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A ROBUST NONPARAMETRIC SLOPE ESTIMATION IN LINEAR FUNCTIONAL RELATIONSHIP MODEL

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ABSTRACT

This paper proposed a robust nonparametric method to estimate the slope parameter of a linear functional relationship model in which both parameters are subject to error. The method is an improvement to the nonparametric method as proposed by Al-Nasser (2005). The performance of the proposed method is compared to the traditional maximum likelihood method using Monte Carlo simulation study. The mean square error of both these two methods gave somewhat similar results when no outlier exists, however as the percentage of outlier increases, the maximum likelihood method seems quite unreliable as its mean square error breaks down easily and became huge. Based on these findings, we can conclude that as the percentage of outlier increases, our proposed method gave significantly smaller mean square error than the nonparametric method (Al-Nasser (2005)). Application of the proposed method is illustrated using two published datasets.

KEYWORDS

Maximum likelihood method, Mean square error, Outlier, Linear Functional Relationship Model.

1. INTRODUCTION

A linear regression model can be explained in the form

$$y = \alpha + \beta x, \tag{1}$$

where x is the explanatory variable and y is the dependent variable. In linear regression model, the explanatory variable is assumed to be fixed and measured without error. However, in reality, for example in life sciences, biology, economics and ecology, it involves variables that could not be recorded exactly. Therefore, this assumption often does not exist and measurement errors may arise in the observations. The study of "errors-in-variables" models (EIVM) have been developed over the years with quite a large number of literatures, such as by Cheng and Van Ness (1994), Madansky (1959), Kendall and Stuart (1979) and Fuller (1987). The linear functional relationship model is one of the families of the EIVM which also includes structural and ultra-structural relationship model, as mentioned by Moran (1971).

Linear Functional Relationship Model (LFRM) can be expressed by

$$Y = \alpha + \beta X \ , \tag{2}$$

where both these two variables X and Y are observed with errors, with α and β is the intercept and slope parameter respectively. For any fixed X_i , we observe x_i and y_i from continuous linear variable subject to errors δ_i and ε_i respectively, i.e.

$$x_i = X_i + \delta_i \text{ and } y_i = Y_i + \varepsilon_i \tag{3}$$

where the error terms δ_i and ε_i are assumed to be mutually independent and normally distributed random variables, i.e.

$$\delta_i \sim N\left(0, \sigma_\delta^2\right) \text{ and } \varepsilon_i \sim N\left(0, \sigma_\varepsilon^2\right) \tag{4}$$

There are $(n+4)$ parameters that need to be estimated, namely $\alpha, \beta, \sigma_\delta^2, \sigma_\varepsilon^2$ and the incidental parameters $X_1, X_2, ..., X_n$. However, with these incidental parameters, it leads to inconsistencies of the estimators (Fuller (1987)). Some information is needed to overcome the inconsistencies of the estimators, which is, either one of the variances or the ratio of the two variances is known.

Several methods of estimation of linear functional relationship model (LFRM) have been suggested by these authors; Kendall and Stuart (1979), Fuller (1987), Cheng and Van Ness (1999), Huwang and Yang (2000) and Al-Nasser (2004). The methods suggested by these authors are mostly based on normality assumption, but it can be erroneous to use the normality assumption when there are outliers in the data set. In other words, when there is outlier, a robust method is necessary to diminish the effect of the outlier. In addition, the functional relationship model for circular data and its application also have been discussed by authors; Hussin and Mamun (2012), Hassan et al. (2010) and Hussin et al. (2010).

In this paper, we proposed a new robust nonparametric estimation of the slope parameter based on the nonparametric method which was proposed by Al-Nasser (2005). The paper is organized as following: Section 2 elaborates on the estimation methods for slope parameter of LFRM. Next, a simulation study is conducted in Section 3 to compare the performance of maximum likelihood estimation (MLE) method, the nonparametric method (Al-Nasser (2005)), and our proposed method. Section 4 highlights the application of our method by using two published data sets. Conclusion and discussion are presented in Section 5.

2. ESTIMATION METHOD FOR SLOPE PARAMETER IN LINEAR FUNCTIONAL RELATIONSHIP MODEL

2.1 Maximum Likelihood Estimation (MLE) Method

Maximum likelihood estimation (MLE) method is commonly used in LFRM. Based on Sprent (1969), when the λ is known, or when the ratio of error variance $\left(\lambda = \frac{\sigma_\varepsilon^2}{\sigma_\delta^2}\right)$ is assumed to be known, the MLE of the slope by assuming normality is:

$$\hat{\beta}_{MLE} = \frac{S_{yy} - \lambda S_{xx} + \left\{\left(S_{yy} - \lambda S_{xx}\right)^2 + 4\lambda S_{xy}^2\right\}^{\frac{1}{2}}}{2S_{xy}} \tag{5}$$

where $\bar{x} = \frac{1}{n}\sum x_i$, $\bar{y} = \frac{1}{n}\sum y_i$,

$S_{xx} = \sum\left(x_i - \bar{x}\right)^2$, $S_{yy} = \sum\left(y_i - \bar{y}\right)^2$

and $S_{xy} = \frac{1}{n}\sum\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)$.

