ON EXPONENTIATED MOMENT EXPONENTIAL DISTRIBUTION

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ABSTRACT

In this paper, an exponentiated moment exponential (EME) distribution is proposed. Various properties of this distribution including conditional-based characterization are drawn. Finally, to illustrate the flexibility and tactability of the proposed distribution against the parent distribution some artificial and real data set are taken. Maximum likelihood estimators are derived and fit it to artificial and real data set.

1. INTRODUCTION

If $H_X(x)$ is a cumulative distribution function (cdf), then $[H_X(x)]^\alpha$, $\alpha > 0$ is defined as the exponentiated distribution (ED) function, where $\alpha$ is the exponentiated parameter. Gompertz (1825) used the function $F(t) = (1 - \rho e^{-\rho t})^\alpha$, $\alpha > 0$ to compare the human mortality tables and population growth model.

Later Gupta et al. (1998) introduced the distribution by substituting $\rho = 1$ to study its theoretical properties and compared it with the characteristics of gamma and the Weibull distribution (See also Gupta and Kundu, 1999, 2000, 2001, 2003a, 2003b, 2004, 2005).

Raja and Mir (2011) conducted the empirical study of the eight distributions namely gamma, Weibull, lognormal, Gumble, exponentiated Weibull, exponentiated exponential, exponentiated lognormal and exponentiated Gumble distributions using two real life data sets. The first data set is the failure times of the air conditioning system of an airplane, the second data set of the runs scored by a cricketer, under exponentiated lognormal and exponentiated exponential distributions.

If $X$ is a non-negative random variable then the weighted distribution is defined as $w(x, \beta)f(x)$ (see Patil, 2000). If $w(X) = X^m$ then it is defined as moment size-biased distributions of order $m$. Dara and Ahmad (2012) studied various moment distribution, and developed some basic properties like moments, skewness, kurtosis, moment generating function and hazard function. In this paper, we develop a new family of distributions named as exponentiated moment exponential (EME) distribution or exponentiated weighted exponential (EWE) distributions and obtained various properties.
2. EXPONENTIATED MOMENT EXPONENTIAL DISTRIBUTIONS

The cumulative distribution function of two parameter exponentiated moment exponential distribution is as follows

\[ G_X(x) = \left[ 1 - \frac{x + \frac{\beta}{\alpha} e^{-x/\beta}}{\beta} \right]^\alpha, \quad x > 0, \quad \alpha > 0, \quad \beta > 0. \tag{2.1} \]

where \( \alpha \) is the shape parameter and \( \beta \) is the scale parameter and exponentiated moment exponential distribution is denoted as EMED(\( \alpha, \beta \)). The probability density function (pdf) is defined as

\[ g(x) = \frac{\alpha}{\beta^2} \left[ 1 - \left(1 + \frac{x}{\beta}\right) e^{-x/\beta} \right]^{\alpha-1} x^{-\frac{\alpha}{\beta}} e^{-x/\beta}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0 \tag{2.2} \]

The graph of \( g(x) \) for various values of \( \alpha \) and \( \beta \) are

![Graph of g(x) for various values of alpha and beta](image)

Suppose \( X \) denotes the EME random variable with parameters \( \alpha \) and \( \beta \), then the \( r^{th} \) raw moment of \( X \) becomes

\[ E\left(X^r\right) = \frac{\alpha}{\beta^2} \int_0^x \left[1 - \left(1 + \frac{x}{\beta}\right) e^{-x/\beta} \right]^{\alpha-1} x^r e^{-x/\beta} dx, \]

Let \( t = \frac{x}{\beta} \) then

\[ E\left(X^r\right) = \alpha \beta^r \int_0^\infty \left[1 - (1 + t) e^{-t} \right]^{\alpha-1} t^{r+1} e^{-t} dt, \]

By using the binomial series expansion, after simplification it reduces to

\[ E\left(X^r\right) = \alpha \beta^r \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} \left(-1\right)^i \binom{i}{j} \frac{\Gamma\left(j + r + 2\right)}{\Gamma\left(1 + i\right)\Gamma\left(j + i + r + 2\right)}. \tag{2.3} \]

The series is convergent for \( r \geq 0 \), and for all values of \( \alpha \).
2.1 Factorial Moments

The factorial moments of EME distribution random variable $X$ are as follows

$$E(X(X-1)(X-2)...(X-r+1)) = \sum_{k=0}^{r} S(r,k) E(X^k) \text{ for } r \in \mathbb{Z}^+$$

where $S(r,k)$ is the Stirling number of first kind and $E(X^k)$ is defined at (2.3).

