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ON EXPONENTIATED MOMENT EXPONENTIAL DISTRIBUTION

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ABSTRACT

In this paper, an exponentiated moment exponential (EME) distribution is proposed. Various properties of this distribution including conditional-based characterization are drawn. Finally, to illustrate the flexibility and tactability of the proposed distribution against the parent distribution some artificial and real data set are taken. Maximum likelihood estimators are derived and fit it to artificial and real data set.

1. INTRODUCTION

If $H_X(x)$ is a cumulative distribution function (cdf), then $\left[H_X(x)\right]^{\alpha}, \alpha > 0$ is defined as the exponentiated distribution (ED) function, where α is the exponentiated parameter. Gompertz (1825) used the function $F(t) = \left(1 - \rho e^{-t\lambda}\right)^{\alpha}, \alpha > 0$ to compare the human mortality tables and population growth model.

Later Gupta et al. (1998) introduced the distribution by substituting $\rho = 1$ to study its theoretical properties and compared it with the characteristics of gamma and the Weibull distribution (See also Gupta and Kundu, 1999, 2000, 2001, 2003a, 2003b, 2004, 2005).

Raja and Mir (2011) conducted the empirical study of the eight distributions namely gamma, Weibull, lognormal, Gumble, exponentiated Weibull, exponentiated exponential, exponentiated lognormal and exponentiated Gumble distributions using two real life data sets. The first data set is the failure times of the air conditioning system of an airplane, the second data set of the runs scored by a cricketer, under exponentiated lognormal and exponentiated exponential distributions.

If X is a non-negative random variable then the weighted distribution is defined as $\dfrac{w(x,\beta)\, f(x)}{E\left[w(X,\beta)\right]}$ (see Patil, 2000). If $w(X) = X^m$ then it is defined as moment size-biased distributions of order m. Dara and Ahmad (2012) studied various moment distribution, and developed some basic properties like moments, skewness, kurtosis, moment generating function and hazard function. In this paper, we develop a new family of distributions named as exponentiated moment exponential (EME) distribution or exponentiated weighted exponential (EWE) distributions and obtained various properties.

2. EXPONENTIATED MOMENT EXPONENTIAL DISTRIBUTIONS

The cumulative distribution function of two parameter exponentiated moment exponential distribution is as follows

$$G_X(x) = \left[1 - \frac{x+\beta}{\beta} e^{-x/\beta}\right]^{\alpha}, \; x > 0, \; \alpha > 0, \; \beta > 0 . \tag{2.1}$$

where α is the shape parameter and β is the scale parameter and exponentiated moment exponential distribution is denoted as $\text{EMED}(\alpha, \beta)$. The probability density function (pdf) is defined as

$$g(x) = \frac{\alpha}{\beta^2}\left[1 - (1 + x/\beta) e^{-x/\beta}\right]^{\alpha-1} x \, e^{-x/\beta}, \quad x > 0, \; \alpha > 0, \; \beta > 0 \tag{2.2}$$

The graph of g(x) for various values of α and β are

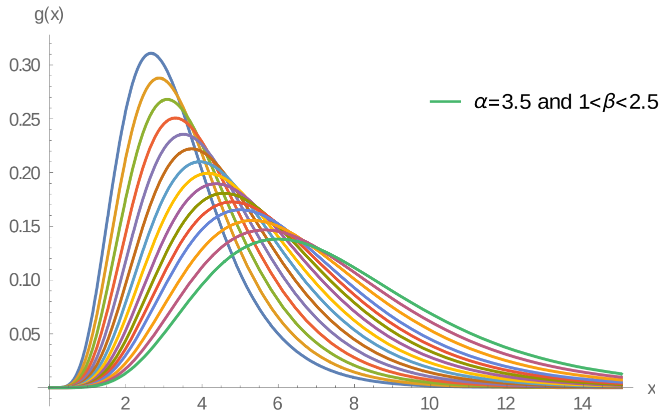

Fig. 1:

Suppose X denotes the EME random variable with parameters α and β, then the r^{th} raw moment of X becomes

$$E\left(X^r\right) = \frac{\alpha}{\beta^2} \int_0^{\infty} \left[1 - (1 + x/\beta) e^{-x/\beta}\right]^{\alpha-1} x^{r+1} e^{-x/\beta} dx ,$$

Let $t = x/\beta$ then

$$E\left(X^r\right) = \alpha\beta^r \int_0^{\infty} \left[1 - (1 + t) e^{-t}\right]^{\alpha-1} t^{r+1} e^{-t} dt ,$$

By using the binomial series expansion, after simplification it reduces to

$$E\left(X^r\right) = \alpha\beta^r \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} (-1)^i \binom{\alpha-1}{\mathrm{i}} \binom{i}{j} \frac{\Gamma(j + r + 2)}{(1+i)^{j+r+2}} . \tag{2.3}$$

The series is convergent for $r \geq 0$, and for all values of α.

