DISCRETE INVERSE RAYLEIGH DISTRIBUTION

Tassaddaq Hussain\textsuperscript{1,2} and Munir Ahmad\textsuperscript{2}

\textsuperscript{1} Department of Statistics, Government Post Graduate College, Rawalakot (AJK), Pakistan. Email: taskho2000@yahoo.com

\textsuperscript{2} National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

Rayleigh distribution is one of the well-known continuous distribution developed by Lord Rayleigh and J.W. Strutt (1880, 1919) used in modeling lifetime data. A reciprocal transformation of Rayleigh variable generates inverse Rayleigh distribution derived by Voda (1972) which is also being used in lifetime experiments. While keeping in mind the famous of Rayleigh and inverse Rayleigh distribution, we hereby proposed the discrete version of continuous inverse Rayleigh distribution by adopting the simple approach and presented as an appropriate lifetime model for discrete data. Non-monotonicity of the hazard function of discrete Inverse Rayleigh distribution is studied, suitability of the model in over dispersed data is highlighted with real lifetime data examples, basic mathematical properties, order statistics and characterization issues of the model are also presented.

KEY WORDS

Inverse Rayleigh distribution; reliability parameters; negative moments; discretized version; generating functions.

1. INTRODUCTION

Generally one associates the lifetime of the product with continuous non-negative lifetime distributions however, in some situations lifetime can be best described through non-negative integer valued random variables e.g. life of a switch is measured by the number of strokes, life of equipment is measured by the number of cycles it completes or the number of times it is operated prior to failure, life of a weapon is measured by the number of rounds fired until failure, number of years of a married couple successfully completed. Although continuous lifetime distributions are playing their roles in reliability analysis very well, yet in certain scenarios, when measured data is discrete and realized from continuous set up, researchers are trying to search out a proper alternate. For this purpose they developed discretized version of continuous lifetime distributions. This development is generally based on discrete lifetime phenomena which are expressed through grouping or finite precision measurement of continuous time phenomena. Such discretized versions are too much functional in small set of discrete type data and have applications in reliability theory in situation where clock time is not an appropriate scale for measuring the reliability of the product. The discretization approaches have expanded the scope of reliability modeling and provides methods for approximating integrals.
coming out of the continuous phenomena. The reliability of the discrete or counted data is measured on the bases of success or failure and number of successes or failures are modeled through binomial, negative binomial, geometric and Poisson distributions. Such models yield imprecise results for small samples. However with the initiation of discretization approach, the discrete success-failure data based on small and to some extent large samples can efficiently be modeled through discretized version of continuous lifetime distributions.

The discrete success-failure data is realized from continuous set up in two common situations (i) a product is scrutinized only once a time period i.e. a day, an hour and a month etc. and observation is made on the number of time period successfully completed prior to failure of the product (ii) an equipment operates in cycles and researcher observes the number of cycles successfully completed prior to failure of the device. If the observed data values are very large e.g. thousands of revolutions, cycles, blows etc. then for modeling such a data it is better to use a continuous counterpart. In order to get an appropriate model for the success-failure data various discretizing approaches exist in the literature which are (i) Moment equalization approach (ii) Discrete concentration approach (iii) Failure rate approach (see Roy and Ghosh, 2009) (iv) Discrete differential equation approach (see Sreehari, 2008) (v) Time discretization approach. Due to these approaches discretized distributions are finding their way into survival analysis. In this regard, an initial attempt was made by Nakagawa and Osaki (1975) who discretized the Weibull distribution. Later on, a number of researchers like Stein and Dattero (1984), Khan et al. (1989), Szablowski (2001), Bracquemond and Gaudoin (2003), Roy (2003, 2004), Kemp (1997, 2004, 2006), Inusah and Kozubowski (2006), Kozubowski and Inusah (2006), Krishna and Pundir (2007, 2009), Sreehari (2008), Roy and Ghosh (2009) and Jazi et al. (2010), Gómez-Déniz and Calderín-Ojeda (2011), Chakraborty and Chakravarty (2012), Hussain and Ahmad (2012) and Al-Huniti and Al-Dayian (2012) and Nekoukhou et al. (2012) developed discretized version of continuous lifetime distributions and applied them on discrete sets of data in various disciplines of life like engineering, social sciences, medical sciences, and forestry etc.. Classifications of discrete distribution have been made by number of researchers like Khalique (1989) and Kemp (2004). In order to develop reliability theory in discrete discipline various attempts have been initiated in multiple directions. We hereby made an attempt to develop suitable discrete lifetime model in terms of discrete inverse Rayleigh distribution which is defined and discussed in section two along with failure rate function and related conditions, mathematical properties, order statistics and the link between discrete inverse Rayleigh and continuous distributions like Rayleigh and inverse Rayleigh and in section three the parameter’s estimation and goodness of fit with real data examples are studied.