2.2 Negative Moments of EME Distribution

Let $X$ be an exponentiated moment exponential (EME) random variable with parameters $\alpha$ and $\beta$, then its negative $r^{th}$ moment is

$$E(X^{-r}) = \frac{\alpha}{\beta^2} \int_0^{\infty} \left[1-(1+x/\beta)e^{-x/\beta}\right]^{\alpha-1} x^{-r+1} e^{-x/\beta} dx,$$

$$= \alpha \beta^{-r} \sum_{i=2}^{\infty} \sum_{j=r-2}^{i} (-1)^{i} \binom{i-1}{i} \binom{i}{j} \frac{\Gamma(j-r+2)}{(1+i)^{j-r+2}}.$$  \hspace{1cm} (2.4)

2.3 Maximum Likelihood Estimator of EME Distribution

Let $X_1, X_2, ..., X_n$ be a random sample from the pdf, then the likelihood function is

$$L(\alpha, \beta; x_1, x_2, ..., x_n) = \frac{\alpha^n}{\beta^{2n}} \prod_{j=1}^{n} \left[1-(1+x/\beta)e^{-x/\beta}\right]^{\alpha-1} x e^{-x/\beta}$$

or

$$lnL(\alpha, \beta; x_1, x_2, ..., x_n) = n \ln \alpha - 2n \ln \beta + \sum x - \frac{\sum x}{\beta} + (\alpha-1) \sum \ln \left[1-(1+x/\beta)e^{-x/\beta}\right]$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum \ln \left[1-(1+x/\beta)e^{-x/\beta}\right]$$ \hspace{1cm} (2.5)

Equating (2.6) to zero we have the following MLE $\alpha$ becomes

$$\hat{\alpha} = \frac{-n}{\sum \ln \left[1-(1+x/\hat{\beta})e^{-x/\hat{\beta}}\right]}.$$

Again differentiate (2.6) w.r.t $\alpha$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2}. \hspace{1cm} (2.7)$$

Then $Var(\hat{\alpha}) = \frac{\alpha^2}{n}$.

Now differentiate equation (2.5) w.r.t $\beta$, we obtain
\[
\frac{\partial \ln L}{\partial \beta} = \frac{-2n}{\beta} + \frac{n \bar{x}}{\beta^2} + \frac{(\alpha - 1)}{\beta^3} \sum_x -x^2 e^{-x/\beta} \left[ 1 - (1 + x/\beta)e^{-x/\beta} \right]. 
\] (2.8)

Again differentiating equation (2.8) w.r.t. \( \beta \)

\[
\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{2n}{\beta^2} - \frac{2n \bar{x}}{\beta^3} - \frac{(\alpha - 1)}{\beta^4} \sum_x -x^2 e^{-x/\beta} \left[ \left\{ 1 - (1 + x/\beta)e^{-x/\beta} \right\} \left\{ x/\beta - 3 \right\} + x^2/\beta \right] \left[ 1 - (1 + x/\beta)e^{-x/\beta} \right]^2
\] (2.9)

\[
\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_x -x^2 e^{-x/\beta}/\beta^2 \left[ 1 - (1 + x/\beta)e^{-x/\beta} \right]
\] (2.10)

\[
\text{COV}(\hat{\alpha}, \hat{\beta}) = \sum_x \frac{1 - (1 + x/\beta)e^{-x/\beta}}{x^2 e^{-x/\beta}/\beta^2}.
\]

The MLE (Maximum Likelihood Estimate) of \( \Theta = (\alpha, \beta) \), say \( \hat{\Theta} \), is obtained by solving the nonlinear system. The solution of this nonlinear system of equations does not have a closed form, but can be found numerically by using software such as SAS.

