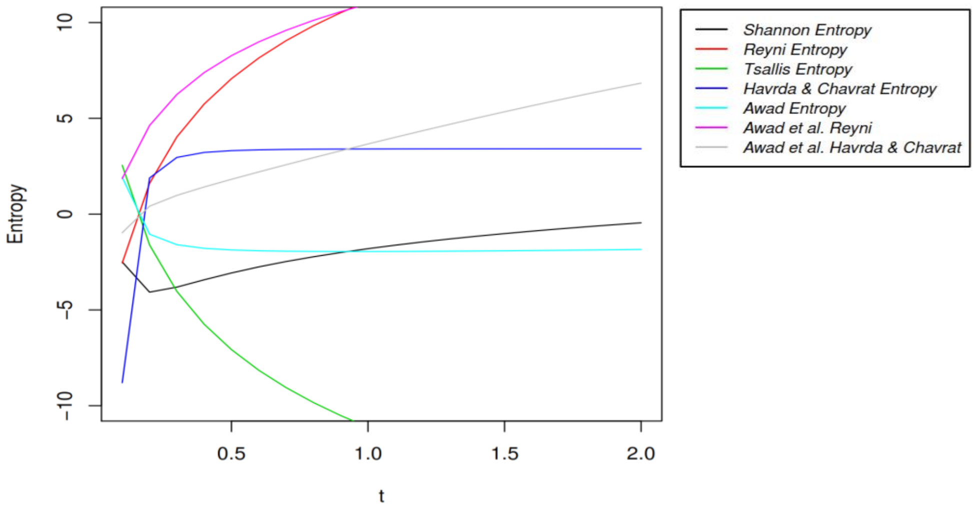
(a) Entropy Measure for $\theta = 1$ and $\alpha = 0.5$



(b) Entropy Measure for $\theta = 0.1$ and $\alpha = 0.5$



(c) Entropy Measure for $\theta = 1$ and $\alpha = 1.5$



(d) Entropy Measure for $\theta = 0.1$ and $\alpha = 1.5$

Figure 1: Different Entropy Measures.

2. PRELIMINARIES

Let us assume that a single component item whose life time X is distributed as Rayleigh with mean $\frac{\sqrt{\pi}}{2\sqrt{\theta}}$.

Since the range of X is [0, ∞), one would like to find a point t within the interval [0, ∞) such that, if it is assumed that the lifetime distribution of X has a truncated Rayleigh distribution on [0,t), the loss of entropy is smaller than a given positive number ε.

Let X be a Rayleigh random variable whose cumulative distribution function (cdf) and probability density function (pdf) are given by

$$F(x;\theta) = 1 - e^{-\theta x^2} \tag{2.1}$$

and

$$f(x;\theta) = 2\theta x\, e^{-\theta x^2} \tag{2.2}$$

respectively.

Let Y be a truncated Rayleigh random variable whose cumulative distribution function (cdf) and probability density function (pdf) are given by

$$F(y;t;\theta) = \frac{1-e^{-\theta y^2}}{F(t;\theta)} = \frac{1-e^{-\theta y^2}}{1-e^{-\theta t^2}}; \tag{2.3}$$

and

$$f(y;t;\theta) = \frac{2\theta y e^{-\theta y^2}}{F(t;\theta)} = \frac{2\theta y e^{-\theta y^2}}{1-e^{-\theta t^2}}; \tag{2.4}$$

respectively.

If $D(X)$ and $D(Y)$ are the two corresponding entropies, then the relative loss of entropy in using Y instead of X is defined as

$$S_D(t) = \frac{[D(X)-D(Y)]}{D(X)}.$$

3. DIFFERENT ENTROPY MEASURES

In this section, we will obtain the relative loss of entropy in using Y instead of X based on different measures of entropy.

1. Shannon Entropy of X and Y are given by

$$H(X) = -\int_0^\infty f(x;\theta)\ln f(x;\theta)dx$$
$$= 1 - ln2 - \frac{1}{2}\,ln\theta - \frac{1}{2}\,\gamma'(1)$$

$$H(Y) = -\int_0^t f(y;t;\theta)\ln f(y;t;\theta)dy$$

$$= 1 - ln2 - \frac{1}{2}\,ln\theta + \ln F(t;\theta) - \frac{\frac{1}{2}\,\gamma'(1,\theta t^2) + \theta\, t^2 e^{-\theta t^2}}{F(t;\theta)}$$

