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CHARACTERIZATIONS OF LEVY DISTRIBUTION VIA SUB-INDEPENDENCE OF THE RANDOM VARIABLES AND TRUNCATED MOMENTS

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ABSTRACT

The concept of sub-independence is based on the convolution of the distributions of the random variables. It is much weaker than that of independence, but is shown to be sufficient to yield the conclusions of important theorems and results in probability and statistics. It also provides a measure of dissociation between two random variables which is much stronger than uncorrelatedness. Following Ahsanullah and Nevzorov (2014), we present certain characterizations of Levy distribution based on: (i) the sub-independence of the random variables; (ii) a simple relationship between two truncated moments; (iii) conditional expectation of certain function of the random variable. In case of independence, characterization (i) reduces to that of Ahsanullah and Nevzorov (2014).

KEYWORD

Sub-independence, Characterization, Levy distribution, Truncated moments.

1. INTRODUCTION

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. Consequently, the investigator relies on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers, resulting in various characterizations reported in many different directions. The present work deals with the characterizations of Levy distribution. These characterizations are based on: (i) the sub-independence of random variables; (ii) a simple relationship between two truncated moments; (iii) conditional expectation of certain function of the random variable.

For the sake of completeness we will state below a few definitions related to the concept of sub-independence. The concept of sub-independence is stated as follows: The

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$rv's$ (random variables) X and Y with $cdf's$ (cumulative distribution functions) F_X and F_Y are $s.i.$ (sub-independent) if the cdf of $X+Y$ is given by

$$F_{X+Y}(z) = (F_X * F_Y)(z) = \int_{\mathbb{R}} F_X(z-y)\, dF_Y(y), \quad z \in \mathbb{R}, \tag{1.1}$$

or equivalently if and only if

$$\varphi_{X+Y}(t) = \varphi_{X,Y}(t,t) = \varphi_X(t)\varphi_Y(t), \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

where φ_X, φ_Y, φ_{X+Y} and $\varphi_{X,Y}$ are $cf's$ (characteristic functions) of X, Y, $X+Y$ and (X,Y), respectively.

The equations (1.1) and (1.2) above are in terms of cdf and cf. The definition of sub-independence in terms of events, similar to that of independence, is as follows. Let $(X,Y):\Omega \to \mathbb{R}^2$ be a discrete random vector with range $\Re(X,Y) = \{(x_i, y_j): i, j = 1, 2, ...\}$ (finitely or infinitely countable). Consider the events

$$A_i = \{\omega \in \Omega : X(\omega) = x_i\},\ B_j = \{\omega \in \Omega : Y(\omega) = y_j\}$$

and

$$A^z = \{\omega \in \Omega : X(\omega) + Y(\omega) = z\},\ z \in \Re(X+Y),$$

where, as usual, $X(\omega)$ is the value of the random variable X at the point ω in the sample space Ω.

Definition 1.1

The discrete $rv's$ X and Y are $s.i.$ if for every $z \in \Re(X+Y)$

$$P(A^z) = \sum_{i, j, x_i + y_j = z} \sum P(A_i) P(B_j). \tag{1.3}$$

For the continuous case, we observe that the half-plane $H = \{(x,y): x+y<0\} (\subseteq \mathbb{R}^2)$ can be written as a countable disjoint union of rectangles:

$$H = \cup_{i=1}^{\infty} E_i \times F_i,$$

where E_i and F_i are intervals. Now, let $(X,Y):\Omega \to \mathbb{R}^2$ be a continuous random vector and for $c \in \mathbb{R}$, let

$$A_c = \{\omega \in \Omega : X(\omega) + Y(\omega) < c\}$$

and

$$A_i^{(c)} = \left\{\omega \in \Omega : X(\omega) - \frac{c}{2} \in E_i\right\}, B_i^{(c)} = \left\{\omega \in \Omega : Y(\omega) - \frac{c}{2} \in F_i\right\}.$$

Definition 1.2

The continuous $rv's$ X and Y are $s.i.$ if for every $c \in \mathbb{R}$

$$P\left(A_c\right) = \sum_{i=1}^{\infty} P\left(A_i^{(c)}\right) P\left(B_i^{(c)}\right). \tag{1.4}$$

Remarks 1.3

(a) The discrete $rv's$ X, Y and Z are $s.i.$ if (1.3) holds for any pair and

$$P\left(A^s\right) = \sum_{i,j,k,} \sum_{x_i+y_j+z_k=s} \sum P\left(A_i\right) P\left(B_j\right) P\left(C_k\right). \tag{1.5}$$

For p variate case we need $2^p - p - 1$ equations of the above form. (b) The representation (1.2) can be extended to the multivariate case as well. (c) For a detailed treatment of the concept of sub-independence, we refer the interested reader to Hamedani (2013).