For interval estimation and hypothesis tests on the parameters, we require the 2×2 information matrix containing second partial derivatives of (2.7), (2.9) and (2.10). Under the regularity conditions stated in Cox and Hinkly (1974), that are fulfilled for our model whenever the parameters are in the interior of the parameter space, we have that the asymptotic distribution of \( \sqrt{n}(\hat{\Theta} - \Theta) \) to be a multivariate normal \( \mathcal{N}_2\left(0, A^{-1}(\Theta)\right) \), where \( A^{-1}(\Theta) = \lim_{n \to \infty} I_n(\Theta) \) is the information matrix.

2.4 Mode of EME Distribution

Let the random variable \( X \) has the pdf (2.2). Differentiate equation (2.2) w.r.t. \( x \) and equating it to zero, we obtain

\[
\frac{\alpha}{\beta^4} e^{-x/\beta} \left[ 1 - (1 + x/\beta)e^{-x/\beta} \right]^{\alpha - 2} \left[ -x \left( x\alpha e^{-x/\beta} - \beta \right) + \beta^2 \left( 1 - e^{-x/\beta} \right) \right] = 0
\]

\[
\text{let } A = \frac{\alpha}{\beta^4} e^{-x/\beta} \left[ 1 - (1 + x/\beta)e^{-x/\beta} \right]^{\alpha - 2}
\]

Since \( A > 0 \), then \( \left[ -x\left( x\alpha e^{-x/\beta} - \beta \right) + \beta^2 \left( 1 - e^{-x/\beta} \right) \right] = 0 \), provides mode for different values of \( \alpha \) and \( \beta \).
Table 1
Mode of EME Distribution for Different Values of \( \alpha \) and \( \beta \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \alpha )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1.94</td>
<td>3.87</td>
<td>5.81</td>
<td>7.75</td>
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<tr>
<td>3</td>
<td>2.46</td>
<td>4.93</td>
<td>7.39</td>
<td>9.86</td>
<td></td>
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<tr>
<td>4</td>
<td>2.83</td>
<td>5.66</td>
<td>8.49</td>
<td>11.32</td>
<td></td>
</tr>
</tbody>
</table>

2.5 Median of EME Distribution

By using the definition of median

\[
\left[1 - \frac{M + \beta}{\beta} e^{-M/\beta} \right]^{\alpha} = \frac{1}{2}
\]

\[
\left(1 + \frac{M}{\beta}\right) - e^{M/\beta} \left(1 - 2 - \frac{1}{\alpha}\right) = 0, \text{ will provide median.}
\]

This table represents the different values of median at different values of \( \alpha \) and \( \beta \).

Table 2:
Values of Median for Different Values of \( \alpha \) and \( \beta \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \alpha )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
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<td>2.47</td>
<td>2.95</td>
<td>3.30</td>
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<td>2</td>
<td>3.36</td>
<td>4.95</td>
<td>5.91</td>
<td>6.59</td>
<td>7.12</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5.04</td>
<td>7.42</td>
<td>8.86</td>
<td>9.89</td>
<td>10.69</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6.71</td>
<td>9.89</td>
<td>11.81</td>
<td>13.18</td>
<td>14.25</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>8.39</td>
<td>12.36</td>
<td>14.76</td>
<td>16.48</td>
<td>17.81</td>
<td></td>
</tr>
</tbody>
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2.6 Hazard Rate Function

The hazard rate function is a significant quantitative exploration of life phenomenon. The hazard rate function measures the conditional instantaneous rate of failure at time \( x \), given survival to time \( x \). In the literature Barlow et al. (1963) introduced hazard rate function, in studying the relationship between properties of a distribution function (or density function) and corresponding hazard rate function. It has significant value in the study of reliability analysis, survival analysis, actuarial sciences and demography, in extreme value theory and in duration analysis in economics and sociology.

For the EME distribution it takes the form

\[
h(x) = \frac{g(x)}{1-G(x)} = \frac{\alpha}{\beta^2} \left[1 - (1+x/\beta)\exp\left(-x/\beta\right)\right]^{\alpha-1}xe^{-x/\beta} \cdot \left(1 - (1+x/\beta)\exp\left(-x/\beta\right)\right)^{\alpha-1}.
\] (2.11)
2.6.1 Graph of EME Hazard Rate

For $\alpha < 0.5$ indicates the hazard rate function $h(x)$ decreases with time at first (Burn-in), then remains constant with respect to time (useful-life). At $\alpha = 0.5$ monotonically increasing and for $\alpha > 0.5$ indicates the hazard rate function $h(x)$ increases with time at first (wear-out), then remains constant with respect to time (useful-life).