Thus the relative loss of the entropy in using Y instead of X is

$$S_H(t) = \frac{\frac{\frac{1}{2}\,\gamma'(1,\theta t^2) + \theta\, t^2 e^{-\theta t^2}}{F(t;\theta)} - \ln F(t;\theta) - \frac{1}{2}\,\gamma'(1)}{1 - ln2 - \frac{1}{2}\,ln\theta - \frac{1}{2}\,\gamma'(1)}$$

where $\gamma(p,w) = \int_0^w y^{p-1}\, e^{-y}$ dy

2. Réyni Entropy of X and Y are given by

$$H_\alpha(X) = \frac{1}{1-\alpha}\ln\int_0^\infty [f(x;\theta)]^\alpha\ dx\ ;\ \alpha > 0, \alpha \neq 1$$

$$= \frac{1}{\alpha-1}\left[(\alpha-1)ln2 + \frac{\alpha-1}{2}\ln\left(\frac{\theta}{\alpha}\right) - ln\alpha + \ln\gamma\left(\frac{\alpha+1}{2}\right)\right]$$

$$H_\alpha(Y) = \frac{1}{1-\alpha}\ln\int_0^t [f(y;t;\theta)]^\alpha\ dy\ , \alpha > 0, \alpha \neq 1$$

$$= \frac{1}{\alpha-1}\left[(\alpha-1)ln2 + \frac{\alpha-1}{2}\ln\left(\frac{\theta}{\alpha}\right) - ln\alpha - \alpha\ln F(t;\theta) + \ln\gamma\left(\frac{\alpha+1}{2}, \alpha\theta t^2\right)\right]$$

3. Tsallis Entropy of X and Y are given by

$$T_\alpha(X) = \frac{1}{\alpha-1}\ln\int_0^\infty [f(x;\theta)]^\alpha\ dx\ ;\ \alpha > 0, \alpha \neq 1 \tag{3.1}$$

$$T_\alpha(Y) = \frac{1}{\alpha-1}\ln\int_0^t [f(y;\theta)]^\alpha\ dx\ ;\ \alpha > 0, \alpha \neq 1 \tag{3.2}$$

Thus the relative loss of the entropy in using Y instead of X is

$$S_T(t) = \frac{H_\alpha(x) - H_\alpha(Y)}{H_\alpha(X)} \tag{3.3}$$

4. Havrda and Charvat Entropy of X and Y are given by

$$H_\alpha(X) = \frac{1}{2^{1-\alpha}-1}\left(\int_0^\infty [f(x;\theta)]^\alpha\ dx - 1\right), \alpha > 0, \alpha \neq 1$$

$$= \frac{1}{2^{1-\alpha}-1}\left[\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha}\ \gamma\left(\frac{\alpha+1}{2}\right) - 1\right]$$

$$H_\alpha(Y) = \frac{1}{2^{1-\alpha}-1}\left(\int_0^t [f(y;t;\theta)]^\alpha \, dy - 1\right), \alpha > 0, \alpha \neq 1$$

$$= \frac{1}{2^{1-\alpha}-1}\left[\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha F^\alpha(t;\theta)} \gamma\left(\frac{\alpha+1}{2}; \alpha\theta t^2\right) - 1\right]$$

$$S_{HC}(t) = \frac{\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha} \gamma\left(\frac{\alpha+1}{2}\right) - \frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha F^\alpha(t;\theta)} \gamma\left(\frac{\alpha+1}{2}; \alpha\theta t^2\right)}{\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha} \gamma\left(\frac{\alpha+1}{2}\right) - 1}$$

5. Awad Entropy of X and Y are given by

$$A(X) = -\int_0^\infty f(x;\theta) ln \frac{f(x;\theta)}{\delta} dx$$

$$= ln\delta + 1 - ln2 - \frac{1}{2} ln\theta - \frac{1}{2}\gamma'(1)$$

$$A(Y) = -\int_0^\infty f(y;t;\theta) ln \frac{f(y;t;\theta)}{\delta} dy$$

$$= ln\delta + 1 - ln2 - \frac{1}{2} ln\theta + lnF(t;\theta) - \frac{\frac{1}{2}\gamma'(1,\theta t^2) + \theta t^2 e^{-\theta t^2}}{F(t;\theta)}$$

Thus the relative loss of entropy in using Y instead of X is

$$S_A(t) = \frac{\frac{\frac{1}{2}\gamma'(1,\theta t^2) + \theta t^2 e^{-\theta t^2}}{F(t;\theta)} - lnF(t;\theta) - \frac{1}{2}\gamma'(1)}{ln\delta + 1 - ln2 - \frac{1}{2} ln\theta - \frac{1}{2}\gamma'(1)}$$