2.1 Factorial Moments

The factorial moments of EME distribution random variable X are as follows

$$E\left(X\left(X-1\right)\left(X-2\right)...\left(X-r+1\right)\right)=\sum_{k=0}^{r}S\left(r,k\right)E\left(X^{k}\right) \text{ for } r\in Z^{+}$$

where $S\left(r,k\right)$ is the Stirling number of first kind and $E\left(X^{k}\right)$ is defined at (2.3).

2.2 Negative Moments of EME Distribution

Let X be an exponentiated moment exponential (EME) random variable with parameters α and β, then its negative r^{th} moment is

$$E\left(X^{-r}\right)=\frac{\alpha}{\beta^{2}}\int_{0}^{\infty}\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]^{\alpha-1}x^{-r+1}e^{-x/\beta}dx\,,$$

$$=\alpha\beta^{-r}\sum_{i}^{\infty}\sum_{j=r-2}^{i}\left(-1\right)^{i}\binom{\alpha-1}{\mathrm{i}}\binom{i}{j}\frac{\Gamma\left(j-r+2\right)}{\left(1+i\right)^{j-r+2}}. \tag{2.4}$$

2.3 Maximum Likelihood Estimator of EME Distribution

Let $X_1,\ \mathrm{X}_2,\ldots,X_n$ be a random sample from the pdf, then the likelihood function is

$$L\left(\alpha,\beta;x_1,x_2,\ldots,x_n\right)=\frac{\alpha^{n}}{\beta^{2n}}\prod_{j=1}^{n}\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]^{\alpha-1}xe^{-x/\beta}$$

or

$$lnL\left(\alpha,\beta;x_1,x_2,...,x_n\right)=n\ln\alpha-2n\ln\beta+\sum\ln x-\frac{\Sigma x}{\beta}+(\alpha-1)\Sigma\ln\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right] \tag{2.5}$$

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}+\sum\ln\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right] \tag{2.6}$$

Equating (2.6) to zero we have the following MLE α becomes

$$\hat{\alpha}=\frac{-n}{\Sigma\ln\left[1-\left(1+x/\hat{\beta}\right)e^{-x/\hat{\beta}}\right]}.$$

Again differentiate (2.6) w.r.t α

$$\frac{\partial^{2}\ln L}{\partial\alpha^{2}}=\frac{-n}{\alpha^{2}}. \tag{2.7}$$

Then $Var(\hat{\alpha})=\alpha^{2}/n.$

Now differentiate equation (2.5) w.r.t. β, we obtain

$$\frac{\partial \ln L}{\partial \beta} = \frac{-2n}{\beta} + \frac{n\overline{x}}{\beta^2} + \frac{(\alpha-1)}{\beta^3}\sum_x \frac{-x^2 e^{-x/\beta}}{\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]}. \tag{2.8}$$

Again differentiating equation (2.8) w.r.t. β

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{2n}{\beta^2} - \frac{2n\overline{x}}{\beta^3} - \frac{(\alpha-1)}{\beta^5}\sum_x \frac{-x^2 e^{-x/\beta}\left[\left\{1-\left(1+x/\beta\right)e^{-x/\beta}\right\}\left\{x/\beta-3\right\}+x^2/\beta\right]}{\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]^2} \tag{2.9}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_x \frac{-x^2 e^{-x/\beta}/\beta^2}{\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]} \tag{2.10}$$

$$COV\left(\hat{\alpha},\hat{\beta}\right) = \sum_x \frac{1-\left(1+x/\beta\right)e^{-x/\beta}}{x^2 e^{-x/\beta}/\beta^2}.$$

The *MLE* (Maximum Likelihood Estimate) of $\Theta = (\alpha, \beta)$, say $\hat{\Theta}$, is obtained by solving the nonlinear system. The solution of this nonlinear system of equations does not have a closed form, but can be found numerically by using software such as SAS.