Ahsanullah and Nevzorov (2014) consider the Levy distribution with parameters 0 and $\sigma\left(Lev(0,\sigma)\right)$ whose pdf (probability density function) and cf are respectively given by

$$f(x) = f(x;0,\sigma) = \sqrt{\frac{\sigma}{2\pi}} x^{-3/2} e^{-\frac{\sigma}{2x}}, \quad x > 0, \tag{1.6}$$

and

$$\varphi(t) = e^{-\sqrt{-2i\sigma t}}.$$

They presented the following characterization of $Lev(0,\sigma)$ distribution.

Theorem 1.4

Let X_1, X_2, X_3 be independent and identically distributed absolutely continuous random variables with cdf $F(x)$ and pdf $f(x)$. We assume $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. Then X_1 and $\left(X_2 + X_3\right)/4$ are identically distributed if and only if $F(x)$ has the $Lev(0,\sigma)$ distribution.

In subsection 2.1 below, we present similar characterization based on the concept of sub-independence which in turn is stronger than Theorem 1.4.

2. CHARACTERIZATION RESULTS

In this section we present characterizations of $Lev(0,\sigma)$ distribution in three different directions (i)-(iii) mentioned in the Introduction.

2.1 Characterizations via Sub-Independence of the Random Variables

We present here two Lemmas characterizing $Lev(0,\sigma)$ distribution.

Lemma 2.2.1

Let X_1, X_2, X_3 be identically distributed continuous random variables with *cdf* $F(x)$ and *pdf* $f(x)$. We assume $F(0)=0$ and $F(x)>0$ for all $x>0$. If $d)$ $X_1 \sim Lev(0,\sigma)$ and $e)$ the random variables $\frac{X_2}{4}$ and $\frac{X_3}{4}$ are $s.i.$, then $(X_2+X_3)/4 \sim Lev(0,\sigma)$.

Proof:

In view of $e)$, we have

$$\varphi_{(X_2+X_3)/4}(t) = \left(\varphi_{(X_2)/4}(t)\right)^2 = \left(\varphi_{X_1}(t/4)\right)^2 = \left(e^{-\frac{1}{2}\sqrt{-2i\sigma t}}\right)^2 = e^{-\sqrt{-2i\sigma t}},$$

i.e., $(X_2+X_3)/4 \sim Lev(0,\sigma)$.

Lemma 2.2.2.

Let X_1, X_2, X_3 be identically distributed continuous random variables with *cdf* $F(x)$ and *pdf* $f(x)$. We assume $F(0)=0$ and $F(x)>0$ for all $x>0$. If $f)$ the random variables X_1 and $(X_2+X_3)/4$ are identically distributed and $i)$ for each $n=1,2,...,$ the random variables $\frac{X_2}{2^{2n}}$ and $\frac{X_3}{2^{2n}}$ are $s.i.$, then $X_1 \sim Lev(0,\sigma)$.

Proof:

In view of $f)$ and $i)$ (for $n=1$), we have

$$\varphi_{X_1}(t) = \left(\varphi_{X_1}\left(t/2^2\right)\right)^2,$$

and in view of $i)$, we arrive at

$$\varphi_{X_1}(t) = \left(\varphi_{X_1}\left(t/2^2\right)\right)^2 = ... = \left(\varphi_{X_1}\left(t/2^{2n}\right)\right)^{2^n}, \quad n=1,2,....$$

Now, the rest follows as in the proof of Theorem 1.4 (Ahsanullah and Nevzotov (2014), page 211).

