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GEOMETRIC PROPERTIES OF FRACTIONAL DIFFUSION EQUATION OF THE PROBABILITY DENSITY FUNCTION IN A COMPLEX DOMAIN

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ABSTRACT

In this study, we introduce a new class of fractional analytic function in the open unit disk. We deal with the subordination and superordination of a class, which involves the fractional differential operator in the sense of Srivastava-Owa operators. The fractional probability density function of nonlinear stochastic system is studied. By applying the new class to find approximate solutions of the time fractional master equation in the complex domain.

KEYWORDS

Fractional calculus; fractional diffusion equation; subordination and superordination; unit disk; analytic function.

1. INTRODUCTION

In the last few decades, the type of fractional differential equations has been cerebrated to be felicitous models of real life phenomenon. One of the fundamental implementations of the fractional calculus is formulated by the intermediate physical process. A very worthy class is the wave equations and fractional diffusion. It has been established that the acoustic, diversity of the universal electromagnetic, mechanical and responses can be formed accurately employing fractional diffusion-wave equations equations (Podlubny, 1999), (Hilfer, 2000), (Kilbas *et al.*, 2006), (Sabatier *et al.*, 2007), (Lakshmikantham *et al.*, 2009), (Jumarie, 2013).

Different seeking of the fractional diffusion equations have been introduced in various fractional operators such as the Riemann-Liouville, Caputo and Rize fractional differential operators (He *et al.*, 2012), (Guo, *et al.*, 2012). In addition, the authors in (Ibrahim, 2012) and (Ibrahim and Jalab, 2013) studied a maximal solution of the time-space fractional heat equation in a complex domain. The fractional time is considered in the sense of the Riemann-Liouville operator, while the fractional space is introduced in the sense of Srivastava-Owa operator for complex variables.

The fractional order plays a significant role in of the dynamical critical exponent. It was shown that there is a relation between the fractional equation and continuous time. Therefore, the characteristic waiting time density is found by utilizing the Mettage-Leffler function (Hilfer,1994).

The concepts of the subordination and superordination are useful tools to describe the upper and lower solutions of fractional differential equations in a complex domain (Ibrahim and Darus, 2008a) and (Ibrahim and Darus, 2008b). These concepts are defined in the geometric function theory, univalent function theory and analytic function theory. Moreover, they provided a geometric explanation for these solutions.

In this investigation, we suggest a new class of fractional analytic function in the open unit disk. We study the subordination and superordination of a class that involves the fractional differential operator in the sense of Srivastava-Owa operators. The fractional probability density function of nonlinear stochastic system is studied. By employing this class to find approximate solutions of the fractional Tim- Space master equation in the complex domain.

2. MATERIALS AND METHODS

Let H be the class of functions analytic in the unit disk $U=\{\xi:|\xi|<1\}$ and for $\alpha\in C$ and $n\in N$. Let $H[\alpha,n]$ be the subclass of H consisting of functions of the form

$$\varphi(\xi)=\alpha+\alpha_n\xi^n+\alpha_{n+1}\xi^{n+1}+\ldots.$$

Let A be the class of functions ϕ, analytic in U and normalized by the conditions $\varphi(0)=\varphi'(0)-1=0.$ Then a function $\phi\in A$ is called convex or starlike if it maps U into a convex or starlike region, respectively. Corresponding classes are symbolized by K and S^*: It is well known that $K\subset S^*$; that both are subclasses of the class of univalent functions and have the following analytical representations

$$\varphi\in S^*\Leftrightarrow\Re\left\{\frac{\xi\varphi'(\xi)}{\varphi(\xi)}\right\}>0,\xi\in U$$

and

$$\varphi\in K\Leftrightarrow\Re\left\{1+\frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right\}>0,\xi\in U.$$

Let φ be analytic in U, ψ analytic and univalent in U and $\varphi(0)=\psi(0)$. Then, by the symbol $\varphi(\xi)\prec\psi(\xi)$ (φ subordinate to ψ) in U, we shall mean $\varphi(U)\subset\psi(U)$.

Let $\phi:C^2\to C$ and let η be univalent in U. If $\wp$ is analytic in U and satisfies the differential subordination $\phi(\wp(\xi)),\xi\wp'(\xi))\prec\eta(\xi)$ then $\wp$ is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $\wp\prec q$. If $\wp$ and $\phi\left(\wp(\xi)\right),\xi\left(\wp'(\xi)\right)$ are univalent in U and satisfy the differential superordination $\eta(\xi)\prec\phi\left(\wp(\xi)\right),\xi\left(\wp'(\xi)\right)$ then $\wp$ is called a solution of the differential superordination. An analytic function q is called subordinate of the solution of the differential superordination if $q\prec\wp$.