2.2 Proposed Method

The method we proposed for estimating the slope parameter is by considering the nonparametric estimation method as proposed by Al-Nasser (2005). The advantage of using nonparametric method is that it does not require normality assumption (Hollander et al. (2013)). In this paper, we will extend the idea proposed by Al-Nasser (2005). The extension that we made is in Step 4 and Step 5 whereby we also arranged the data for y in ascending order, and we also find the median of all the slopes when x is arranged in ascending order, followed by when y is arranged in ascending order. The following are the steps involved in our proposed method.

Firstly, we arranged the observed pairs $(x_i, y_i)'s$, where $i = 1, 2, ..., n$; according to the magnitude of x value, by taking into account that all the values of x are distinct. Next, we sort these observations into several groups to obtain all the possible paired of slopes. Later on, we determine another possible paired of slopes by arranging the observed pairs according to the magnitude of y value. The steps are listed down in detail:

Step 1:

Arrange the observations in ascending order, base on x value, i.e., $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$. The associated values of y which may not be in ascending order are taken, i.e., $y_{[1]}, y_{[2]}, \ldots, y_{[n]}$. The new pairs will be $\left(x_{(i)}, y_{[i]}\right)$.

Step 2:

All the data are divided into m-subsamples. These subsamples contains r elements, such that $m*r=n$. The samples are arranged in the following form:

$$\begin{matrix} \left(x_{(1)},y_{[1]}\right) & \left(x_{(2)},y_{[2]}\right) & \cdots & \left(x_{(r)},y_{[r]}\right) \\ \left(x_{(r+1)},y_{[r+1]}\right) & \left(x_{(r+2)},y_{[r+2]}\right) & \cdots & \left(x_{(2r)},y_{[2r]}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(x_{((m-1)*(r+1))},y_{[(m-1)*(r+1)]}\right) & \cdots & \cdots & \left(x_{(mr)},y_{[mr]}\right) \end{matrix}$$

where m is the maximum divisor of n, such that $m\le r$. As an example, if $n=50$, then $m=5$ and $r=10$ respectively.

Step 3:

Find all the possible paired slopes.

$$\left\{b_x(k)_{ij}=\frac{y_{[j]}-y_{[i]}}{x_{(j)}-x_{(i)}};i=1,2,\ldots,j-1;j=2,3,\ldots,r\right\};k=1,2,\ldots m$$

Step 4:

Repeat Steps 1 to 3 by interchanging y and x to get another possible paired slopes of $b_y(k)_{ij}$.

$$\left\{b_y(k)_{ij}=\frac{y_{(j)}-y_{(i)}}{x_{[j]}-x_{[i]}};i=1,2,\ldots,j-1;j=2,3,\ldots,r\right\};k=1,2,\ldots m$$

Step 5:

Find the median of all these slopes.

$$\hat{\beta}_{new}=median\left\{b_x(k)_{ij},b_y(k)_{ij}\right\}.$$

This gives us the new estimated value of the slope parameter. The reason we use median of all the slopes in Step 5 is that the median is found to be more robust than using the mean.

3. SIMULATION STUDY

A simulation study is performed to compare our proposed method of estimation with the MLE method and the nonparametric method (Al-Nasser (2005)). We took into account when there is no outlier and also when there are different percentages of outliers. We begin by simulating our observations from our LFRM,

$$Y_i=1+X_i\ ,\ x_i=X_i+\delta_i,\ y_i=Y_i+\varepsilon_i, \tag{6}$$

where $X_i = 10\frac{i}{n}$ and $\delta_i, \varepsilon_i \sim N(0, 0.1$. Later on we contaminated the data for different levels of contamination by replacing the original observation by contaminated observations. The contamination observations were generated using the given relationship where $\delta_i, \varepsilon_i \sim N(0, 25)$. The properties of these methods were examined by looking at the mean square error (MSE) of the slope in 10,000 trials. For each simulation, we generated a sample size with $n = 20$, 50 and 100 from the sampling distribution as in (6). In order to investigate the robustness of our proposed method, we also consider the non-normal error terms whereby the error terms δ_i and ε_i were generated from three different distributions namely; symmetric Beta distribution with parameters (3, 3), right skewed Beta distribution (2, 9) and left skewed Beta distribution (9, 2) using the same above relationships.