For $\alpha \geq 1$ the graphs of different shapes of hazard function show an increasing function of $x$. Furthermore for $\alpha > 1$ when $x \to 0$, $h(x) \to 0$ and when $x \to \infty$, $h(x) \to 1/\beta$. 
2.7 Survival Function
The branch of statistics that deals with the failure in mechanical systems is called survival analysis. In engineering it is called as reliability analysis or reliability theory. In fact the survival function is the probability of failure by time $y$, where $y$ represents survival time. We use of the survivor function to predict quantiles of the survival time.

By definition of survival function

$$S(x) = 1 - G(x) = 1 - \left[1 - (1 + x/\beta)e^{-x/\beta}\right]^\alpha$$

2.7.1 Graph of Survival Function
The graph of survival function is drawn for different values of $\alpha$ and $\beta$.

![Graph of Survival Function](image_url)

The survivals curves show the decreasing rate.

2.8 Entropy
Entropy is used as a measure of information or uncertainty, which present in a random observation of its actual population. There will be the greater uncertainty in the data if the value of entropy is large. Entropy is an important factor in communication, which limits both data compression and channel capacity. Signal processing techniques and analysis based on Entropy is very successful in a dissimilar set of applications ranging from ecological system monitoring to crystallography. If entropy-based approaches are applied successfully then it is predictable to have analytical expressions for the entropy of a given signal model, especially in the communication area. For some probability distributions expressions the differential entropy is considered mostly effective. An extension of the Shannon (1948) entropy for the true continuous random variable $X$ is defined as

$$\text{Entropy} = -E(\ln f(X)) = -\int_0^\infty \ln (f(x)) f(x) dx$$
is given by using equation (2.2)

\[
\text{Entropy} = -\int_0^\infty \left( \ln \alpha/\beta^2 + (\alpha - 1) \ln \left[ 1 -(1 + x/\beta) e^{-x/\beta} \right] \right) \frac{\alpha}{\beta^2} \left[ 1 -(1 + x/\beta) e^{-x/\beta} \right]^{\alpha-1} x e^{-x/\beta} \, dx
\]

since theoretical result of entropy is not in a closed form, some of the numerical values of entropy for different parameters are given below

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \alpha )</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
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<td>1</td>
<td>1.5637</td>
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<td>1.7200</td>
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<tr>
<td>4</td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>3.1855</td>
<td>3.30036</td>
<td>3.3294</td>
<td>3.3406</td>
<td>3.3455</td>
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</tr>
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</table>

2.9 Information Function

The information function is defined as (see

\[
\text{IF} = \frac{\alpha \beta \beta^{1-s}}{s+1} \int_0^{\infty} \left[ 1 -(1 + t/s) \exp \left( -t/s \right) \right]^{n-s} e^{-t} \, dt.
\]

By using the binomial expansion, after simplification it reduces to

\[
\text{IF} = \alpha \beta \beta^{1-s} \sum_{j=0}^{s} \sum_{i=0}^{r} \left( \begin{array}{c} s \alpha - 1 \\ j \end{array} \right) \left( \begin{array}{c} r \alpha - 1 \\ j \end{array} \right) \left( \begin{array}{c} j + s + 1 \\ r \end{array} \right) \Gamma \left( j + s + 1 \right) / (r+s)^{j+s+1}.
\]

2.10 Characterization

An important area of statistical theory is characterization of probability distributions. Different methods are used for the characterizations of continuous distributions. Characterization based on conditional expectations is one of them. Raqab (2002) characterized generalized exponential distribution and some other distributions based on conditional expectations of record values.