6. Awad et al Entropy of X and Y is given by

$${}_\alpha A(X) = \frac{1}{1-\alpha} \ln \int_0^\infty \left[\frac{f(x;\theta)}{\delta}\right]^{\alpha-1} f(x;\theta) dx$$

$$= ln\delta + \frac{1}{1-\alpha}\left[(\alpha-1)ln2 + \frac{\alpha-1}{2}\ln\left(\frac{\theta}{\alpha}\right) - ln\alpha + ln\gamma\left(\frac{\alpha+1}{2}\right)\right]$$

$${}_\alpha A(Y) = \frac{1}{1-\alpha} \ln \int_0^t \left[\frac{f(y;t;\theta)}{\delta}\right]^{\alpha-1} f(y;t;\theta) dy$$

$$= ln\delta + \frac{1}{1-\alpha}\Big[(\alpha-1)ln2 + \frac{\alpha-1}{2}\ln\left(\frac{\theta}{\alpha}\right) - ln\alpha - \alpha F(t;\theta) + ln\gamma\left(\frac{\alpha+1}{2}, \alpha\theta t^2\right)\Big]$$

Thus the relative loss in entropy in using Y instead of X is

$$S_{AER}(t) = \frac{\frac{1}{1-\alpha}\left[ln\gamma\left(\frac{\alpha+1}{2}\right) + \alpha F(t;\theta) - \; ln\gamma\left(\frac{\alpha+1}{2}, \alpha\theta t^2\right)\right]}{ln\delta + \frac{1}{1-\alpha}\left[(\alpha-1)ln2 + \frac{\alpha-1}{2}\ln\left(\frac{\theta}{\alpha}\right) - ln\alpha + ln\gamma\left(\frac{\alpha+1}{2}\right)\right]}$$

7. Awad et al. (1987) version of Havrda and Charvat Entropy of X and Y are given by

$$A^{\alpha}(X) = \frac{1}{2^{1-\alpha}-1}\left(\int_0^{\infty}\left[\frac{f(x;\theta)}{\delta}\right]^{\alpha-1} f(x;\theta)dx - 1\right)$$

$$= \frac{1}{2^{1-\alpha}-1}\left[\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha\delta^{\alpha-1}}\gamma\left(\frac{\alpha+1}{2}\right) - 1\right]$$

$$A^{\alpha}(Y) = \frac{1}{2^{1-\alpha}-1}\left(\int_0^{t}\left[\frac{f(y;t;\theta)}{\delta}\right]^{\alpha-1} f(y;t;\theta)dy - 1\right)$$

$$= \frac{1}{2^{1-\alpha}-1}\left[\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha\delta^{\alpha-1}F^{\alpha}(t;\theta)}\gamma\left(\frac{\alpha+1}{2}; \alpha\theta t^2\right) - 1\right]$$

Thus the relative loss in entropy in using Y instead of X is

$$S_{AEH}(t) = \frac{\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha}\gamma\left(\frac{\alpha+1}{2}\right) - \frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha F^{\alpha}(t;\theta)}\gamma\left(\frac{\alpha+1}{2}, \alpha\theta t^2\right)}{\left[\frac{2^{\alpha-1}\left(\frac{\theta}{\alpha}\right)^{\frac{\alpha-1}{2}}}{\alpha}\gamma\left(\frac{\alpha+1}{2}\right) - \delta^{\alpha-1}\right]}.$$

4. NUMERICAL RESULTS AND DISCUSSIONS

The results presented in Tables 1-6 are by no means comprehensive but hopefully will pave the way for reexamining the popular measures of entropy in a different, more general and rigorous setting. The following are the observations we made from Tables 1-6:

1. The relative loss of Shannon Entropy $\left[S_H(t)\right]$ decreases when the truncation time t increases for the Rayleigh distributed random variable with parameter θ. This is in the same line as in the case of exponential distribution (see Awad et al. (1987)). This result is natural, since the entropy increases as t increases.

2. The relative loss of Awad Entropy increases when the truncation time t increases; this is not natural because the information must increase as t increases.