2. DISCRETE INVERSE RAYLEIGH DISTRIBUTION

As discretization of continuous lifetime distribution is an emerging issue of discrete reliability theory, so various discretization approaches exist in the literature. However, these approaches are used by various researchers under different circumstances. For example while using the discrete concentration approach researchers considered the discrete time space either as \( N = \{0,1,2,3,...\} \) or as \( N = \{1,2,3,...\} \) which was usually based on the support of continuous random variable i.e. if support of continuous random
variable is \([0, \infty)\) or \(x \geq 0\) then the support for discretized random variable will be as 
\(N = \{0, 1, 2, 3, \ldots\}\) with survival function 
\(S(x) = P_r(X \geq x)\) (see Nakagawa and Osaki (1975), Roy (2003, 2004), Krishna and Pundir (2007, 2009), Gómez-Déniz and Calderín-Ojeda (2011), Chakaraborty and Chakravarty (2012), Hussain and Ahmad (2012) and Al-Huniti and Al-Dayian (2012)) and if support of continuous random variable is \((0, \infty)\) or \(x > 0\), then discretized random variable will based on 
\(N = \{1, 2, 3, \ldots\}\) with survival function
\[S_x = \sum_{j \geq x} p_j q^j, p_x = S_x - S_{x+1} = q^x - q^{x+1}, 0 < \exp(-\lambda) = q < 1, x = 0, 1, 2, 3, \ldots\]

Since there is one to one correspondence between survival function of geometric distribution and exponential distribution, so a number of researchers considered the geometric distribution as discrete exponential distribution with lack of memory property. Moreover, if the survival functions of discretized distribution retain the same functional forms as their continuous counterparts then many reliability measures and class properties under series, parallel and coherent structures will remain unchanged (see Roy, 2004). In view of the above characteristics we have adopted this approach for discretizing the inverse Rayleigh distribution. The Inverse Rayleigh distribution is a special case of inverse Weibull distribution i.e. if \(Y \sim W(\theta, \beta)\) then
\[X \left(\frac{1}{Y}\right) \sim IW(\theta, \beta)\] and for \(\beta = 2\) we have \(X \sim IR(\theta)\) with survival function as \(S(x) = P_r(X \geq x) = 1 - \exp\left(-\theta / x^2\right), \theta > 0, x \geq 0\). Although most of the continuous distribution exhibit monotonic failure rate yet the inverse Rayleigh distribution is among the rare distributions which is being effectively used in the area of reliability studies where failure rate exhibits non-monotonic behavior. It is used in lifetime experiments (see Voda, 1972), record values from Inverse Rayleigh distribution are being used for prediction purposes in real life problems like weather, economic and support data (see Soliman et al., 2010) and used in acceptance sampling plans (see Rosaiah and Kantam (2005) and Aslam and Jun, 2009) etc. The important feature of this distribution is that its variance and higher order moments do not exist. However its \(r^\text{th}\) moment, mean, mode and failure rate function are expressed as
\[\mu_r = \theta^{r/2} \Gamma\left(1 - \frac{r}{2}\right), \text{mean} = \sqrt{\pi} \theta, \text{mode} = \sqrt{3/2}\]
and
\[h(x) = \frac{f(x)}{S(x)} = 20x^{-3} \left(\exp\left(\theta / x^2\right) - 1\right)^{-1} \cdot\]
It was first considered by Voda (1972), Mukherjee and Saran (1984) stated that a single parameter inverse Rayleigh distribution possessed increasing failure rate (IFR) for \( x < 1.0694543 \sqrt{\theta} \) and decreasing failure rate (DFR) for \( x > 1.0694543 \sqrt{\theta} \).

### 2.1 Definition

A random variable \( Y \) is said to have discrete inverse Rayleigh distribution with parameter \( 0 < q < 1 \), denoted by \( dIR(q) \), if

\[
P_Y(Y = x) = p_x = S_x - S_{x+1} = q^{\frac{1}{2}(x+1)^2} - q^{\frac{1}{2}x^2}, \quad 0 < q < 1, \quad x = 0, 1, 2, 3, ... \quad (1)
\]

where \( S_x \) is the preserved survival function of inverse Rayleigh distribution at integers expressed as

\[
S_x = P_Y(Y \geq x) = \sum_{j \geq x} p_j = 1 - q^{\frac{1}{2}x^2}, \quad 0 < q < 1, \quad x = 0, 1, 2, 3, ..., \text{ where } q^\infty = 0, S_0 = 1,
\]

\( Y = \lfloor X \rfloor \) denote the observed discrete random variable i.e. \( Y \) is equal to the greatest integer less than or equal to \( X \). If \( Y \) is a random variable denoting the number of times a product fail in any given week/month/year and \( q \) denotes the probability of failure of a product in any given week/month/year than \( P(Y = 0) \) gives the probability of no failure in any given week/month/year.