For interval estimation and hypothesis tests on the parameters, we require the 2×2 information matrix containing second partial derivatives of (2.7), (2.9) and (2.10). Under the regularity conditions stated in Cox and Hinkly (1974), that are fulfilled for our model whenever the parameters are in the interior of the parameter space, we have that the asymptotic distribution of $\sqrt{n}\left(\hat{\theta}-\theta\right)$ to be a multivariate normal $N_2\left(0, A^{-1}(\Theta)\right)$, where $A^{-1}(\Theta) = \lim_{n\to\infty} I_{\text{n}}(\Theta)$ is the information matrix.

2.4 Mode of EME Distribution

Let the random variable X has the pdf (2.2). Differentiate equation (2.2) w.r.t. x and equating it to zero, we obtain

$$\frac{\alpha}{\beta^4} e^{-x/\beta}\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]^{\alpha-2}\left[x\left(x\alpha e^{-x/\beta}-\beta\right)+\beta^2\left(1-e^{-x/\beta}\right)\right]=0$$

$$let \quad A = \frac{\alpha}{\beta^4} e^{-x/\beta}\left[1-\left(1+x/\beta\right)e^{-x/\beta}\right]^{\alpha-2}$$

Since A>0, then $\left[x\left(x\alpha e^{-x/\beta}-\beta\right)+\beta^2\left(1-e^{-x/\beta}\right)\right]=0$, provides mode for different values of α and β.

Table 1
Mode of EME Distribution for Different Values of α and β

β	α			
	1	2	3	4
1	1	2	3	4
2	1.94	3.87	5.81	7.75
3	2.46	4.93	7.39	9.86
4	2.83	5.66	8.49	11.32

2.5 Median of EME Distribution

By using the definition of median

$$\left[1-\frac{M+\beta}{\beta}e^{-M/\beta}\right]^{\alpha}=\frac{1}{2}$$

$\left(1+\frac{M}{\beta}\right)-e^{M/\beta}\left(1-2^{-\frac{1}{\alpha}}\right)=0$, will provide median.

This table represents the different values of median at different values of α and β.

Table 2:
Values of Median for Different Values of α and β

β	**Median**				
	α				
	1	2	3	4	5
1	1.68	2.47	2.95	3.30	3.56
2	3.36	4.95	5.91	6.59	7.12
3	5.04	7.42	8.86	9.89	10.69
4	6.71	9.89	11.81	13.18	14.25
5	8.39	12.36	14.76	16.48	17.81

2.6 Hazard Rate Function

The hazard rate function is a significant quantitative exploration of life phenomenon. The hazard rate function measures the conditional instantaneous rate of failure at time x, given survival to time x. In the literature Barlow et al. (1963) introduced hazard rate function, in studying the relationship between properties of a distribution function (or density function) and corresponding hazard rate function. It has significant value in the study of reliability analysis, survival analysis, actuarial sciences and demography, in extreme value theory and in duration analysis in economics and sociology.

For the EME distribution it takes the form

$$h(x)=\frac{g(x)}{1-G(x)}=\frac{\frac{\alpha}{\beta^2}\left[1-(1+x/\beta)\exp\left(-x/\beta\right)\right]^{\alpha-1}xe^{-x/\beta}}{1-\left(1-(1+x/\beta)\exp\left(-x/\beta\right)\right)^{\alpha}}. \qquad (2.11)$$

