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MODIFIED BURR III DISTRIBUTION, PROPERTIES AND APPLICATIONS

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ABSTRACT

Many researchers have attempted to develop appropriate data-friendly models to analyze statistical data. Even new generalizations of models with numerous parameters are derived to handle complex data. The Burr III distribution has been widely used in forestry, reliability quality control, mechanical factors, life distributions, risk analysis, weather forecasting, consumer prices etc. Lindsay et al. (1996), Wingo (1993), Wang et al. (1996) Zimmer et al. (1998), Watkins (1999), Shao (2000 and 2004). In this paper, we propose a modification in Burr III (MB III) distribution. The proposed distribution is more flexible and tractable than its parent Burr type III (B III) distribution. Some properties of the proposed distribution are derived. To illustrate its application we use fracture toughness data from the material Alumina (Al_2O_3) and compare with Burr III and its sub-model.

1. INTRODUCTION

Burr (1942) developed a system of twelve types of distribution functions based on generating the Pearson differential equation. The density function has a range of shapes that is applicable to a wide area of applications. From the system of Burr distributions, the Burr XII distribution is widely used model. The Burr III distribution has been used in a variety of setting for the purpose of statistical modeling. Some examples include such as in forestry by Gove et al. (2008) and Lindsay et al. (1996), in fracture roughness by Nadarajah and Kotz (2006 and 2007), in life testing by Wingo (1983 and1993), Operational risk Chernobai et al. (2007), in option market price distributions by Sherrick et al. (1996), in meteorology by Mielke (1973), in modeling crop rice by Tejeda and Goodwin (2008), and in reliability by Abdel-Ghaly et al. (1997). The performance of the Burr III over distributions has not been received much attention of researchers.

The inverse distribution of Burr XII is Burr III with the cumulative distribution function

$$G(x)=\left[1+x^{-\beta}\right]^{-\alpha}, \quad \alpha,\beta>0, x>0, \tag{1.1}$$

where α,β are the shape parameters. Various fields of science used the B III distribution. It is also called the Dagum distribution in studies of income, wage and wealth distribution (see Dagum, 1977). In the actuarial literature, it is known as the inverse Burr distribution (see Kleiber and Kotz, 2003) and the kappa distribution in the meteorological literature (see Mielke, 1973). Benjamin et al. (2013) used Dagum distribution to model the maximum daily levels of tropospheric ozone. Quantile of Dagum distribution was used to analyze ozone data from Pedregal Station for the period 2001-2008. Shao et al. (2008) used extended B III distribution in low-flow frequency analysis where its lower tail was of main interest.

2. MODIFIED BURR III (MB III) DISTRIBUTION

2.1 MB III Distribution:

The modified Burr III distribution is defined as

$$F(x)=\left[1+\gamma x^{-\beta}\right]^{-\frac{\alpha}{\gamma}},\ \alpha,\beta,\gamma>0,\ x>0, \tag{2.1}$$

with the probability density function

$$f(x)=\alpha\beta x^{-\beta-1}\left[1+\gamma x^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}, \tag{2.2}$$

where $\alpha,\beta,$ and γ are the shape parameters of MB III distribution. The limiting distribution for $\gamma\to 0$ of MB III distribution is a generalized inverse Weibull distribution (see Gusmao et al. 2011). When $\gamma=1$, the MB III distribution reduces to B III distribution (see Burr, 1942). When $\alpha=\gamma=1$ and adding a scale parameter, the MB III distribution leads to log logistic distribution (see Shoukri et al., 1988).