**Fig. (2.2.1): Discrete Inverse Rayleigh Distribution**

Fig. (2.2.1) shows the probability plots for discrete Inverse Rayleigh distribution for different values of the parameter \( q \), which portrays that as \( q \to 0 \) the mode of the distribution shifted towards the right and as \( q \) increases the mode of the distribution shifted towards the left and distribution shows a reverse J-shaped.
2.2 Hazard Function

Let $Y$ be a discrete random variable with probability mass function $p_X = P(Y = x)$ and reliability function $S_X = P_r(Y \geq x)$ then the failure rate function of $Y$ is defined as the conditional probability that failure is observed at $x$ given that the product has not failed before $x$ and expressed as

$$h_x = \frac{p_X}{S_X} = \frac{S_X - S_{x+1}}{S_X}, \quad x = 0, 1, 2, 3, ...$$

(2)

The hazard function defined in equation (2) has some misconception i.e.

i) It is bounded i.e. $h_x \leq 1$. This may add some confusion in industry that failure rate and failure probability are sometimes mixed up (see Xie et al., 2002).

ii) Suppose that if we have $m$ discrete component connected independently in series then their failure rate is not additive i.e. $h_x = 1 - \prod_{i=1}^{m}(1 - h_{ix}) \neq \sum_{i=1}^{m} h_{ix}$, (see Xie et al., 2002).

iii) The cumulative hazard function $H_x = \sum_{x=1}^{k} h_x \neq -l nS_x$. (see Xie et al., 2002).

Due to these problems an alternative hazard function for the discrete random variable is defined as $h_x^* = l n\left(\frac{S_x}{S_{x+1}}\right)$, which is based on the fact that the hazard function defined in the continuous case and expressed as $h(x) = \frac{p(x)}{S(x)} = -\frac{d(l nS(x))}{dx}$ can be defined into discrete case by replacing $h(x)$ by $h_x^*$ and $-\frac{d(l nS(x))}{dx}$ by $-(l nS_{x+1} - l nS_x)$ so

$$h_x^* = l n\left(\frac{S_x}{S_{x+1}}\right), \quad x = 0, 1, 2, 3,... \text{ which is not bounded like } h(x), \text{ and shows the same monotonicity as shown by } h_x, H_x^* = -l nS_x \text{ and additive in series system (see Xie et al., 2002). Roy and Gupta (1999) named this failure rate function as second failure rate function. Now by using the above definitions the failure rate and second failure rate functions for discrete inverse Rayleigh distribution are defined as}$$

$$h_x = \frac{p_X}{S_X} = \left[q^{(x+1)^{-2}} - q^{(x)^{-2}}\right]\left[1 - q^{(x)^{-2}}\right]^{-1}, \quad 0 < q < 1, \quad x = 0, 1, 2, ...$$
Discrete Inverse Rayleigh Distribution

Fig. (2.2.2): Failure Rate of Discrete Inverse Rayleigh Distribution

\[ h_x^* = \ln \left( \frac{S_x}{S_{x+1}} \right) = \ln \left( \frac{1-q^{x+1}}{1-q^{x}} \right), \quad 0 < q < 1, \ x = 0, 1, 2, \ldots. \]

Fig. (2.2.3): Second Failure rate of discrete inverse Rayleigh distribution
Mukherjee and Saran (1984) stated that a single parameter inverse Rayleigh distribution possessed increasing failure rate (IFR) for \( x < 1.0694543 \sqrt{\theta} \) and decreasing failure rate (DFR) for \( x > 1.0694543 \sqrt{\theta} \). The same is true with the discrete inverse Rayleigh distribution which has discrete increasing failure rate (dIFR) when \( x < 1.0694543 \sqrt{-\ln q} \) and discrete decreasing failure rate when (dDFR) \( x > 1.0694543 \sqrt{-\ln q} \) at integers. However certain features of this non-monotonic behavior of hazard function are also explored like

i) The hazard function of discrete Inverse Rayleigh distribution is an upside down bath tub function of \( x \) with change points either at \( x=1 \) or at \( x=2 \) for \( 0 < q < 0.75 \). However as \( q \to 0 \) the change point appears at \( x=2 \) see Fig. (2.2.2 and 2.2.3).

ii) The hazard function of discrete Inverse Rayleigh distribution is an upside down bath tub function of \( x \) with constant failure rate at \( x=1 \) and \( x=2 \) for \( 0.0636 < q < 0.0673 \) see Fig. (2.2.2 and 2.2.3).

iii) The hazard function of discrete Inverse Rayleigh distribution is a decreasing function of \( x \) for \( 0.71 < q < 1.00 \) see Fig. (2.2.2 and 2.2.3).