2.6.1 Graph of EME Hazard Rate

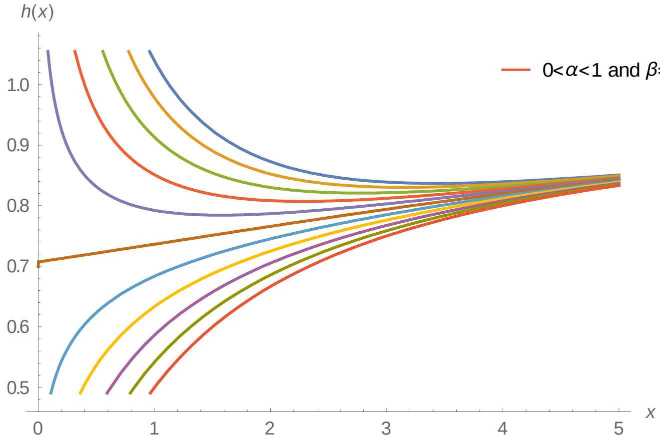

Fig. 2: When α < 1

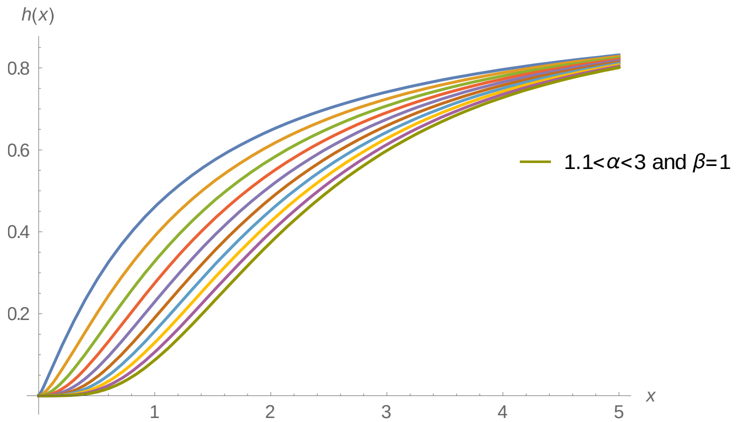

Fig. 3: When α >1

For $\alpha < 0.5$ indicates the hazard rate function $h(x)$ decreases with time at first (Burn-in), then remains constant with respect to time (useful-life). At $\alpha = 0.5$ monotonically increasing and for $\alpha > 0.5$ indicates the hazard rate function $h(x)$ increases with time at first (wear-out), then remains constant with respect to time (useful-life).

For $\alpha \geq 1$ the graphs of different shapes of hazard function show an increasing function of x. Furthermore for $\alpha > 1$ when $x \to 0$, $h(x) \to 0$ and when $x \to \infty$, $h(x) \to 1/\beta$.

2.7 Survival Function

The branch of statistics that deals with the failure in mechanical systems is called survival analysis. In engineering it is called as reliability analysis or reliability theory. In fact the survival function is the probability of failure by time y, where y represents survival time. We use of the survivor function to predict quantiles of the survival time.

By definition of survival function

$$S(x)=1-G\left(x\right)=1-\left[1-(1+x/\beta)e^{-x/\beta}\right]^{\alpha}$$

2.7.1 Graph of Survival Function

The graph of survival function is drawn for different values of α and β.

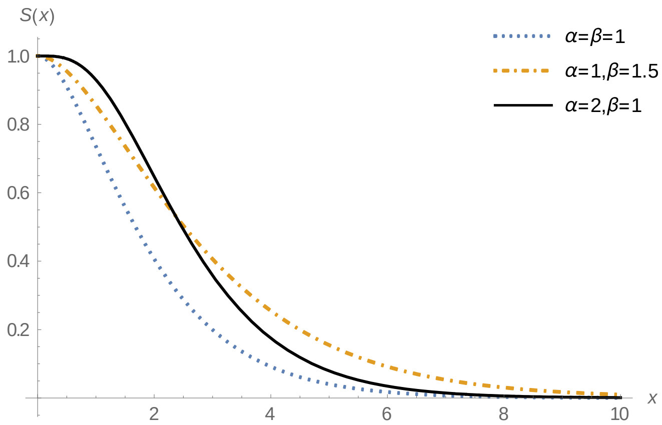

Fig. 4: The Graph of Survival Function

The survivals curves show the decreasing rate.

2.8 Entropy

Entropy is used as a measure of information or uncertainty, which present in a random observation of its actual population. There will be the greater uncertainty in the data if the value of entropy is large. Entropy is an important factor in communication, which limits both data compression and channel capacity. Signal processing techniques and analysis based on Entropy is very successful in a dissimilar set of applications ranging from ecological system monitoring to crystallography. If entropy-based approaches are applied successfully then it is predictable to have analytical expressions for the entropy of a given signal model, especially in the communication area. For some probability distributions expressions the differential entropy is considered mostly effective. An extension of the Shannon (1948) entropy for the true continuous random variable X is defined as

$$\text{Entropy}=-E\left(\ln f(X)\right)=-\int_{0}^{\infty}\ln\left(f(x)\right)f(x)dx$$

is given by using equation (2.2)

$$\text{Entropy} = -\int_0^{\infty}\begin{pmatrix} \ln \alpha/\beta^2 + (\alpha-1)\ln\left[1-(1+x/\beta)e^{-x/\beta}\right] \\ + \ln x - x/\beta \end{pmatrix}\frac{\alpha}{\beta^2}\left[1-(1+x/\beta)e^{-x/\beta}\right]^{\alpha-1} xe^{-x/\beta}dx$$

since theoretical result of entropy is not in a closed form, some of the numerical values of entropy for different parameters are given below