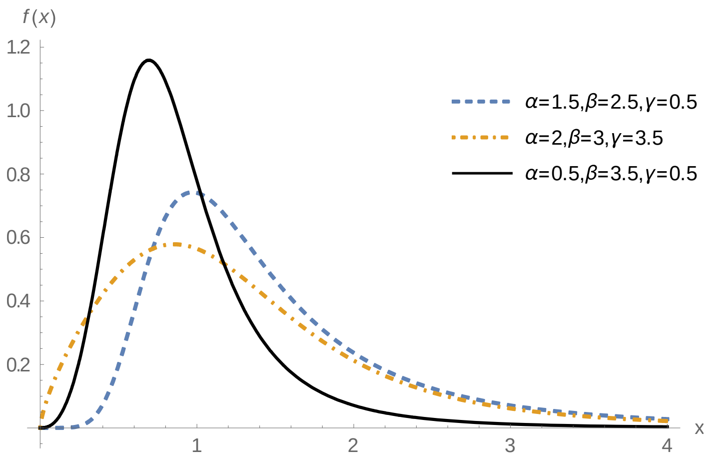

Fig. (2.1): PDF Plot of MB III Distribution

3. PROPERTIES OF MB III DISTRIBUTION

Some properties of MB III distribution have been derived which are quite useful in distribution theory to understand the application scope of a distribution. We derive moments, percentile and random number generation, mode, reliability and in last we characterize MB III distribution.

3.1 Moments of MB III Distribution:

The *rth* moments of MB III distribution are obtained as

$$\mu_r' = \int_0^\infty x^r f(x)dx = \alpha\gamma^{\frac{r}{\beta}-1} B\left(A_r, B_r\right),\ r = 1, 2, \cdots \tag{3.1.1}$$

where $B\left(A_r, B_r\right)$ is a beta function, $A_r = 1 - \frac{r}{\beta}$, $B_r = \frac{\alpha}{\gamma} + \frac{r}{\beta}$ and $r < \beta$.

The negative moments are

$$\mu_{-r}' = \frac{\alpha}{\gamma^{\frac{r}{\beta}+1}} B\left(\frac{r}{\beta}+1, \frac{\alpha}{\gamma} - \frac{r}{\beta}\right),\ r = 1, 2, \cdots \tag{3.1.2}$$

for $\alpha\beta > r\gamma$.

The mean and variance of MB III distribution are given as

$$E(X) = \frac{\alpha}{\gamma^{\frac{1}{\beta}+1}} B\left(\frac{1}{\beta}+1, \frac{\alpha}{\gamma} - \frac{1}{\beta}\right), \tag{3.1.3}$$

$$V\text{ar}(X) = \frac{\alpha}{\gamma^{\frac{2}{\beta}+1}} \left[B\left(\frac{2}{\beta}+1, \frac{\alpha}{\gamma} - \frac{2}{\beta}\right) - \frac{\alpha}{\gamma} B^2\left(\frac{1}{\beta}+1, \frac{\alpha}{\gamma} - \frac{1}{\beta}\right)\right]. \tag{3.1.4}$$

The central moments are

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \alpha^{j+1} \gamma^{\frac{r}{\beta}-j-1} \left[B\left(A_1, B_1\right)\right]^j B\left(A_{r-j}, B_{r-j}\right). \tag{3.1.5}$$

The cumulants is

$$K_r = \alpha\gamma^{\frac{r}{\beta}-1} B\left(A_r, B_r\right) - \sum_{j=1}^{r-1} \binom{r-1}{j-1} K_j \gamma^{-\frac{j}{\beta}} B\left(A_{r-j}, B_{r-j}\right). \tag{3.1.6}$$

The mode is

$$\hat{x} = \left[\frac{\alpha\beta-\gamma}{\beta+1}\right]^{\frac{1}{\beta}}. \tag{3.1.7}$$

The coefficient of skewness is

$$\beta_1 = \frac{\sum_{j=0}^{3}\binom{3}{j}(-1)^j \alpha^{j+1}\gamma^{\frac{3}{\beta}-j-1}\left(B\left(A_1,B_1\right)\right)^j B\left(A_{3-j},B_{3-j}\right)}{\left[\sum_{j=0}^{2}\binom{2}{j}(-1)^j \alpha^{j+1}\gamma^{\frac{2}{\beta}-j-1}\left(B\left(A_1,B_1\right)\right)^j B\left(A_{2-j},B_{2-j}\right)\right]^{\frac{3}{2}}}, \tag{3.1.8}$$

and the coefficient of kurtosis is

$$\beta_2 = \frac{\sum_{j=0}^{4}\binom{4}{j}(-1)^j \alpha^{j+1}\gamma^{\frac{4}{\beta}-j-1}\left(B\left(A_1,B_1\right)\right)^j B\left(A_{4-j},B_{4-j}\right)}{\left[\sum_{j=0}^{2}\binom{2}{j}(-1)^j \alpha^{j+1}\gamma^{\frac{2}{\beta}-j-1}\left(B\left(A_1,B_1\right)\right)^j B\left(A_{2-j},B_{2-j}\right)\right]^{2}}. \tag{3.1.9}$$

Further we plot β_1 and β_2 -3 for the values of α, β and γ .