**Theorem 2.1.1:**

Let \( Y = \lfloor X \rfloor \) be an integer valued random variable which follows the discrete Inverse Rayleigh distribution with parameter \( q \) i.e. \( Y \sim dIR(q) \). Then expectation for \( Y = \varphi(x) \) is expressed as

\[
E(\varphi(x)) = \sum_{x=1}^{\infty} \{ \varphi(x) - \varphi(x-1) \} \left( 1 - q^{-x^2} \right) + \varphi(0),
\]

where \( P_r(Y \geq x) = 1 - q^{-x^2} \) and \( \exp(-\theta) = q, \ 0 < q < 1, \ x = 0,1,2,3,.... \)

**Proof:**

We have by definition \( E(\varphi(x)) = \sum_{x=0}^{\infty} \varphi(x)P(Y = x) \)

Consider \( \{ \varphi(x) - \varphi(x-1) \} P_r(Y \geq x) = \varphi(x)P_r(Y \geq x) - \varphi(x-1)P_r(Y \geq x) \),

Taking summation over \( x \) from 1 to \( \infty \), we get

\[
\sum_{x=1}^{\infty} \{ \varphi(x)P_r(Y \geq x) - \varphi(x-1)P_r(Y \geq x) \} = \varphi(1)P_r(Y \geq 1) - \varphi(0)P_r(Y \geq 1) + \varphi(2)P_r(Y \geq 2) - \varphi(1)P_r(Y \geq 2) + \varphi(3)P_r(Y \geq 3) - \varphi(2)P_r(Y \geq 3) + ... ...
\]

\[
\sum_{x=0}^{\infty} \varphi(x)P(Y = x) = \sum_{x=1}^{\infty} \{ \varphi(x)P_r(Y \geq x) - \varphi(x-1)P_r(Y \geq x) \} + \varphi(0)
\]

where \( P_r(Y \geq 0) = 1 \).
\[ E(\varphi(x)) = \sum_{i=1}^{\infty} \{\varphi(x) - \varphi(x-1)\} P_r(Y \geq x) + 1 \Rightarrow E(\varphi(x)) \]

\[ = \sum_{i=1}^{\infty} \{\varphi(x) - \varphi(x-1)\} \left(1-q^{x^2}\right) + \varphi(0). \]

where \( P_r(Y \geq x) = 1 - q^{x^2}, x = 0,1,2,3,..... \)

The series converges i.e. as \( x \to \infty \) the tail probabilities approaches zero. This completes the proof.

**Corollary:**

If \( \varphi(x) = t^x \) then resulting expression is probability generating function of discrete inverse Rayleigh distribution.

**Corollary:**

If \( \varphi(x) = e^{tx} \) then resulting expression is the moment generating function of discrete inverse Rayleigh distribution.

**Corollary:**

If \( \varphi(x) = x^r \) then resulting expression is the \( r^{th} \) moment about origin of discrete inverse Rayleigh distribution.

Its mean and variance are

\[ \mu'_1 = \sum_{x=1}^{\infty} \left(1-q^{x^2}\right) \quad \text{and} \quad \text{Var}(Y) = \frac{1}{4} + 2 \sum_{x=1}^{\infty} x \left(1-q^{x^2}\right) - \frac{(1+2\mu'_1)^2}{4}, \]

**Corollary:**

If \( \varphi(x) = (x+a)^{-1} \) then resulting expression is the first order negative moment of discrete inverse Rayleigh distribution.

**Corollary:**

If \( \varphi(x) = (x+a)^{-s} \) then resulting expression is the \( s^{th} \) order negative moment of discrete inverse Rayleigh distribution.

**Corollary:**

If \( \varphi(x) = \frac{1}{(x+a)^s} \) then resulting expression is the \( s^{th} \) order negative factorial moment of discrete inverse Rayleigh distribution, where

\[ (x+a)^s = (x+a)(x+a+1)\ldots(x+a+s-1), a > 0, 0 < q < 1 \quad \text{and} \quad \exp(-\theta) = q. \]
In order to check the suitability of distribution for specific type of data we define immediately the index of dispersion and showed mean and variance of the distribution in table-1.