Table 3
Entropy

		α				
		1	2	3	4	5
β	1	1.5637	1.6908	1.7200	1.7322	1.7361
	2	2.2653	2.3841	2.4131	2.4243	2.4293
	3	2.6731	2.7895	2.8186	2.8298	2.8347
	4	2.9179	3.0772	3.1063	3.1175	3.1224
	5	3.1855	3.30036	3.3294	3.3406	3.3455

2.9 Information Function

The information function is defined as (see

$$\text{IF} = \frac{\alpha^s\beta^{1-s}}{s^{s+1}}\int_0^{\infty}\left[1-(1+t/s)\exp(-t/s)\right]^{s(\alpha-1)} t^s e^{-t}dt\ .$$

By using the binomial expansion, after simplification it reduces to

$$IF = \alpha^s\beta^{1-s}\sum_{i=0}^{s(\alpha-1)}\sum_{j=0}^{r}(-1)^i\binom{s(\alpha-1)}{\mathrm{i}}\binom{r}{j}\frac{\Gamma(j+s+1)}{(r+s)^{j+s+1}}\ .$$

2.10 Characterization

An important area of statistical theory is characterization of probability distributions. Different methods are used for the characterizations of continuous distributions. Characterization based on conditional expectations is one of them. Raqab (2002) characterized generalized exponential distribution and some other distributions based on conditional expectations of record values.

Theorem 2.1

Let X be a non-negative random variable having an absolute continuous distribution function $F_X(x)$ with $F_X(0)=0$ and $0 < F_X(x) \le 1$ for all $x > 0$, then its distribution function is

$$F_X(x) = \left[1-(1+x/\beta)\exp(-x/\beta)\right]^{\alpha},\quad x>0,\ \alpha>0,\ \beta>0,$$

if and only if

$$E\left[(1+x/\beta)\exp(-x/\beta)\mid \mathrm{X}\le t\ \right] = \left[1+\alpha(1+t/\beta)\exp(-t/\beta)\right]/(\alpha+1).$$

Proof:

The necessary part follows as

$$E\left[\left(1+x/\beta\right)e^{-\frac{x}{\beta}}\middle|\, \mathrm{X}\le t\right]=\frac{1}{F(t)}\int_0^t\left[\left(1+x/\beta\right)\exp\left(-x/\beta\right)\right]$$

$$\left[1-\left(1+x/\beta\right)\exp\left(-x/\beta\right)\right]^{\alpha-1}\frac{\alpha}{\beta^2}\,x\exp\left(-x/\beta\right)dx$$

By performing integration by parts, we obtain

$$=\frac{1}{F(t)}\left[\left(1+x/\beta\right)\exp\left(-x/\beta\right)\right]\left[1-\left(1+x/\beta\right)\exp\left(-x/\beta\right)\right]^{\alpha}\Big|_0^t$$

$$+\int_0^x\left[1-\left(1+x/\beta\right)\exp\left(-x/\beta\right)\right]^{\alpha}\frac{x}{\beta^2}\exp\left(-x/\beta\right)dx$$

$$=\frac{1}{F(t)}\left[\left(1+t/\beta\right)\exp\left(-t/\beta\right)F(t)\right]+\frac{1}{F(t)}\frac{\left[1-\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]^{\alpha+1}}{\alpha+1}$$

$$=\left(1+t/\beta\right)\exp\left(-t/\beta\right)+\frac{\left[1-\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]}{\alpha+1}$$

$$=\left[1+\alpha\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]\big/(\alpha+1).$$

For sufficiency case

$$E\left[\left(1+X/\beta\right)\exp\left(-X/\beta\right)\mid \mathrm{X}\le t\right]=\left[1+\alpha\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]\big/(\alpha+1).$$

$$\frac{1}{F(t)}\int_0^t\left(1+x/\beta\right)\exp\left(-x/\beta\right)f(x)dx=\left[1+\alpha\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]\big/(\alpha+1)$$