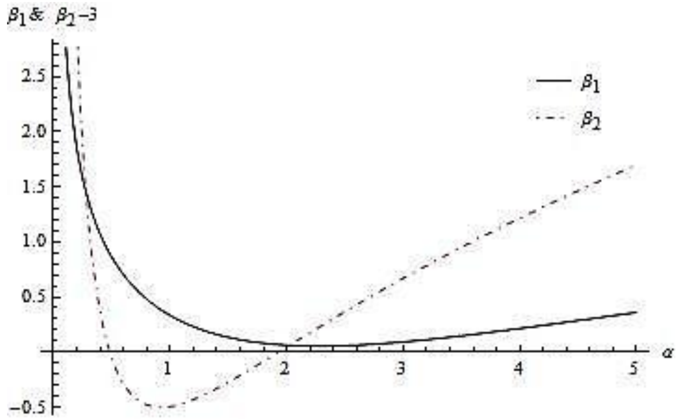

Fig. 3.1(a) ($\beta=\gamma=10$**)**

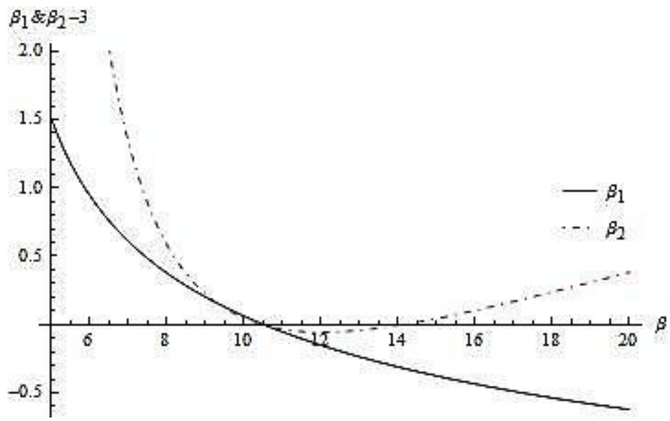

Fig. 3.1(b) ($\alpha=2,\gamma=10$**)**

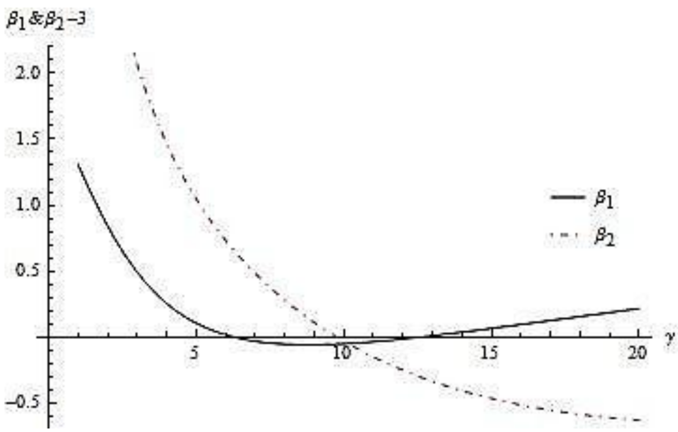

Fig. 3.1(c) ($\alpha=2,\beta=11$**)**

In Fig. 3.1(a) both β_1 and $\beta_2 - 3$ approximately touch the x-axis. So for β =10 and γ =2, the MBIII distribution is approximately symmetrical. Similarly in Fig. 3.1(b) we fix α and γ at 2 and 10 respectively and vary β. For β =11 both β_1 and β_2 -3 touch the x-axis. So at $\alpha = 2, \beta = 11$ and $\gamma = 10$, MBIII distribution is symmetrical. And in Fig. 3.1(c) we fix α and β at 2 and 11 and vary γ. For γ =10 both β_1 and β_2 -3 touch the x-axis. So at $\alpha = 2$, $\beta = 11$ and $\gamma = 10$, the MBIII distribution is symmetrical.