2.2 Characterizations based on Two Truncated Moments

In this subsection we present characterizations of Levy distribution in terms of a simple relationship between two truncated moments. We like to mention here the works of Glänzel (1987,1990), Glänzel and Hamedani (2001) and Hamedani (2002,2006) in this direction. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem 2.2.1 below). The advantage of the characterizations given here is that, cdf F need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 2.2.1

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a given probability space and let $I = [a,b]$ be an interval for some $a < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to I$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on I such that

$$\mathrm{E}\left[g(X) \mid X \geq x\right] = \mathrm{E}\left[h(X) \mid X \geq x\right] \eta(x),\ x \in I\ ,$$

is defined with some real function η. Assume that g, $h \in C^1(I)$, $\eta \in C^2(I)$ and F is twice continuously differentiable and strictly monotone function on the set I. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of I. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)h(u) - g(u)}\right| \exp(-s(u))\,du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to make $\int_I dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions g_n, h_n and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 2.2.1 and let $g_n \to g$, $h_n \to h$ for some continuously differentiable real functions g and h. Let, finally, X be a random variable with distribution F. Under the condition that $g_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E\left[g(X) \mid X \geq x\right]}{E\left[h(X) \mid X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions g, h and η, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani (2001).

Remarks 2.2.2

In Theorem 2.2.1, the interval I need not be closed since the condition is only on the interior of I.

Proposition 2.2.3

Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $h(x)=x^{-1/2}$ and $g(x)=x^{-1/2}e^{-\frac{\sigma}{2x}}$ for $x\in(0,\infty)$. The *pdf* of X is (1.6) if and only if the function η defined in Theorem 2.2.1 has the form

$$\eta(x)=\frac{1}{2}\left(1+e^{-\frac{\sigma}{2x}}\right),\quad x>0.$$

Proof:

Let X have density (1.6), then

$$(1-F(x))\mathrm{E}\left[h(X)\mid X\geq x\right]=\sqrt{\frac{2}{\sigma\pi}}\left(1-e^{-\frac{\sigma}{2x}}\right),\quad x>0,$$

and

$$(1-F(x))\mathrm{E}\left[g(X)\mid X\geq x\right]=\sqrt{\frac{1}{2\sigma\pi}}\left(1-e^{-\frac{\sigma}{x}}\right),\quad x>0,$$

and finally

$$\eta(x)h(x)-g(x)=\frac{1}{2}x^{-1/2}\sqrt{\frac{2}{\sigma\pi}}\left(1-e^{-\frac{\sigma}{2x}}\right)>0\quad for\ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)h(x)}{\eta(x)h(x)-g(x)}=\frac{\frac{\sigma}{2}x^{-2}e^{-\frac{\sigma}{2x}}}{\left(1-e^{-\frac{\sigma}{2x}}\right)},\quad x>0,$$

and hence

$$s(x)=-\log\left\{\left(1-e^{-\frac{\sigma}{2x}}\right)\right\},\quad x>0.$$

Now, in view of Theorem 2.2.1, X has density (1.6).

Corollary 2.2.4

Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 2.2.3. The *pdf* of X is (1.6) if and only if there exist functions g and η defined in Theorem 2.2.1 satisfying the differential equation

$$\frac{\eta'(x)h(x)}{\eta(x)h(x)-g(x)}=\frac{\frac{\sigma}{2}x^{-2}e^{-\frac{\sigma}{2x}}}{\left(1-e^{-\frac{\sigma}{2x}}\right)},\quad x>0.$$

Remarks 2.2.5

A) The general solution of the differential equation in Corollary 2.2.4 is

$$\eta(x)=\left(1-e^{-\frac{\sigma}{2x}}\right)^{-1}\left[-\int\frac{\sigma}{2}x^{-3/2}e^{-\frac{\sigma}{2x}}g(x)\,dx+D\right],$$

for $x>0$, where D is a constant. One set of appropriate functions is given in Proposition 2.2.3 with $D=\frac{1}{2}$.

B) Clearly there are other triplets of functions (h,g,η) satisfying the conditions of Theorem 2.2.1.

2.3 Characterizations based on Certain Function of the Random Variable

We assume, without loss of generality, that the parameter σ of Levy distribution is 1. We start with the following simple lemma.