Differentiate both sides w.r.t. t, we have

$$\left(\alpha+1\right)\left[\left(1+t/\beta\right)\exp\left(-t/\beta\right)f(t)\right]$$

$$=f(t)\left[1+\alpha\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]-\alpha\left(t/\beta^2\right)\exp\left(-t/\beta\right)F(t)$$

$$\left(1+t/\beta\right)\exp\left(-t/\beta\right)f(t)=f(t)-\alpha\left(t/\beta^2\right)\exp\left(-t/\beta\right)F(t)$$

after simplification

$$\frac{f(t)}{F(t)}=\frac{\frac{\alpha t}{\beta^2}\exp\left(-t/\beta\right)}{1-\left(1+t/\beta\right)\exp\left(-t/\beta\right)}.$$

Integrating the above function

$$\ln F(t) = \alpha \ln\left[1-\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]$$

which proves that

$$F(t) = \left[1-\left(1+t/\beta\right)\exp\left(-t/\beta\right)\right]^{\alpha}. \quad t>0,\ \alpha>0,\ \beta>0.$$

2.11 Percentiles

Percentage points of this new distribution are computed with pdf given in (2.2). For any $0<p<1$, the 100 pth percentile (also called the quantile of order p) is a number x_p such that the area under the curve of the pdf given in (2.2) to the left of x_p is p. Provided x_p is the root of the equation

$$G\left(x_p\right) = \left(1-\left(1+\frac{x_p}{\beta}\right)e^{-\frac{x_p}{\beta}}\right)^{\alpha-1} = p \tag{2.12}$$

By solving the equation (2.12) numerically, the percentage points x_p are computed for some selected values of the parameters. These are provided in the Tables 3 and 4.

Table 4

Percentage Points for $\alpha=1,\ \beta=1,\ 2,\ 3,\ 4,\ 5$

	75%	**80%**	**85%**	**90%**	**95%**	**99%**
$\beta=1$	2.693	2.994	3.372	3.890	4.744	6.638
$\beta=2$	5.385	5.989	6.745	7.779	9.488	13.277
$\beta=3$	8.078	8.983	10.117	11.669	14.232	19.915
$\beta=4$	10.771	11.977	13.489	15.558	18.975	26.553
$\beta=5$	13.463	14.971	16.862	19.449	23.719	33.192

Table 5

Percentage Points for $\beta=1, \alpha=2,\ 3,\ 4,\ 5$

		75%	**80%**	**85%**	**90%**	**95%**	**99%**
α	2	3.518	3.821	4.199	4.714	5.557	7.427
	3	4.002	4.304	4.679	5.189	6.027	7.885
	4	4.344	4.645	5.018	5.526	6.359	8.208
	5	4.608	4.908	5.280	5.785	6.615	8.457

2.12 Simulation and Application

To check the flexibility of the proposed distribution, we discuss simulation study of proposed distribution. For this purpose we generate artificial population of size 25, 50, 100 and 200. Several measures of goodness of fit test are carried out to check the distribution of artificial data. Parameters estimates are derived using maximum likelihood method in such a way that they maximize the likelihood function of the proposed distribution. The computed values of goodness of fit, parameters estimates and likelihood functions are given below

Table 6:
Parameter Estimates, Likelihood and Goodness of Fit Tests for $n = 25, 50, 100, 200$

		Size			
		25	**50**	**100**	**200**
Parameters Estimates	$\hat{\alpha}$	18.7383	36.0318	20.7172	14.5074
	$\hat{\beta}$	0.4853	0.4023	0.4451	0.4927
Log Likelihood		-24.7435	-39.8371	-91.0975	-205.943
Goodness of Fit Test (P-Value)	**Anderson-Darling**	0.2738	0.3068	0.5130	0.7863
	Cramer-von Mises	0.2421	0.3131	0.5833	0.7342
	Kolmogorov-Smirnov	0.3466	0.1376	0.5948	0.6585
	Kuiper	0.4359	0.0211	0.3002	0.6887
	Pearson χ^2	0.4629	0.6993	0.0648	0.6344
	Watson U^2	0.2811	0.0817	0.2786	0.7681

The plots of empirical data and estimated CDF of proposed distribution for different sample sizes are given below