In (Srivastava and Owa, 1989) defined fractional operators (derivative and integral) in the complex z-plane C as follows:

Definition 2.1

The fractional derivative of order μ is defined, for a function $\varphi(\xi)$ by

$$D_{\xi}^{\mu}\varphi(\xi) := \frac{1}{\Gamma(1-\mu)}\frac{d}{d\xi}\int_0^{\xi}\frac{\varphi(\zeta)}{(\xi-\zeta)^{\mu}}d\zeta;\ 0 \le \mu < 1,$$

where the function $\varphi(\xi)$ is analytic in simply-connected region of the complex z-plane C involving the origin and the multiplicity of $(\xi-\zeta)^{-\mu}$ is extracted by demanding $log(\xi-\zeta)$ to be real when $(\xi-\zeta) > 0.$

Definition 2.2

The fractional integral of order α is defined, for a function $\varphi(\xi),$ by

$$I_{\xi}^{\mu}\varphi(\xi) := \frac{1}{\Gamma(\mu)}\int_0^{\xi}\varphi(\zeta)(\xi-\zeta)^{\mu-1}d\zeta;\ \mu > 0,$$

where the function $\varphi(\xi)$ is analytic in simply-connected region of the complex z-plane (C) including the origin and the multiplicity of $(\xi-\zeta)^{\mu-1}$ is removed by demanding $log(\xi-\zeta)$ to be real when $(\xi-\zeta) > 0.$

Remark 2.1

$$D_{\xi}^{\mu}\xi^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\xi^{\lambda-\mu},\ \lambda > -1; 0 \le \mu < 1$$

and

$$I_{\xi}^{\mu}\xi^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\xi^{\lambda+\mu},\ \lambda > -1; \mu > 0.$$

Note that the above fractional operators are the extended operators for the Riemann-Liouville fractional operators (Miller and Mocanu, 2003)

$$_{a}I_t^{\nu} f(t) = \int_a^t \frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)} f(\tau)d\tau.$$

The fractional (arbitrary) order differential of the function f of order $\nu > 0$ is given by

$$_{a}D_t^{\nu} f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\nu}}{\Gamma(1-\nu)} f(\tau)d\tau.$$

When $a = 0$, we shall denote ${}_0D_t^\nu f(t) := D_t^\nu f(t)$ and ${}_0I_t^\nu f(t) := I_t^\nu f(t)$ in the follow-up.

Definition 2.3 (Miller and Mocanu, 2003)

The set Θ of all functions $\varphi(\xi)$ that are analytic and univalent on $\overline{U} - \varepsilon(\varphi)$ where $\varepsilon(\varphi) := \left\{\zeta \in \partial U : lim_{\xi\to\zeta}\varphi(\xi) = \infty\right\}$ and are such that $\varphi'(\zeta) \neq 0$ for $\zeta \in \partial U - \varepsilon(\varphi)$.

Lemma 2.1 (Shanmugam *et al.*, 2006)

Let $\rho(\xi) \in K$ and ψ and $\gamma \in C\setminus\{0\}$ with $\Re\left\{1 + \frac{\xi\rho''(\xi)}{\rho'(\xi)} + \frac{\psi}{\gamma}\right\} > 0.$ If $\wp(\xi) \in H$ and $\psi\wp(\xi) + \gamma\xi\wp'(\xi) \prec \psi\rho(\xi) + \gamma\xi\rho'(\xi),$ then $\wp(\xi) \prec \rho(\xi)$ and ρ is the best dominant.

Lemma 2.2 (Miller and Mocanu, 2003)

Let $\rho(\xi) \in K$ and $\gamma \in C$ with $\Re\{\gamma\} > 0.$ If $\wp(\xi) \in H\left[\rho(0),1\right] \cap \Theta,$ with $\wp(\xi) + \gamma\xi\wp'(\xi) \in S$ then $\rho(\xi) + \gamma\xi\rho'(\xi) \prec \wp(z) + \gamma\xi\wp'(\xi)$ leads to $\rho(\xi) \prec \wp(\xi)$ and $\rho(\xi)$ is the best subordinant.