3. a) For $\theta < 1$ and $\alpha < 1$, i) for $t < 1$, $S_{HC}(t) < S_R(t)$ or $S_T(t) < S_H(t)$
   whereas ii) for $t > 1$, $S_R(t)$ or $S_T(t) < S_{HC}(t) < S_H(t)$

   b) For θ<1 and α>1, i) for $t < 1$, $S_H(t) < S_R(t)$ or $S_T(t) < S_{HC}(t)$
   whereas ii) for $t > 1$, $S_{HC}(t) < S_H(t) < S_R(t)$ or $S_T(t)$

4. As θ increases $S_H(t)$, $S_R(t)$ or $S_T(t)$ and $S_{HC}(t)$ increases, but no such pattern is followed for remaining entropy measures.

Theoretically, it will be tedious to show which relative loss of entropy is better than others since the expressions in the relative loss of entropy function involves incomplete gamma functions and its derivative.

5. CONCLUDING REMARKS

In this article, we have compared different entropy measures for the Rayleigh distributed life time model in terms of relative information loss. For exponentially distributed lifetime model, Awad’s entropy measures are found to be better from loss of information point of view (see Awad et al. (1987)). However, in our case, it is noteworthy that Awad’s entropy measures and their extended versions are no way better than that of Shannon, Réyni or Tsallis, and Havrda and Charvat measures. Again, Shannon, Réyni or Tsallis, and Havrda and Charvat entropy measures are relatively better than one another depending on different settings or situations

ACKNOWLEDGEMENT

The author would like to thank the referee for very careful reading of the manuscript and making a number of nice suggestions which improved the earlier version of the manuscript.

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Table 1
The Relative Loss of Entropies

t	θ=0.1		θ=0.5		θ=1		θ=2		θ=3	
	$S_H(t)$	$S_A(t)$	$S_H(t)$	$S_A(t)$	$S_H(t)$	$S_A(t)$	$S_H(t)$	$S_A(t)$	$S_H(t)$	$S_A(t)$
0.25	1.9038	0.7689	2.6725	0.6417	3.6396	0.6882	7.2854	0.8005	34.7396	0.9273
0.5	1.5056	0.8921	1.9248	0.7019	2.4396	0.7835	4.3454	1.0235	18.6044	1.4260
0.75	1.2713	1.0316	1.4760	0.7525	1.7082	0.8555	2.5476	1.2278	9.0139	2.4858
1.0	1.1036	1.2057	1.1481	0.7867	1.1741	0.8695	1.3289	1.2282	3.4122	11.3379
1.25	0.9721	1.4389	0.8875	0.7931	0.7661	0.7856	0.5757	0.8487	0.9127	-1.5084
1.5	0.8634	1.7767	0.6731	0.7597	0.4623	0.6025	0.1968	0.3627	0.1636	-0.2097
1.75	0.7702	2.3222	0.4956	0.6809	0.2523	0.3819	0.0516	0.1019	0.0194	-0.0241
2.0	0.6884	3.3739	0.3511	0.5630	0.1224	0.2002	0.0103	0.0206	0.0015	-0.0019

Table 2
The Relative Loss of Entropies

t	θ=0.1				
	α	SR(t) or ST(t)	SHC(t)	SAER(t)	SAEH(t)
0.25	0.5	1.789545	1.332083	0.7602652	0.9153671
0.5		1.424882	1.209082	0.8737204	0.9418415
0.75		1.210829	1.114291	0.998081	0.9989794
1		1.058198	1.033875	1.147254	1.086237
1.25		0.9390648	0.9624619	1.336774	1.211845
1.5		0.8410185	0.8972982	1.591356	1.393532
1.75		0.7574692	0.836761	1.957316	1.667157
2		0.6845139	0.7798141	2.5359	2.112877
0.25	1.5	1.978316	3.22995	0.7772074	0.5718739
0.5		1.560961	2.054676	0.9074911	0.8368388
0.75		1.314872	1.53093	1.05755	1.095278
1		1.138256	1.216066	1.248922	1.391277
1.25		0.9992888	0.9989516	1.513187	1.771353
1.5		0.8838784	0.8368195	1.914101	2.320257
1.75		0.7845927	0.7092998	2.613116	3.246991
2		0.6970897	0.6053434	4.18365	5.28927