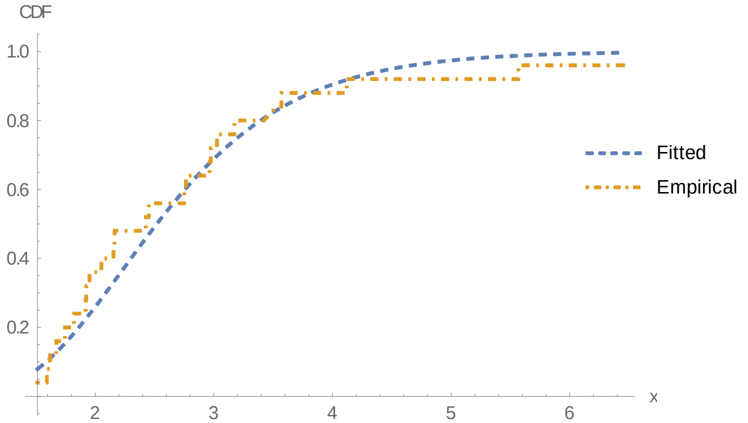

Fig. 5: For $n = 25$

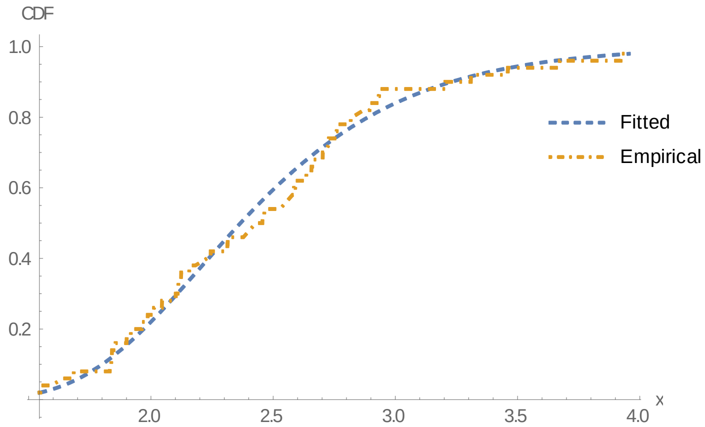

Fig. 6: For $n = 50$

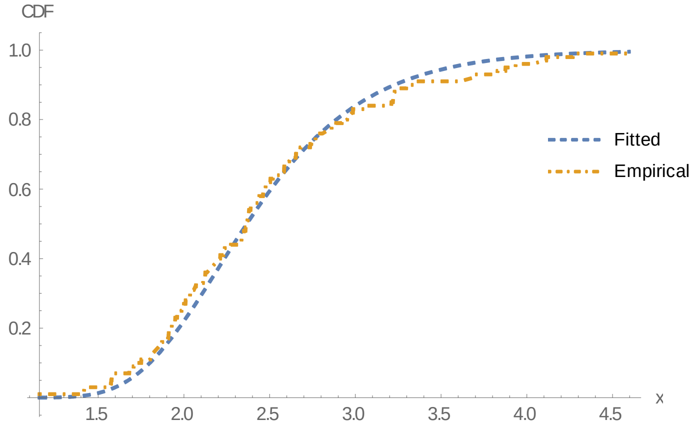

Fig. 7: For $n = 100$

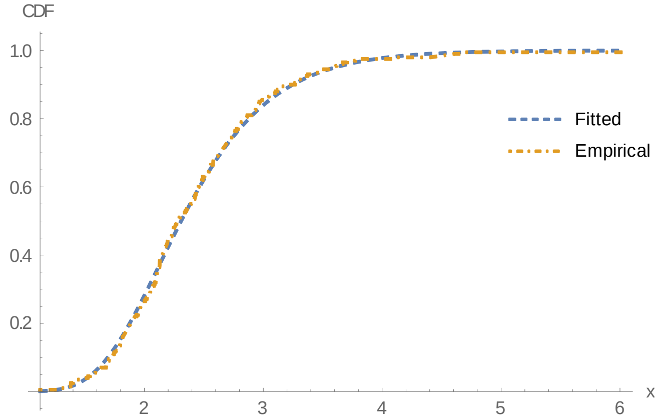

Fig. 8: For $n = 200$

From above figures 5 to 8 and table 6, we can see that as we increases sample sizes the fitted probabilities and goodness of fit test provided us good fits. It is due to flexibility of the proposed distribution.