3.2 Quantile Function and Random Numbers Generation:

A distribution function in general is a non-decreasing function. The quantile function of F is a generalized inverse given by $F^{-1}(p) = \inf\{x : F(x) \geq p\}$. The quantile function is left continuous. Its range is [0,1]. The quantile function of MB III distribution is

$$x_q = \left[\frac{q^{-\frac{\gamma}{\alpha}} - 1}{\gamma} \right]^{-\frac{1}{\beta}}.$$

For $q = 0.5$, we get the median of MB III distribution. Random sample from MB III distribution can be generated, where $0 < q < 1$.

3.3 Reliability of MB III Distribution:

Reliability, some-times is defined to have the strength of a device to perform under certain stress until it wears out. Some of the devices designs include strength that tolerates stress under constant operating conditions. The stress-strength relationship is studied in many branches of science such as engineering, medicine, sociology, etc. If deals with estimation of the probabilities of the type P(X<Y), P(X<Z), etc. However, this set of probabilistic models is referred to the "stress-strength" or "reliability" models.

Let X_1 be the strength and X_2 be the stress of a random component then stress-strength model describes the life of the random component. The component fails when stress applied to it exceeds the strength and the component will function satisfactory whenever $X_2 < X_1$, hence

$$R = P\left(X_2 < X_1\right) = \int_0^\infty f_1(x) F_2(x) dx, \tag{3.3.1}$$

where 'R' is a measure of reliability.

Let X_1 and X_2 be independent MB III distributions as X_1 follows $MBIII(\alpha_1, \beta, \gamma)$ and X_2 follow $MBIII(\alpha_2, \beta, \gamma)$. Then from equations (2.2.1) and (2.2.2) the reliability is determined as

$$R = \alpha_1 \beta \int_0^\infty x^{-(\beta+1)} \left[1 + \gamma x^{-\beta}\right]^{-\frac{1}{\gamma}(\alpha_1 + \alpha_2) - 1} dx = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \tag{3.3.2}$$

3.4 Order Statistics of MB III Distribution:

Order statistics play an important role in statistical modeling and have been extensively investigated in the literature regarding its properties and statistical applications. Based on i.i.d. random variables, $X_1, X_2, \ldots, X_n$, with absolutely continuous distribution function and density of MBIII, the i^{th} order statistic of MBIII is defined as

$$f_{X_{(i)}}(x) = \frac{\alpha\beta n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} x^{-\beta-1} \left(1+\gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}(i+j)-1}. \quad (3.4.1)$$

The ith order moments is

$$E\left(X_{(i)}^r\right) = \frac{\alpha\beta n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \int_0^\infty x^{r-\beta-1} \left(1+\gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}(i+j)-1} dx$$

$$= \frac{\alpha n!}{(i-1)!(n-i)!\gamma^{-\frac{r}{\beta}+1}} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} B\left(1-\frac{r}{\beta}, \frac{r}{\beta}+\frac{\alpha}{\gamma}(i+j)\right). \quad (3.4.2)$$

3.5 Characterization of MB III Distribution:

Theorem 3.5.1

Let $f(x)$ be the pdf of the distribution of X defined on $(0,\infty)$. Then X follows the modified Burr III distribution if and only if

$$E\left(X^{-\beta} \middle| X < t\right) = t^{-\beta} - \frac{1+\gamma t^{-\beta}}{(\gamma-\alpha)}, \quad (3.5.1)$$

where $\alpha \neq \gamma$.