Lemma 2.3.1

Let $X:\Omega\to(0,\infty)$ be a continuous positive random variable with twice differentiable and strictly increasing *cdf* $F(x)$ and corresponding *pdf* $f(x)$ with $f(0)>0$. Let $h_1(x)$ be a function on $(0,\infty)$ with $E\left[h_1(X)\right]<\infty$ and $h_2(x)$ be a differentiable function on $(0,\infty)$ such that $\int_0^\infty\frac{h_1(x)-h_2'(x)}{h_2(x)}dx$ exists. Then

$$E\left[h_1(X)\,|\,X\le x\right]=h_2(x)\left(\frac{f(x)}{F(x)}\right),\quad x>0$$

implies

$$f(x)=c\exp\left\{\int_0^x\frac{h_1(u)-h_2'(u)}{h_2(u)}du\right\},\quad x>0$$

where c is chosen so the $\int_0^\infty f(x)dx=1$.

Proof:

The proof is straightforward and will be omitted.

Proposition 2.3.2

Let $X:\Omega \to (0,\infty)$ be a positive continuous random variable with twice differentiable and strictly increasing *cdf* $F(x)$ and corresponding *pdf* $f(x)$ with $f(0)>0$. Then

$$E\left[X^{-\left(k+\frac{1}{2}\right)} \mid X \le x\right] = h_2(x)\left(\frac{f(x)}{F(x)}\right), \quad x>0, \tag{2.3.1}$$

where $h_2(x) = 2^{k-\frac{1}{2}}\sum_{j=0}^{k}\frac{k!}{(k-j)!}(2x)^{-\left(k-j-\frac{3}{2}\right)}$ if and only if $f(x)=\frac{1}{\sqrt{2\pi}}x^{-3/2}e^{-\frac{1}{2x}}$, $x>0$.

Proof:

If X has *pdf* $f(x)=\frac{1}{\sqrt{2\pi}}x^{-3/2}e^{-\frac{1}{2x}}$, $x>0$, then

$$F(x)E\left[X^{-\left(k+\frac{1}{2}\right)} \mid X \le x\right] = \int_0^x \frac{1}{\sqrt{2\pi}}u^{-\left(k+\frac{1}{2}\right)}u^{-3/2}e^{-\frac{1}{2u}}du$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^x u^{-(k+2)}e^{-\frac{1}{2u}}du$$

$$= \frac{2^{k+1}}{\sqrt{2\pi}}\int_{\frac{1}{2x}}^{\infty} t^k e^{-t}dt$$

$$= \frac{2^{k+1}}{\sqrt{2\pi}}e^{-\frac{1}{2x}}\left[\sum_{j=0}^{k}\frac{k!}{(k-j)!}\left(\frac{1}{2x}\right)^{k-j}\right]$$

$$= h_2(x)\left(\frac{1}{\sqrt{2\pi}}x^{-3/2}e^{-\frac{1}{2x}}\right),$$

where $h_2(x) = 2^{k-\frac{1}{2}}\sum_{j=0}^{k}\frac{k!}{(k-j)!}(2x)^{-\left(k-j-\frac{3}{2}\right)}$. Thus, (2.3.1) holds.

Now, suppose (2.3.1) holds with

$$h_2(x) = 2^{k-\frac{1}{2}}\sum_{j=0}^{k}\frac{k!}{(k-j)!}(2x)^{-\left(k-j-\frac{3}{2}\right)}.$$

Then

$$h_2'(x) = -2^{k-\frac{1}{2}} \sum_{j=0}^{k} \frac{k!}{(k-j)!}\left[-2\left(k-j-\frac{3}{2}\right)\right](2x)^{-\left(k-j-\frac{1}{2}\right)}$$

$$= -2^{k+\frac{1}{2}} \left\{ \sum_{j=0}^{k} \frac{k!}{(k-j)!}(k-j)(2x)^{-\left(k-j-\frac{1}{2}\right)} - \frac{3}{2}\sum_{j=0}^{k-1} \frac{k!}{(k-j)!}(2x)^{-\left(k-j-\frac{1}{2}\right)} \right\}$$

$$= \frac{3}{2x} h_2(x) - (2x)^2 h_2(x) + x^{-\left(k+\frac{1}{2}\right)},$$

after a couple of simplification steps. Thus, we have

$$x^{-\left(k+\frac{1}{2}\right)} - h_2'(x) = \left(-\frac{3}{2x} + \frac{1}{2x^2}\right) h_2(x).$$

Now, using Lemma 2.3.1 and the fact that $\int_0^\infty f(x)\,dx = 1$, we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-\frac{1}{2x}}, \quad x > 0.$$

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