Let A_μ be a class of analytic functions, which approximate by the infinite power series

$$h(\xi) = \frac{1}{\Gamma(2+\mu)}\xi^{1+\mu} + \sum_{n=2}^{\infty}\beta_n\xi^{n+\mu},$$

$$\left(0 \le \mu < 1, \beta_1 := \frac{1}{\Gamma(2+\mu)}, \xi \in U\right).$$

We have the following result:

Theorem 2.1

Let $\varphi \in A_\mu, D_\xi^\mu\varphi(\xi) + \gamma\xi D_\xi^{'\mu}\varphi(\xi) \in S$ and $\gamma \in C\setminus\{0\}.$ If the subordinations

$$\rho_1(\xi) + \gamma\rho_{1'}(\xi) \prec D_\xi^\mu\varphi(\xi) + \gamma\xi D_\xi^{'\mu}\varphi(\xi) \prec \rho_2(\xi) + \gamma\rho_{2'}(\xi),$$

where ρ_1 and ρ_2 satisfying the conditions of Lemma 2.2 and Lemma 2.1 respectively then

$$\rho_1(\xi) \prec D_\xi^\mu\varphi(\xi) \prec \rho_2(\xi), \quad \forall\xi \in U. \tag{1}$$

Proof.

Putting $\wp(\xi) := D_\xi^\mu\varphi(\xi), \varphi \in A_\mu.$ By using Remark 2.1, yields that $D_\xi^\mu\varphi(\xi) \in A$ and $\wp(0) = \rho_1(0) = \rho_2(0) = 0.$ By letting $\psi = 1$ in Lemma 2.1 and utilizing Lemma 2.2, the assertion (1) is followed directly.

3. RESULTS AND DISCUSSION

The fractional diffusion equation of the probability density function $P(t,r)$ can be expressed by

$$D_t^{\nu}P(t,r)=(\varepsilon_1 D_r^{\mu}m(r)+\varepsilon_2 D_r^{1+\mu}\sigma^2(r))P(t,r), \tag{2}$$

where $\varepsilon_1,\varepsilon_2$ are real constants, r is the amplitude of the system, $m(r)=\sum_n a_n r^n$ and $\sigma^2(r)=\sum_n b_n r^n$. In virtue of Theorem 2.1, the operator D_r^{μ} upper and lower bound by convex functions, therefore, we may obtain the following fractional equation:

$$D_t^{\nu}P(r,t)=\sum_{n=1}^{\infty}\omega_n(r)P(r,t),\quad 0\le\nu<1, \tag{3}$$
$$(r_0,t_0)=(r_0,0)=\upsilon(r_0),$$

where $P(t,r)$ is the probability density of computing diffusing entity at the position r at time t and $\omega_n(r)$ is the fractional transition rate. Our aim is to approximate the solution of (3) in the open unit disk in term of time $t\in J:=[0,T]$. Eq. (3) has various applications not only in mathematics but in different fields. It represented to fractional Brownian motion, electrochemical response and the river flow system (Ibrahim *et al.*, 2015). We need the following results in the sequel. The proof is similar to Theorems 7,8 in (Ibrahim and Jalab, 2013).

Lemma 3.1

Let $P(t,\xi)$ be univalent function in the unit disk for all $t\in J$. Then

$$\left|D_{\xi}^{\mu}P(t,\xi)\right|\le\frac{r^{1-\mu}}{\Gamma(1-\mu)}\Big(rF\big((2)_n,(1)_n;(1-\mu)_n;rt\big)\Big)',\quad 0\le\mu<1 \tag{4}$$
$$\left(':=\frac{d}{d\xi},r=|\xi|;\xi\in U\setminus\{0\}\right),$$

where the equality holds true for the Koebe function.

Lemma 3.2

Let $P(t,\xi)$ be convex function in the unit disk for all $t\in J$. Then

$$\left|D_{\xi}^{\mu}P(t,\xi)\right|\le\frac{r^{1-\mu}}{\Gamma(1-\mu)}F\big((2)_n,(1)_n;(1-\mu)_n;rt\big),\quad 0\le\mu<1 \tag{5}$$
$$\big(r=|\xi|;\xi\in U\setminus\{0\}\big).$$

Lemma 3.3

Let $P(t,\xi)$ be univalent function in the unit disk for all $t\in J$ and let $\beta:=1+\mu, 0<\mu<1$. Then

$$\left|D_{\xi}^{\beta}P(t,\xi)\right| \le \frac{r^{-\mu}}{\Gamma(1-\mu)}\Big(rF\big((2)_n,(1)_n;(1-\mu)_n;rt\big)\Big)' \tag{6}$$

$$\Big(' := \frac{d}{d\xi}, r = |\xi|; \xi \in U\setminus\{0\}\Big),$$

where the equality holds true for the Koebe function.