### 2.3 Index of Dispersion

Index of dispersion (ID) for any distribution is defined as the ratio between variance to mean which indicate whether the distribution is suitable for over or under dispersed data. If \( ID > 1 \) the distribution is over-dispersed (under-dispersed) (see Chakraborty and Chakravarty, 2012). Table-I portrays the dispersion pattern of discrete Inverse Rayleigh distribution in which upper values indicate mean and lower values indicate variance of the distribution against particular value of the parameter \( q \). It is observed that discrete Inverse Rayleigh distribution is over dispersed for all values of the parameter \( q \). Moreover it can also be seen that as mean and variance decreases the value of the parameter increases and vice versa.

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**Theorem 2.1.2:**

Let \( Y_1 \leq Y_2 \leq \ldots \leq Y_n \) denote an order sample of size \( n \) drawn identically and independently from the discrete inverse Rayleigh distribution whose distribution function can also be written as \( F_{x+1} = \sum_{j=0}^{x} p_j = q^{(x-1)^2} \), \( S_x = 1 - F_{x+1} \), \( 0 < q < 1 \), then the probability function of \( i^{th} \) order statistics is...
\[ P_r \left( Y_{(i)} = x \right) = K_i \left\{ q^{ix^2} \binom{n}{i} F_1 \left( -n+i,i;i+1;q^{x^2} \right) - q^{(i-1)x^2} \binom{n}{i} F_1 \left( -n+i,i;i+1;q^{(x-1)^2} \right) \right\}, \]

the recurrence relation between \( i^{th} \) order statistics’s probabilities are

\[ (i+1) P_r \left( X_{(i+1)} = x \right) = i P_r \left( X_{(i)} = x \right) - \binom{n}{i} \left\{ q^{ix^2} \left( 1-q^{x^2} \right)^{n-i} - q^{(i-1)x^2} \left( 1-q^{(x-1)^2} \right)^{n-i} \right\}, \]

and

\[ (i+1) P_r \left( X_{(i+1)} = x \right) = i P_r \left( X_{(i)} = x \right) - \binom{n}{i} \left\{ F_x \left( S_{x+1} \right)^{n-i} - F_{x-1} \left( S_x \right)^{n-i} \right\}, \]

where

\[ K_i = \frac{1}{i} \binom{n}{i}, \quad K_{i+1} = \frac{1}{i+1} \binom{n}{i+1} \quad \text{and} \quad F_2(\alpha_1, \alpha_2; \beta_1; z) = \sum_{n=0}^{\infty} \frac{\left( \alpha_1 \right)_n (\alpha_2)_n z^n}{(\beta_1)_n n!}. \]

Proof:

By definition the probability function of \( i^{th} \) order statistics is

\[ P_r \left( Y_{(i)} = x \right) = P_r \left( Y_{(i)} \leq x \right) - P_r \left( Y_{(i)} \leq x-1 \right), \]

\[ P_r \left( Y_{(i)} \leq x \right) = P_r \left( \text{at least } i \text{ of } Y's \text{ are } \leq x \right), \]

\[ P_r \left( X_{(i)} \leq x \right) = \sum_{j=i}^{n} \binom{n}{j} P_r \left( X_j \leq x \right) \left( 1 - P_r \left( X_j \leq x \right) \right)^{n-j}, \]

(since \( X_1, X_2, ..., X_n \) are i.i.d)

where

\[ \sum_{j=i}^{n} \binom{n}{j} (F_x)^j (1-F_x)^{n-j} = \int_0^F \frac{1}{B(i,n-i+1)} u^{i-1} (1-u)^{n-i} \, du = I_{(F_x)}(i,n-i+1), \]

\( I_{(F_x)}(i,n-i+1) \) is the incomplete beta function.

Therefore

\[ P_r \left( X_{(i)} = x \right) = \int_0^F \frac{1}{B(i,n-i+1)} u^{i-1} (1-u)^{n-i} \, du - \int_0^{F-x^2} \frac{1}{B(i,n-i+1)} u^{i-1} (1-u)^{n-i} \, du, \]

\[ P_r \left( X_{(i)} = x \right) = I_{\left( q^{x^2}, q^{(x-1)^2} \right)}(i,n-i+1), \] (where \( F_x = q^{x^2} \))

\[ P_r \left( X_{(i)} = x \right) = \frac{n!}{i!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \left( -1 \right)^j \binom{\binom{j+1}{2} - \binom{j}{2}}{i+j} q^{(i+j)(x-1)^2} - q^{(i+j)(x-1)^2}, \]