**Theorem 2.1**

Let \( X \) be a non-negative random variable having an absolute continuous distribution function \( F_X(x) \) with \( F_X(0) = 0 \) and \( 0 < F_X(x) \leq 1 \) for all \( x > 0 \), then its distribution function is

\[
F_X(x) = \left[ 1 -(1 + x/\beta) \exp \left( -x/\beta \right) \right]^{\alpha}, \quad x > 0, \ \alpha > 0, \ \beta > 0,
\]

if and only if

\[
E \left[ (1 + x/\beta) \exp \left( -x/\beta \right) | X \leq t \right] = \left[ 1 + \alpha (1 + t/\beta) \exp \left( -t/\beta \right) \right] / (\alpha + 1).
\]
Proof:
The necessary part follows as
\[
E \left[ (1+x/\beta) e^{-\frac{x}{\beta}} \mid X \leq t \right] = \frac{1}{F(t)} \left[ \int (1+x/\beta) \exp(-x/\beta) \right]
\]
\[
\left[ 1-(1+x/\beta) \exp(-x/\beta) \right]^{\alpha-1} \frac{\alpha}{\beta^2} x \exp(-x/\beta) dx
\]
By performing integration by parts, we obtain
\[
= \frac{1}{F(t)} \left[ (1+x/\beta) \exp(-x/\beta) \right] \left[ 1-(1+x/\beta) \exp(-x/\beta) \right]^{\alpha-1}
\]
\[
+ \int_0^x \left[ 1-(1+x/\beta) \exp(-x/\beta) \right]^{\alpha-1} \frac{x}{\beta^2} \exp(-x/\beta) dx
\]
\[
= \frac{1}{F(t)} \left[ (1+t/\beta) \exp(-t/\beta) \right] + \frac{1}{F(t)} \left[ 1-(1+t/\beta) \exp(-t/\beta) \right]^{\alpha+1}
\]
\[
= (1+t/\beta) \exp(-t/\beta) + \frac{1-(1+t/\beta) \exp(-t/\beta)}{\alpha+1}
\]
\[
= \frac{1+\alpha(1+t/\beta) \exp(-t/\beta)}{(\alpha+1)}.
\]
For sufficiency case
\[
E \left[ (1+X/\beta) \exp(-X/\beta) \mid X \leq t \right] = \left[ 1+\alpha(1+t/\beta) \exp(-t/\beta) \right]/(\alpha+1).
\]
\[
\frac{1}{F(t)} \int_0^t (1+x/\beta) \exp(-x/\beta) f(x) dx = \left[ 1+\alpha(1+t/\beta) \exp(-t/\beta) \right]/(\alpha+1)
\]
Differentiate both sides w.r.t. \(t\), we have
\[
(\alpha+1) \left[ (1+t/\beta) \exp(-t/\beta) f(t) \right]
\]
\[
= f(t) \left[ 1+\alpha(1+t/\beta) \exp(-t/\beta) \right] - \alpha \left( t/\beta^2 \right) \exp(-t/\beta) F(t)
\]
\[
(1+t/\beta) \exp(-t/\beta) f(t) = f(t) - \alpha \left( t/\beta^2 \right) \exp(-t/\beta) F(t)
\]
after simplification
\[
\frac{f(t)}{F(t)} = \frac{\alpha t}{\beta^2} \exp(-t/\beta).
\]
Integrating the above function

\[ \ln F(t) = \alpha \ln \left[ 1 - (1 + t/\beta) \exp\left( -t/\beta \right) \right] \]

which proves that

\[ F(t) = \left[ 1 - (1 + t/\beta) \exp\left( -t/\beta \right) \right]^\alpha. \quad t > 0, \alpha > 0, \beta > 0. \]

### 2.11 Percentiles

Percentage points of this new distribution are computed with pdf given in (2.2). For any \( 0 < p < 1 \), the 100 \( p \)th percentile (also called the quantile of order \( p \)) is a number \( x_p \) such that the area under the curve of the pdf given in (2.2) to the left of \( x_p \) is \( p \). Provided \( x_p \) is the root of the equation

\[ G(x_p) = \left( 1 - \left( 1 + \frac{x_p}{\beta} \right) e^{x_p/\beta} \right)^{\alpha-1} = p \]  \hspace{1cm} (2.12)

By solving the equation (2.12) numerically, the percentage points \( x_p \) are computed for some selected values of the parameters. These are provided in the Tables 3 and 4.