Table 3
The Relative Loss of Entropies

t	θ=0.5				
	α	SR(t) or ST(t)	SHC(t)	SAER(t)	SAEH(t)
0.25	0.5	2.365692	1.720427	0.6374797	0.83638
0.5		1.729561	1.450821	0.6939127	0.8225628
0.75		1.351834	1.239966	0.7412545	0.828353
1		1.079054	1.058031	0.7743396	0.8360976
1.25		0.8642396	0.894263	0.784239	0.8323249
1.5		0.6875993	0.7443242	0.7620752	0.8049491
1.75		0.5396179	0.6070023	0.703613	0.7461959
2		0.4154287	0.4828147	0.6131216	0.6565689
0.25	1.5	2.885063	4.584324	0.6480274	0.4240886
0.5		2.065154	2.665367	0.7120118	0.5760179
0.75		1.56898	1.794218	0.7657766	0.6812475
1		1.203168	1.261484	0.8015333	0.7469368
1.25		0.9108718	0.8923328	0.8061559	0.766264
1.5		0.6706422	0.6219497	0.7659597	0.7310117
1.75		0.4743145	0.4208855	0.6744967	0.6406313
2		0.3188126	0.2732977	0.540888	0.5092183

Table 4
The Relative Loss of Entropies

t	θ=1				
	α	SR(t) or ST(t)	SHC(t)	SAER(t)	SAEH(t)
0.25	0.5	2.99126	2.153706	0.6812679	0.8357694
0.5		2.054643	1.716776	0.7691638	0.8493007
0.75		1.4932	1.370667	0.8351265	0.8777915
1		1.087296	1.07072	0.8513953	0.8800393
1.25		0.7739607	0.8057036	0.7894421	0.8189786
1.5		0.5298828	0.5764729	0.6494141	0.6837952
1.75		0.3448066	0.388055	0.4712696	0.5056319
2		0.2115136	0.2439861	0.305391	0.3334875
0.25	1.5	4.138756	6.492185	0.6968957	0.5257581
0.5		2.743081	3.499028	0.7987243	0.7219069
0.75		1.881814	2.123962	0.8761712	0.84405
1		1.249165	1.292065	0.8899841	0.871403
1.25		0.7715678	0.7482748	0.7942552	0.7732478
1.5		0.4299678	0.3985196	0.5874659	0.5644276
1.75		0.2111323	0.1901575	0.3466099	0.3289295
2		0.0900383	0.079826	0.1630398	0.153457

Table 5
The Relative Loss of Entropies

t	θ=2				
	α	**SR(t) or ST(t)**	**SHC(t)**	**SAER(t)**	**SAEH(t)**
0.25	0.5	4.678289	3.335932	0.785031	0.8738594
0.5		2.918732	2.432775	0.9772146	0.983321
0.75		1.859887	1.711455	1.121749	1.100309
1		1.123693	1.110156	1.06997	1.0623
1.25		0.6181836	0.6419627	0.7697858	0.7840328
1.5		0.3021221	0.3238239	0.4196019	0.4374868
1.75		0.129712	0.141469	0.1858524	0.1967428
2		0.0488487	0.0537147	0.0704512	0.0751591
0.25	1.5	10.46149	16.17682	0.8156001	0.7253883
0.5		6.094915	7.656282	1.062212	1.079871
0.75		3.397286	3.774779	1.322308	1.371858
1		1.615016	1.658717	1.442317	1.473691
1.25		0.6055825	0.5953732	1.141855	1.145551
1.5		0.1712998	0.1653064	0.5154569	0.5119188
1.75		0.036171	0.0347047	0.1293459	0.1280076
2		0.0057531	0.0055128	0.0214055	0.0211665

Table 6
The Relative Loss of Entropies

t	**θ=3**				
	α	**SR(t) or ST(t)**	**SHC(t)**	**SAER(t)**	**SAEH(t)**
0.25	0.5	8.316747	5.898827	0.8984711	0.9359211
0.5		4.769321	3.973	1.279369	1.217884
0.75		2.664213	2.454056	1.6707	1.585201
1		1.316437	1.295791	1.564003	1.527299
1.25		0.546644	0.5592498	0.8587552	0.8626589
1.5		0.1868372	0.1946605	0.3130078	0.3195729
1.75		0.0525255	0.0550999	0.0887759	0.0912349
2		0.0122133	0.0128383	0.0206588	0.0212742
0.25	1.5	-48.72722	-74.64187	0.9519103	0.9307491
0.5		-25.11665	-31.2147	1.546344	1.667626
0.75		-11.13043	-12.29024	4.034453	4.321571
1		-3.5791	-3.714168	-1.789071	-1.871562
1.25		-0.7590077	-0.769844	-0.2301902	-0.2378155
1.5		-0.1039898	-0.1049183	-0.0293103	-0.0301819
1.75		-0.0093799	-0.0094564	-0.0026193	-0.0026959
2		-0.0005694	-0.000574	-0.0001589	-0.0001635