Proof:

Let X follows the modified Burr III distribution, then

$$E\left(X^{-\beta} \middle| X < t\right) = \int_0^t x^{-\beta} f(x)dx \Big/ \int_0^t f(x)dx = t^{-\beta} - \frac{1+\gamma t^{-\beta}}{(\gamma-\alpha)} \text{ is trivial.}$$

Conversely we have

$$F^{-1}(t)\int_0^t x^{-\beta} f(x)dx = t^{-\beta} - 1 + \gamma t^{-\beta} / (\gamma-\alpha).$$

Differentiating with respect to 't' and simplifying we get

$$\frac{f(t)}{F(t)} = \frac{\alpha\beta t^{-\beta-1}}{\left(1+\gamma t^{-\beta}\right)}.$$

Integrating with respect to '*t*', we have

$$\ln F(t) = \frac{-\alpha}{\gamma}\ln\left(1+\gamma t^{-\beta}\right) + \ln c$$

or

$$F(t) = c\left(1+\gamma t^{-\beta}\right)^{\frac{-\alpha}{\gamma}}, \text{ with c=1.}$$

Then we can write

$$F(x) = \left(1+\gamma x^{-\beta}\right)^{\frac{-\alpha}{\gamma}}.$$

Hence proved.

3.6 Renyi's Entropy and q-Entropies of MB III Distribution:

An entropy is quantitative value of uncertainty of a system. Larger the entropy, larger the uncertainty in the data. The Renyi's entropy is defined as

$$I_R(\theta) = \frac{1}{1-\theta}\log\left[I(\theta)\right], \tag{3.6.1}$$

where $I(\theta) = \int_S f^{\theta}(x)dx, \theta > 0, \theta \neq 1$ and $S = \left\{x : f(x) > 0\right\}$.

Based on Eq. (2.2),

$$f^{\theta}(x) = \left(\alpha\beta\right)^{\theta} x^{-\theta(\beta+1)}\left[1+\gamma x^{-\beta}\right]^{-\theta\left(\frac{\alpha}{\gamma}+1\right)}. \tag{3.6.2}$$

$$I(\theta) = \left(\alpha\beta\right)^{\theta}\int_0^{\infty} x^{-\theta(\beta+1)}\left[1+\gamma x^{-\beta}\right]^{-\theta\left(\frac{\alpha}{\gamma}+1\right)}dx$$

$$= \alpha^{\theta}\beta^{\theta-1}\gamma^{(1-\theta)\left(1+\frac{1}{\beta}\right)-1} B\left(\theta\left(1+\frac{1}{\beta}\right)+1, \theta\left(\frac{\alpha}{\gamma}+\frac{1}{\beta}\right)-1\right). \tag{3.6.3}$$

3.6.1 q-Entropy:

$H_q(f) = \frac{1}{1-q}\log\left[1-I(q)\right]$, where $I(q)$ is determined from eq. (3.6.3) replacing θ by q.

4. PARAMETER ESTIMATION OF MB III DISTRIBUTION

In this section we fit the proposed distribution along with B III distribution. The maximum likelihood technique is used to find the parameter estimates for real data set. Cordeiro et al. (2013) fit another extended B III distribution on carbon monoxide (CO) measurements made in several brands of cigarettes in USA using the R package.

Shao et al. (2008) proposed an extension of the three-parameter B III distribution and used it to determine low-flow frequency analysis in water resources research. Lindsay et al. (1996) predict timber volume in a forest using B III distribution as an alternative to the Weibull distribution. Using data from 20 permanent sample plots of Pinus radiata, they show that the B III distribution enhances precision by roughly 13%.

The likelihood function of MB III distribution is

$$L(\alpha,\beta,\gamma;x) = n\ln(\alpha\beta) - (\beta+1)\sum_{i=1}^{n}\ln x_i - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left[1+\gamma x_i^{-\beta}\right] \tag{4.1}$$

to find the parameter estimates we partially differentiate above equation with respect to each parameters and equate it to zero, i.e.

$$\frac{n}{\alpha} - \frac{1}{\gamma}\sum_{i=1}^{n}\ln\left[1+\gamma x_i^{-\beta}\right] = 0\,, \tag{4.2}$$

and

$$\frac{n}{\beta} - \sum_{i=1}^{n}\ln x_i + \left(\alpha+\gamma\right)\sum_{i=1}^{n}\frac{\beta x_i^{-\beta-1}}{\left[1+\gamma x_i^{-\beta}\right]} = 0 \tag{4.3}$$

and

$$\left(\alpha+\gamma\right)\sum_{i=1}^{n}\ln\frac{\beta x_i^{-\beta-1}}{\left[1+\gamma x_i^{-\beta}\right]} = 0\,. \tag{4.4}$$