Table 1 shows the values of m and r for different sample size for our proposed estimator. The simulation results are highlighted in Table 2 to 5.

Looking at Table 2 where the errors δ_i and ε_i are normally distributed, the mean square error (MSE) of our proposed method was somewhat similar to that of the MLE and the nonparametric method (Al-Nasser (2005)) when no outlier exists in the data. However, we can observe a great difference when the data gets contaminated. The MSE of slope estimator using MLE method breaks down easily and became huge. The MSE values for our proposed method and the nonparametric method (Al-Nasser (2005)), on the other hand were not much affected by outliers even when the percentage of outliers kept increasing.

Meanwhile, for Table 3 where the errors δ_i and ε_i are skewed to the right with Beta distribution (2, 9), the MSE of our proposed method also showed similar results to that of the MLE and the nonparametric method (Al-Nasser (2005)) in the case when no outlier is present. When the data gets contaminated, we can observe that the MSE of the slope estimator using MLE method breaks down very quickly. However, the MSE values for our proposed method and the nonparametric method (Al-Nasser (2005)) were shown to not being affected by the outliers even when the percentage of outliers kept increasing.

Next, from Table 4, where the errors δ_i and ε_i are skewed to the left with Beta distribution (9, 2), consistent MSE are obtained for the MLE, the nonparametric method (Al-Nasser (2005)) and our proposed method when no outlier is present in the data. However, when there is a single outlier, 10%, 20% or 30% outlier in the data set, the MSE for MLE became very huge. Nevertheless, the MSE for our proposed method and the nonparametric method (Al-Nasser (2005)) were shown to be consistently small and were not much affected by the presence of the outliers.

For Table 5, with errors δ_i and ε_i that are non-normal symmetric case with Beta distribution (3, 3), all three methods also showed somewhat similar MSE values when no outlier exist in the data. However, when there is outlier in the data, the MSE of the slope for MLE method breaks down quickly and became huge. But the MSE of the slope for our proposed method and the nonparametric method (Al-Nasser (2005)) were shown to maintain small and was not affected by the presence of outliers.

4. PRACTICAL EXAMPLE

In this section, we applied our proposed nonparametric technique to real life data and compare the three estimation methods namely the MLE method, the nonparametric method (Al-Nasser (2005)), and our proposed method. We considered a real data set from a study conducted by Goran et al. (1996), where the data set comprises of 97 observations. The study was to examine the accuracy of some widely used body-composition techniques for children between the ages of 4 and 10 years by two different techniques, namely skinfold thickness (ST) and bioelectrical resistance (BR). We assume that the measurement error can occur in either variable of this experiment to make the relationship as in model (2).

It is worthwhile to note that Ghapor et al. (2014) applied *COVRATIO* statistic in which the 45th observation is found to be an outlier. As it is an outlier in that data, we deleted that particular data and now the sample size is 96. In order to apply our proposed method to estimate the slope parameter, we divide these data sets into 8 groups, with each group having 12 elements. Next, to examine the effect on the slope with the presence of outlying observation, we modified Goran et al. (1996) data by inserting several outliers to create different outlier situation. The outliers are inserted by following Kim (2000) and Imon and Hadi (2008) where a certain percentage of the observations are deleted and replaced with the outliers' observation. The contaminated observation were generated base on the given relationship where $\delta_i, \varepsilon_i \sim N(0, 25)$. As an example, in our study, we insert a single outlier, 10%, 20%, and 30% outlier cases. The estimated slopes by using three different methods are shown in Table 6.

Based on Table 6, it clearly shows that our proposed method is more robust compared to the MLE method. We can observe that when there is no outlier in the data set, all the three methods showed somewhat similar results in terms of slope parameter. However, as the percentage of outlier increases, the slope for MLE method changed significantly compared to when there is no outlier. We can conclude that the MLE method breaks down easily with the increase of percentage of outliers, as compared to the nonparametric method (Al-Nasser (2005)) and our proposed method. Looking at the slope parameter of the nonparametric method (Al-Nasser (2005)) and our proposed method, we can conclude that both these estimation method were not much affected by the presence of outliers.

Next, we also consider another data set with smaller sample size, $n = 20$. We used the so-called Pilot-Plant data from Daniel and Wood (1971) where the response variable corresponds to the acid content determined by titration, and the explanatory variable is the organic acid content determined by extraction and weighing.