To show the authenticity of our newly developed EME distribution, we consider the real life data set which already been used by Smith and Naylor (1987). This data represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data is given below

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0,
0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01,
0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24,
0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84,
1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89

Proposed distribution along with moment exponential and exponential distribution have been fitted on real life data set. PROC NLMIXED command is used in SAS for estimating the parameters by employing the method of maximum likelihood estimator. Table 7 showing the estimation of parameters along with MLE's, and likelihood ratio criterion.

Table 7
Maximum Likelihood Estimator and Information Criterion

Model	**Maximum Likelihood Estimates**		**Information Criterion**			
	α	β	**W**	**AIC**	**CAIC**	**BIC**
EME	12.9250(3.6410)	0.3126(0.0258)	60.2	64.2	64.4	68.4
ME	1	0.7534(0.06712)	132.6	134.6	134.7	136.8
E	1	1.5068(0.1898)	177.1	179.7	179.7	181.8

2.13 Concluding Remarks

In this work, we have established a new family of exponentiated moment exponential distribution. We obtained its distribution, density functions with graphs to see how scale and shape parameters influence on its behavior.

We have also outlined some basic important properties of this new family, which has allowed us to outline its complete characterization. For this distribution, we also found its moments, entropy, survival function and hazard rate function, all of which play an important role in the reliability analysis. Numerical study conducted to find the values of mean, median, mode and variance by using Mathematica 10.0. The maximum likelihood estimates of the parameters are discussed. The statistical application of the results to a problem of real as well as artificial data have been provided. It is found that the EME distribution fits better than moment exponential and exponential distribution.

REFERENCES

1. Barlow, R.E., Marshall, A.W. and Proschan, F. (1963) Properties of Probability Distributions with Monotone Hazard Rate. *Ann. Math. Statist.*, 34(2), 375-389.
2. Cox, D.R. and Hinkley, D.V. (1974). *Theoretical Statistics*. Chapman and Hall, London.
3. Dara, S.T. and Ahmad, M. (2012). *Recent Advances in Moment Distributions and their Hazard Rate*. Ph.D. Thesis. National College of Business Administration and Economics, Lahore, Pakistan.
4. Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society London*, 115, 513-585.
5. Gupta, R.C., Gupta, R.D. and Gupta, P.L. (1998). Modeling failure time data by Lehman alternatives. *Commun. in Statist. Theo. and Meth.*, 27(4), 887-904.
6. Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions, *Aust. & N.Z. J. Statist.*, 41(2), 173-188.
7. Gupta, R.D. and Kundu, D. (2000). Generalized exponential distributions; Different Method of Estimations. *J. Statist. Comput. and Simula.*, 69, 315-338.
8. Gupta, R.D. and Kundu, D. (2001). Generalized exponential distributions; An alternative to gamma or Weibull distribution. *Biometrical Journal*, 43, 117-130.
9. Gupta, R.D. and Kundu, D. (2002). Generalized exponential distributions; Statistical Inferences. *J. Statist. Theo. and Applica.*, 1, 101-118.
10. Gupta, R.D. and Kundu, D. (2003a). Closeness between the gamma and generalized exponential distributions. *Commun. in Statist.- Theo. and Meth.*, 32, 705-722.
11. Gupta, R.D. and Kundu, D. (2003b). Discriminating between the Weibull and Generalized exponential distributions. *Computational Statistics and Data Analysis*, 43, 179-196.
12. Gupta, R.D. and Kundu, D. (2004). Discriminating between gamma and the generalized exponential distributions. *J. Statist. Comput. and Simula.*, 74, 107-122.
13. Gupta, R.D. and Kundu, D. (2005). Estimation of $P(Y < X)$ for generalized exponential distribution. *Metrika*, 61, 291-308.
14. Patil, G.P. (2000). Weighted Distributions. *Encyclopedia of Econometrics*, 4, (eds. A.H. El-Shaarawi and W.W. Piegorsch), John Wiley & Sons, Chichester, 2369-2377.
15. Raja, T.A. and Mir, A.H. (2011). On Extension of Some Exponentiated Distributions with Application. *Int. J. Contemp. Math. Sciences*, 6(8), 393-400.
16. Raqab, M.Z. (2002). Inferences for generalised exponential distribution based on record statistics, *J. Statist. Plann. and Infer.*, 104(2), 339-350.
17. Shannon, C. (1948). A mathematical theory of communication. *Bell System Tech. J.*, 27, 379-423.
18. Smith, R.L. and Naylor, J.C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *Applied Statistics*, 36, 358-369.