(6)
since
\[ \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left( \frac{x^2}{i} \right)^{(i+j)} = q^{ix^2}_i 2 F_1 \left( -n+i,i;i+1; q^{x^2} \right), \]
and
\[ \sum_{j=0}^{n-i} \binom{n-i}{j} \left( \frac{x^{(x-1)^2}}{i} \right)^{(i+j)} = q^{i(x-1)^2}_i 2 F_1 \left( -n+i,i;i+1; q^{(x-1)^2} \right), \]
on substituting the values of above expression in (6) we get (3). From equation (3) we have the probability function of \( i^{th} \) and \((i+1)^{th}\) order statistics as
\[ P_r \left( Y_i = x \right) = K_i \left\{ q^{ix^2}_i 2 F_1 \left( -n+i,i;i+1; q^{x^2} \right) - q^{i(x-1)^2}_i 2 F_1 \left( -n+i,i;i+1; q^{(x-1)^2} \right) \right\}, \]
\[ P_r \left( X_{(i+1)} = x \right) = K_{i+1} q^{ix^2}_i q^{x^2}_i 2 F_1 \left( -n+i+1,i+1;i+2; q^{x^2} \right) - K_{i+1} q^{i(x-1)^2}_i q^{(x-1)^2}_i 2 F_1 \left( -n+i+1,i+1;i+2; q^{(x-1)^2} \right), \]
\[ P_r \left( X_{(i+1)} = x \right) = K_{i+1} (A_2 - B_2), \]
where
\[ A_2 = q^{x^2}_i 2 F_1 \left( -n+i+1,i+1;i+2; q^{x^2} \right), B_2 = q^{(x-1)^2}_i 2 F_1 \left( -n+i+1,i+1;i+2; q^{(x-1)^2} \right). \]
Using the Gauss’ recurrence relation for \( A_i \) and \( B_i \) (see Gradshteyn and Ryzhik, 1965).
\[ c_2 F_1 (a,b;c;z) - c_2 F_1 (a,b+1;c;z) + a z_2 F_1 (a+1,b+1;c+1;z) = 0, \]
Let \( a = -n+i, b = i, c = i+1 \) and \( z = q^{(x-1)^2} \) then
\[ (i+1) 2 F_1 \left( -n+i,i;i+1; q^{(x-1)^2} \right) - (i+1) 2 F_1 \left( -n+i,i+1;i+1; q^{(x-1)^2} \right) \]
\[ + (-n+i) q^{(x-1)^2}_i 2 F_1 \left( -n+i+1,i+1;i+2; q^{(x-1)^2} \right) = 0, \]
for \( A_i q^{x^2}_i 2 F_1 \left( -n+i+1,i+1;i+2; q^{x^2} \right) \)
\[ = \frac{(i+1)}{n-i} 2 F_1 \left( -n+i,i;i+1; q^{x^2} \right) - \frac{(i+1)}{n-i} \left( 1 - q^{x^2} \right)^{n-i} , \]
for $B_2$ 

$$q^{(x-1)^2} F_{1} \left( -n + i, i + 1; i + 2; q^{(x-1)^2} \right)$$

$$= \frac{(i + 1)}{n - i} \left[ \frac{(i + 1)}{n - i} \right] F_{1} \left( -n + i, i; i + 1; q^{(x-1)^2} \right) - \frac{(i + 1)}{n - i} \left( 1 - q^{(x-1)^2} \right)^{-i},$$

On substituting above expression for $A_2$ and $B_2$ into equation (7) we get

$$P_{i+1} \left( X_{(i+1)} = x \right) = \frac{1}{i+1} \left[ \frac{n}{i + 1} q^{ix} \left( -n + i, i; i + 1; q^{x^2} \right) \right]$$

$$- q^{i(x-1)^2} F_{1} \left( -n + i, i; i + 1; q^{(x-1)^2} \right)$$

$$- \frac{(i + 1)}{n - i} \left[ q^{ix} \left( 1 - q^{x^2} \right)^{-i} - q^{i(x-1)^2} \left( 1 - q^{(x-1)^2} \right)^{-i} \right],$$

$$P_{i+1} \left( X_{(i+1)} = x \right) = \frac{i}{i + 1} \left[ \frac{1}{i} \left( n \right) q^{ix} \left( -n + i, i; i + 1; q^{x^2} \right) \right]$$

$$- q^{i(x-1)^2} F_{1} \left( -n + i, i; i + 1; q^{(x-1)^2} \right)$$

$$- q^{i(x-1)^2} F_{1} \left( -n + i, i; i + 1; q^{(x-1)^2} \right)$$

and then simplifying it we get (4).