In order to apply our proposed method to estimate the slope parameter, we divide these data sets into 4 groups, with each group having 5 elements. Next, to examine the effect on the slope with the presence of outlying observation, we modified Daniel and Wood (1971) data by inserting several outliers to create different outlier situation. The outliers are inserted by following Kim (2000) and Imon and Hadi (2008) where a certain percentage of the observations are deleted and replaced with the outliers' observation. The contaminated observation were generated base on the given relationship where

$\delta_i, \varepsilon_i \sim N(0, 25)$. In this study, we insert a single outlier, 10%, 20%, and 30% outlier cases. Table 7 presents the estimated slopes by using three different methods.

From Table 7, it also shows that our proposed method is more robust compared to the MLE method. When there is no outlier in the data set, all the three methods showed somewhat consistent results in terms of slope parameter. However, as the percentage of outlier increases, the slope for MLE method changed significantly compared to when there is no outlier. We can conclude that the MLE method breaks down easily with the increase of percentage of outliers, as compared to the nonparametric method (Al-Nasser (2005)) and our proposed method. Looking at the slope parameter of the nonparametric method (Al-Nasser (2005)) and our proposed method, we can conclude that both these estimation method were not much affected by the presence of outliers.

5. CONCLUSION

In this paper, we proposed a robust nonparametric method to estimate the slope parameter of the LFRM. From the simulation study, we conclude that all the three methods showed somewhat similar results when no outlier exists in the data. However, as the percentage of outlier increases, the MLE method is shown to break down quickly compared to the nonparametric method (Al-Nasser (2005)) and our proposed method. Comparing the MSE of the nonparametric method (Al-Nasser (2005)) and our proposed method, we can conclude that our proposed method gave a more satisfactory result in the presence of outliers.

Base on both the real data set, we can conclude that when there are no outliers in the data set, all the three methods showed somewhat similar results in terms of slope parameter. However, as the percentage of outlier increases, the MLE method is shown to break down easily. For the nonparametric method (Al-Nasser (2005)) and our proposed method, we can conclude that the slope parameters were not much affected in the presence of outliers.

To summarize, the extension that we have made on Al-Nasser (2005) nonparametric method of estimating the slope parameter could be considered as the best method as it shown to perform well even with the existence of high percentage of outliers.

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Table 1
Values of m and r

Sample Size, n	m	r
20	4	5
50	5	10
100	10	10

Table 2
MSE of the slope: Normal-Case

Contamination (%)	Methods \ Sample Size	20	50	100
No outlier	MLE	1.1825E-04	4.7854E-05	2.4419E-05
	Al-Nasser (2005)	1.5459E-04	5.7457E-05	2.9538E-05
	Proposed	1.5464E-04	5.5672E-05	2.7677E-05
Single outlier	MLE	4.4369E+01	6.4742E-01	8.7298E-02
	Al-Nasser (2005)	2.2430E-04	7.2124E-05	2.1181E-02
	Proposed	2.2419E-04	6.6974E-05	4.0672E-04
10%	MLE	1.5929E+02	1.6038E+02	1.6043E+02
	Al-Nasser (2005)	4.8663E-04	4.7335E-04	4.4865E-04
	Proposed	4.8642E-04	4.4584E-04	4.0672E-04
20%	MLE	3.9998E+01	4.0067E+01	4.0083E+01
	Al-Nasser (2005)	4.3560E-03	3.5335E-03	3.5644E-03
	Proposed	4.3562E-03	3.3497E-03	3.2682E-03
30%	MLE	3.1452E+01	3.1495E+01	3.1498E+01
	Al-Nasser (2005)	2.6180E+00	3.4945E-02	3.6090E-02
	Proposed	2.6179E+00	3.1784E-02	3.1157E-02

Table 3
MSE of the Slope: Right Skewed Case, Beta (2, 9)

Contamination (%)	Methods \ Sample Size	20	50	100
No outlier	MLE	1.5133E-04	6.0594E-05	3.0176E-05
	Al-Nasser (2005)	1.9075E-04	6.8761E-05	3.4546E-05
	Proposed	1.9064E-04	6.6482E-05	3.2149E-05
Single outlier	MLE	4.4529E+01	6.4764E-01	8.7356E-02
	Al-Nasser (2005)	2.7768E-04	8.6845E-05	4.0448E-05
	Proposed	2.7693E-04	7.9988E-05	3.4972E-05
10%	MLE	1.5966E+02	1.6054E+02	1.6055E+02
	Al-Nasser (2005)	6.0344E-04	5.7011E-04	5.3175E-04
	Proposed	6.0091E-04	5.2728E-04	4.7306E-04
20%	MLE	4.0002E+01	4.0076E+01	4.0081E+01
	Al-Nasser (2005)	5.3034E-03	4.3112E-03	4.2902E-03
	Proposed	5.2779E-03	4.0164E-03	3.8901E-03
30%	MLE	3.1461E+01	3.1494E+01	3.1501E+01
	Al-Nasser (2005)	2.6401E+00	4.3176E-02	4.3854E-02
	Proposed	2.6394E+00	3.8207E-02	3.7091E-02