Lemma 3.4

Let $P(t,\xi)$ be convex function in the unit disk for all $t \in J$ and let $\beta = 1+\mu$ defined. Then

$$\left|D_{\xi}^{\beta}u(t,\xi)\right| \le \frac{r^{-\mu}}{\Gamma(1-\mu)}F\big((2)_n,(1)_n;(1-\mu)_n;rt\big), \tag{7}$$

$$\big(r = |\xi|; \xi \in U\setminus\{0\}\big).$$

We have the following result:

Theorem 3.1

Consider the initial differential equation (2). If $P(\xi,t)$ is univalent then Eq. (2) has an approximate solution to the hypergeometric function

$$P(t,\xi) \approx \frac{(\varepsilon_1 r + \varepsilon_2)}{r^{\mu}\Gamma(1-\mu)\Gamma(1+\nu)} t^{\nu}\Big(rF\big((2)_n,(1)_n,(1)_n;(1-\mu)_n,(1+\nu)_n;rt\big)\Big)'.$$

Proof.

By employing the upper bound of the operator $D_{\xi}^{\mu}P(t,\xi)$ and $D_{\xi}^{1+\mu}P(t,\xi)$ (Lemma 3.1 and Lemma 3.3), we have

$$\begin{aligned} D^{\nu}P(t,\xi) &\approx \frac{\varepsilon_1 r^{1-\mu}}{\Gamma(1-\mu)}\Big(rF\big((2)_n,(1)_n;(1-\mu)_n;rt\big)\Big)' \\ &\quad + \frac{\varepsilon_2 r^{-\mu}}{\Gamma(1-\mu)}\Big(rF\big((2)_n,(1)_n;(1-\mu)_n;rt\big)\Big)' \\ &= \frac{\big(\varepsilon_1 r + \varepsilon_2\big)}{r^{\mu}\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{(2)_n(1)_n}{(1-\mu)_n}\frac{n+1}{n!}(rt)^n \end{aligned} \tag{8}$$

$$\big(0 < |\xi| = r < 1; 0 < \mu < 1\big).$$

Operating (8) by I^{ν} and utilizing Remark 2.1 yield

$$
\begin{aligned}
P(t,\xi) &= \frac{(\varepsilon_1 r+\varepsilon_2)}{r^{\mu}\Gamma(1-\mu)}\sum_{n=0}^{\infty}\frac{(2)_n(1)_n}{(1-\mu)_n}\frac{n+1}{n!}r^n\frac{\Gamma(n+1)}{\Gamma(n+1+\nu)}t^{n+\nu}\\
&= \frac{(\varepsilon_1 r+\varepsilon_2)}{r^{\mu}\Gamma(1-\mu)\Gamma(1+\nu)}t^{\nu}\sum_{n=0}^{\infty}\frac{(2)_n(1)_n(1)_n}{(1-\mu)_n(1+\alpha)_n}\frac{n+1}{n!}(rt)^n \qquad (9)\\
&= \frac{(\varepsilon_1 r+\varepsilon_2)}{r^{\mu}\Gamma(1-\mu)\Gamma(1+\nu)}t^{\nu}\left(rF\left((2)_n,(1)_n,(1)_n;(1-\mu)_n,(1+\nu)_n;rt\right)\right)'
\end{aligned}
$$

Hence the proof.

Theorem 3.2

Consider the initial differential equation (2). If $P(\xi,t)$ is convex then Eq. (2) has an approximate solution to the hypergeometric function

$$
P(t,\xi)\approx\frac{(\varepsilon_1 r+\varepsilon_2)}{r^{\mu}\Gamma(1-\mu)\Gamma(1+\nu)}t^{\nu}\left(rF\left((2)_n,(1)_n,(1)_n;(1-\gamma)_n,(1+\nu)_n;rt\right)\right).
$$

Proof.

By applying the upper bound of the operator $D_{\xi}^{\mu}P(t,\xi)$ and $D_{\xi}^{1+\mu}P(t,\xi)$ (Lemma 3.2 and Lemma 3.4), we attain to

$$
\begin{aligned}
D^{\nu}P(t,\xi) &\approx \frac{\varepsilon_1 r^{1-\mu}}{\Gamma(1-\mu)}F\left((2)_n,(1)_n;(1-\mu)_n;rt\right)\\
&\qquad + \frac{\varepsilon_2 r^{-\mu}}{\Gamma(1-\mu)}F\left((2)_n,(1)_n;(1-\mu)_n;rt\right)\\
&= \frac{(\varepsilon_1 r+\varepsilon_2)}{r^{\mu}\Gamma(1-\mu)}F\left((2)_n,(1)_n;(1-\mu)_n;rt\right) \qquad (10)
\end{aligned}
$$

$(0<|\xi|=r<1; 0<\mu<1).$

Operating (10) by I^{ν} and using again Remark 2.1 imply the desired assertion.

4. CONCLUSION

We conclude that the probability density functions can be studied in view of the geometric function theory in a complex domain. It has been shown that the function is bounded by a generalized hypergeometric function. The method based on fractional differential equation type time-space.

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