### Table 4

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>75%</th>
<th>80%</th>
<th>85%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
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<tr>
<td>( \beta = 1 )</td>
<td>2.693</td>
<td>2.994</td>
<td>3.372</td>
<td>3.890</td>
<td>4.744</td>
<td>6.638</td>
</tr>
<tr>
<td>( \beta = 2 )</td>
<td>5.385</td>
<td>5.989</td>
<td>6.745</td>
<td>7.779</td>
<td>9.488</td>
<td>13.277</td>
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<tr>
<td>( \beta = 3 )</td>
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<td>( \beta = 4 )</td>
<td>10.771</td>
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<td>13.489</td>
<td>15.558</td>
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### Table 5

<table>
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<tr>
<th>( \alpha )</th>
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<th>85%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
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<td>4.908</td>
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### 2.12 Simulation and Application

To check the flexibility of the proposed distribution, we discuss simulation study of proposed distribution. For this purpose we generate artificial population of size 25, 50, 100 and 200. Several measures of goodness of fit test are carried out to check the distribution of artificial data. Parameters estimates are derived using maximum likelihood
method in such a way that they maximize the likelihood function of the proposed distribution. The computed values of goodness of fit, parameters estimates and likelihood functions are given below

| Table 6: Parameter Estimates, Likelihood and Goodness of Fit Tests for $n = 25, 50, 100, 200$ |
|-----------------------------------------------|-----------------|-----------------|-----------------|-----------------|
| Parameters Estimates                          | $\hat{\alpha}$  | $\hat{\beta}$  |
| $n = 25$                                      | 18.7383         | 0.4853          |
| $n = 50$                                      | 36.0318         | 0.4023          |
| $n = 100$                                     | 20.7172         | 0.4451          |
| $n = 200$                                     | 14.5074         | 0.4927          |
| Goodness of Fit Test (P-Value)                | Anderson-Darling | 0.2738          | 0.3068          | 0.5130          | 0.7863          |
|                                              | Cramer-von Mises | 0.2421          | 0.3131          | 0.5833          | 0.7342          |
|                                              | Kolmogorov-Smirnov | 0.3466          | 0.1376          | 0.5948          | 0.6585          |
|                                              | Kuiper           | 0.4359          | 0.0211          | 0.3002          | 0.6887          |
|                                              | Pearson $\chi^2$ | 0.4629          | 0.6993          | 0.0648          | 0.6344          |
|                                              | Watson $U^2$     | 0.2811          | 0.0817          | 0.2786          | 0.7681          |

The plots of empirical data and estimated CDF of proposed distribution for different sample sizes are given below

Fig. 5: For $n = 25$
Fig. 6: For $n = 50$

Fig. 7: For $n = 100$

Fig. 8: For $n = 200$
From above figures 5 to 8 and table 6, we can see that as we increases sample sizes the fitted probabilities and goodness of fit test provided us good fits. It is due to flexibility of the proposed distribution.

To show the authenticity of our newly developed EME distribution, we consider the real life data set which already been used by Smith and Naylor (1987). This data represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data is given below

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89

Proposed distribution along with moment exponential and exponential distribution have been fitted on real life data set. PROC NLMIXED command is used in SAS for estimating the parameters by employing the method of maximum likelihood estimator. Table 7 showing the estimation of parameters along with MLE’s, and likelihood ratio criterion.

<table>
<thead>
<tr>
<th>Model</th>
<th>Maximum Likelihood Estimates</th>
<th>Information Criterion</th>
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<tr>
<td></td>
<td>α</td>
<td>β</td>
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<td>0.3126(0.0258)</td>
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<tr>
<td>ME</td>
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<td>0.7534(0.06712)</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>1.5068(0.1898)</td>
</tr>
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</table>

2.13 Concluding Remarks
In this work, we have established a new family of exponentiated moment exponential distribution. We obtained its distribution, density functions with graphs to see how scale and shape parameters influence on its behavior.

We have also outlined some basic important properties of this new family, which has allowed us to outline its complete characterization. For this distribution, we also found its moments, entropy, survival function and hazard rate function, all of which play an important role in the reliability analysis. Numerical study conducted to find the values of mean, median, mode and variance by using Mathematica 10.0. The maximum likelihood estimates of the parameters are discussed. The statistical application of the results to a problem of real as well as artificial data have been provided. It is found that the EME distribution fits better than moment exponential and exponential distribution.
REFERENCES