Table 4
MSE of the Slope: Left Skewed Case, Beta (9, 2)

Contamination (%)	Methods \ Sample Size	20	50	100
No outlier	MLE	1.5120E-04	6.0583E-05	3.0172E-05
	Al-Nasser (2005)	1.9106E-04	6.8610E-05	3.4436E-05
	Proposed	1.9089E-04	6.5936E-05	3.1775E-05
Single outlier	MLE	4.4658E+01	6.4804E-01	8.7271E-02
	Al-Nasser (2005)	2.7683E-04	8.6440E-05	4.0811E-05
	Proposed	2.7580E-04	7.9212E-05	3.5052E-05
10%	MLE	1.5982E+02	1.6046E+02	1.6037E+02
	Al-Nasser (2005)	5.9557E-04	5.6534E-04	5.3778E-04
	Proposed	5.9307E-04	5.2404E-04	4.7994E-04
20%	MLE	3.9997E+01	4.0071E+01	4.0081E+01
	Al-Nasser (2005)	5.3072E-03	4.2962E-03	4.3022E-03
	Proposed	5.2842E-03	4.0062E-03	3.9089E-03
30%	MLE	3.1455E+01	3.1498E+01	3.1499E+01
	Al-Nasser (2005)	2.6396E+00	4.2998E-02	4.3809E-02
	Proposed	2.6388E+00	3.8164E-02	3.7132E-02

Table 5
MSE of the Slope: Non-Normal Symmetric Case, Beta (3, 3)

Contamination (%)	Sample Size / Methods	20	50	100
No outlier	MLE	4.1847E-04	1.7195E-04	8.6134E-05
	Al-Nasser (2005)	5.7502E-04	2.2126E-04	1.1804E-04
	Proposed	5.6131E-04	2.0281E-04	1.0113E-04
Single outlier	MLE	4.5718E+01	6.4956E-01	8.7330E-02
	Al-Nasser (2005)	8.6448E-04	2.9515E-04	1.4889E-04
	Proposed	8.1974E-04	2.4402E-04	1.1237E-04
10%	MLE	1.6275E+02	1.6150E+02	1.6107E+02
	Al-Nasser (2005)	1.8795E-03	1.8761E-03	1.8190E-03
	Proposed	1.7693E-03	1.5884E-03	1.4830E-03
20%	MLE	4.0043E+01	4.0100E+01	4.0091E+01
	Al-Nasser (2005)	1.5717E-02	1.3294E-02	1.3519E-02
	Proposed	1.4875E-02	1.1492E-02	1.1323E-02
30%	MLE	3.1459E+01	3.1504E+01	3.1501E+01
	Al-Nasser (2005)	2.7694E+00	1.2712E-01	1.2854E-01
	Proposed	2.7570E+00	9.9070E-02	9.6533E-02

Table 6
The Slope Estimates using Three Different Methods from Goran et al. (1997) data

Contamination (%)	Methods	Slopes
No outlier	MLE	1.0988
	Al-Nasser (2005)	1.0016
	Proposed	1.0268
Single outlier	MLE	3.7150
	Al-Nasser (2005)	1.0093
	Proposed	1.0274
10%	MLE	21.3319
	Al-Nasser (2005)	1.0274
	Proposed	1.0579
20%	MLE	27.9859
	Al-Nasser (2005)	1.0961
	Proposed	1.1056
30%	MLE	49.5205
	Al-Nasser (2005)	1.0806
	Proposed	1.1011

Table 7
The Slope Estimates using Three Different Methods from Pilot Plant Data (n=20)

Contamination (%)	Methods	Slopes
No outlier	MLE	0.3218
	Al-Nasser (2005)	0.3198
	Proposed	0.3214
Single outlier	MLE	0.07160
	Al-Nasser (2005)	0.3231
	Proposed	0.3195
10%	MLE	0.0544
	Al-Nasser (2005)	0.3206
	Proposed	0.3143
20%	MLE	0.0377
	Al-Nasser (2005)	0.2783
	Proposed	0.2876
30%	MLE	0.0266
	Al-Nasser (2005)	0.0734
	Proposed	0.1260