The general recurrence relation is

$$(i + 1) P_r (X_{(i+1)} = x) = i P_r (X_{(i)} = x) - \left( \frac{n}{i} \right) \left[ F_x \left( S_{x+1} \right)^n - F_{x-1} \left( S_{x} \right)^n \right].$$

This completes the proof.

**Theorem 2.1.3:**

Let $X$ be non-negative continuous Inverse Rayleigh random variable and $Z = [X]$ be an integer valued random variable. Then $Z \sim dIR(q)$ if $X \sim IR(\theta)$.

**Proof:**

Let $X \sim IR(\theta)$ with $S_X(x) = P_r (X \geq x) = 1 - e^{-\theta x^2}$, $\theta > 0$, $x \geq 0$, then we observe that $\forall x = 0, 1, 2, 3, \ldots$

$S_Z(x) = P_r (Z \geq x)$,

$$= P_r ([X] \geq x),$$
\[
S_2(x) = P_r(X \geq x), \quad S_Z(x) = 1 - \exp\left(\frac{-\theta}{x^2}\right), \quad S_Z(x) = 1 - q^{x^2},
\]

since \([X] \geq Z \iff X \geq Z\)

where \(0 < q < 1\) and \(q = \exp(-\theta)\).

This completes the proof.

**Theorem 2.1.4:**

Let \(X\) be non-negative continuous Rayleigh random variable and \(W = \left\lfloor \frac{1}{X} \right\rfloor\) be an integer valued random variable. Then \(W \sim dIR(q)\) if \(X \sim R(\theta)\).

**Proof:**

Let \(X \sim R(\theta)\) with \(F_X(x) = P_r(X \leq x) = 1 - e^{-\theta x^2}, \ \theta > 0, x \geq 0\), then we observe that \(\forall x = 0, 1, 2, 3, \ldots\)

\[
S_W(x) = P_r(W \geq x), \quad S_W(x) = P_r\left(\left\lfloor \frac{1}{X} \right\rfloor \geq x\right),
\]

\[
S_Z(x) = P_r\left(X \leq \frac{1}{x}\right), \quad S_Z(x) = 1 - \exp\left(\frac{-\theta}{x^2}\right), \quad S_Z(x) = 1 - q^{x^2},
\]

since \([X] \geq Z \iff X \geq Z\)

where \(0 < q < 1\) and \(q = \exp(-\theta)\).

This completes the proof.

3. PARAMETERS’ ESTIMATION, GOODNESS OF FIT TESTS AND APPLICATIONS

In order to estimate the parameter \(q\) of discrete Inverse Rayleigh distribution we have studied three methods like proportions, pseudo-moments (see Khan et al. 1989) and maximum likelihood. Simulation results of these methods are based on 100 replication and presented in table 2.

3.1 Method of Moments

While using the method of moments for estimating the parameter \(q\), we have to first equate the population moment to the corresponding sample moment than solve the equation for \(q\). Since moments are not in closed form so this equation cannot be solved by ordinary techniques. However Jazi et al. (2010) used the
method of pseudo-moments, as proposed by Khan et al. (1989), by minimizing
\[ S(q, \beta) = \left( M_1 - E(X) \right)^2 + \left( M_2 - E(X^2) \right)^2, \]
with respect to \( q \) and \( \beta \) where \( M_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \) and
\[ M_2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2. \]
We have also used this method to estimate \( q \) by minimizing
\[ S(q) = \left( M_1 - E(X) \right)^2 \]
with respect to \( q \). Unlike Jazi et al. (2010), though this method yields smaller variance yet the deviation from the true value is larger as compared to the other estimators for larger \( q \) and \( n \).

3.2 Method of Proportions

The method of proportions, proposed and studied by Khan et al. (1989) and Jazi et al. (2010), are based on the proportions of 1’s and 2’s. Now we proposed the similar method for discrete inverse Rayleigh distribution but based on proportions of 0’s which is outlined below.

Let \( x_1, x_2, ..., x_n \) be a random sample of \( n \) items from discrete inverse Rayleigh distribution, then the indicator function is defined as
\[ I(x_i) = \begin{cases} 
1, & x_i = 0 \\
0, & x_i > 0.
\end{cases} \]

As \( Z = \sum_{i=1}^{n} I(x_i) \) denotes the number of 0’s in the sample so the proportions of zeros i.e. \( \frac{Z}{n} \) estimates the probability \( p_{0,q} = q \). Hence, we have denoted \( \tilde{q} \) as an estimate of \( q \) and \( z \) as observed value of \( Z \) therefore \( \tilde{q} = \frac{z}{n} \). It is known that an empirical cumulative distribution function (cdf) is consistent and an unbiased estimator of the actual cdf the same is true for \( \tilde{q} \) which is an unbiased and consistent estimator of \( P(Y \leq 0) = q \) (see Jazi et al. 2010).