Solving the nonlinear equations (4.2) to (4.4) simultaneously we obtain the maximum likelihood estimates of the MB III distribution. The asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ normal distribution is $N_3\left(0,\mathrm{K}(\theta)^{-1}\right)$, where θ is the vector of parameters and $K(\theta) = E\left\{J(\theta)\right\}$ is the expectation of the information matrix. $J(\theta)$ is defined as

$$J(\theta) = \begin{pmatrix} \dfrac{\partial^2 L(\theta)}{\partial\alpha\partial\alpha} & & \\ \dfrac{\partial^2 L(\theta)}{\partial\beta\partial\alpha} & \dfrac{\partial^2 L(\theta)}{\partial\beta\partial\beta} & \\ \dfrac{\partial^2 L(\theta)}{\partial\gamma\partial\alpha} & \dfrac{\partial^2 L(\theta)}{\partial\gamma\partial\beta} & \dfrac{\partial^2 L(\theta)}{\partial\gamma\partial\gamma} \end{pmatrix}. \tag{4.5}$$

The approximate multivariate normal distribution is $N_3\left(0,J(\hat{\theta})^{-1}\right)$, where $J(\hat{\theta})$ is observed information matrix evaluated at $\theta = \hat{\theta}$.

5. APPLICATION OF MB III DISTRIBUTION

We compare the fitting results of MB III distribution to its sub-model B III, for this purpose we used fracture toughness MPa m$^{1/2}$ data from the material Alumina

(Al_2O_3) (Nadarajah and Kotz, 2007. This data is available online at http://www.ceramics.nist.gov/srd/summary/ftmain.htm. The data set is

5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

The descriptive statistics of data set are given in the following table

Table 5(a)
Descriptive Statistics of Alumina (Al_2O_3)

Data	Mean	Median	S.D	Variance	Skewness	Kurtosis
Fracture Toughness (in the unit MPa $m^{1/2}$)	4.33	4.38	1.012	1.026	-0.42	3.093

Using SAS package with Proc NLMixed commands, we use several initial values to find the best fit for each model. We obtain the following results.

Table 5(b)
Maximum Likelihood Estimator and Information Criterion

Model	Maximum Likelihood Estimates			Information Criterion			
	α	β	γ	W	AIC	CAIC	BIC
MB III	1111.23	4.9484	770.70	373.4	379.4	379.6	387.7
B III	52.0509	3.0604	1	419.5	423.5	423.6	429.1
GIW	48.5844	3.0240	0	421.8	425.8	425.9	431.4

The MB III distribution provides the best fit as compared to B III and GIW distributions, where ‘W’ is -2loglikelihood. The values of all criteria are smaller for MB III distribution.

Fig (5.1) provides the fitted MB III, B III, GIW and histogram of the empirical data.

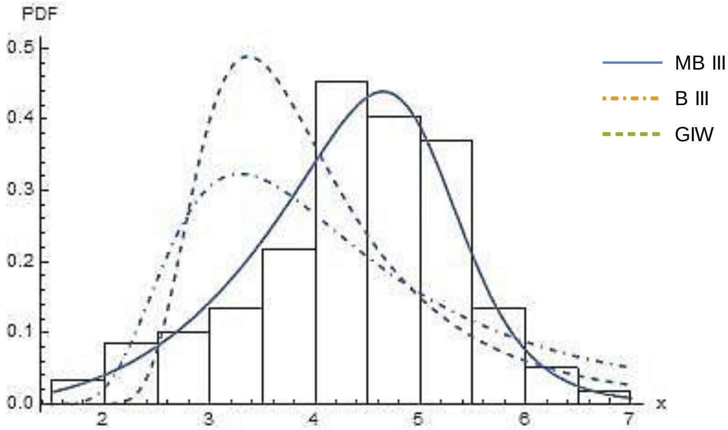

Fig. (5.1): Empirical Histogram and Fitted Distributions

The MB III distribution provides the close fit to the empirical data. Thus the proposed distributions provide a better fit than the B III distribution and GIW distribution.

6. CONCLUDING REMARKS

In this work we have proposed MBIII distribution and developed its various statistical properties. To illustrate its practical application we fitted the proposed distribution to a real data set and showed that it provides the best fit compared to its sub-models.

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