3.3 Maximum Likelihood:

Let \( X_1, X_2, ..., X_n \) be the recorded lifetimes of a random sample of \( n \) items. If these recorded lifetimes identically independently follow the dIR i.e. \( X_i \sim dIR(\tilde{q}) \) then the likelihood function for dIR can be expressed as
\[ L(q) = \prod_{i=1}^{n} p_{x_i} = \prod_{i=1}^{n} \left\{ q^{(x_i+1)^2} - q^{(x_i)^2} \right\}, \]
\[ \frac{\partial \ln L(q)}{\partial q} = \sum_{i=1}^{n} \frac{q^{(x_i+1)^2 - 1} - q^{(x_i)^2 - 1}}{q^{(x_i+1)^2} - q^{(x_i)^2}} = 0. \]

A numerical solution of the above equation will yield the MLEs of \( q \).
As MLEs are generally unbiased and consistent estimators so the asymptotic
distribution of MLE of \( q \) i.e. \( \hat{q} \) is normal with mean \( q \) and variance of \( \hat{q} \) is
\[
\text{Var}(\hat{q}) = (I(\hat{q}))^{-1}
\] i.e. \( \hat{q} \sim \mathcal{N}(q, \frac{1}{I(q)}) \) where \( I(q) \) is the Fisher Information and is
defined as \( I(q) = E(-\frac{\partial^2 \ln L}{\partial q^2}) \) an estimate of \( I(q) \) is \( I(\hat{q}) \) by virtue of invariance
property of MLE and is expressed as \( I(\hat{q}) = -\frac{\partial^2 \ln L}{\partial q^2}\big|_{q=\hat{q}} \) so we have variance of \( \hat{q} \) in
this form \( \text{Var}(\hat{q}) = (I(\hat{q}))^{-1} \).

<table>
<thead>
<tr>
<th>( q ) (PM)</th>
<th>( \hat{q} ) (PM)</th>
<th>( \hat{q} ) (MM)</th>
<th>( \hat{q} ) (ML)</th>
<th>( \hat{q} ) (ML)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.0858</td>
<td>0.0055</td>
<td>0.1243</td>
<td>0.0110</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0907</td>
<td>0.0023</td>
<td>0.1375</td>
<td>0.0249</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0926</td>
<td>0.0045</td>
<td>0.1570</td>
<td>0.0979</td>
</tr>
<tr>
<td>0.30</td>
<td>0.2748</td>
<td>0.0034</td>
<td>0.3167</td>
<td>0.0061</td>
</tr>
<tr>
<td>0.30</td>
<td>0.2953</td>
<td>0.0053</td>
<td>0.3263</td>
<td>0.0129</td>
</tr>
<tr>
<td>0.30</td>
<td>0.2886</td>
<td>0.0105</td>
<td>0.3617</td>
<td>0.0516</td>
</tr>
<tr>
<td>0.60</td>
<td>0.5881</td>
<td>0.0040</td>
<td>0.5993</td>
<td>0.0024</td>
</tr>
<tr>
<td>0.60</td>
<td>0.6021</td>
<td>0.0059</td>
<td>0.6504</td>
<td>0.0052</td>
</tr>
<tr>
<td>0.60</td>
<td>0.6014</td>
<td>0.0120</td>
<td>0.6335</td>
<td>0.0215</td>
</tr>
</tbody>
</table>

Table 2 is based on 100 replication, from this table, it is evident that the estimators
obtained by the method of moments are asymptotically unbiased and consistent for larger
\( q \) and \( n \). It is also observed that method of proportion is much better than the method of
moments in terms of smaller variances for all \( n \) and \( q < 0.40 \) and smaller deviation
about the true value of \( q \) for all \( n \) and \( q \). Finally, the maximum likelihood method is
the most efficient procedure for almost all \( q \) and \( n \).

Now in the next section we deals with the goodness of fit tests which uses the above
developed MLE to test the suitability of discrete Inverse Rayleigh Distribution in over
dispersed data structure.

### 3.3 Goodness of Fit Tests

Generally the goodness of fit (GOF) tests compute the compatibility of a random
sample with a theoretical probability distribution function. In short, these tests measure
the suitability of your data to the distribution you have selected. The general procedure consists of defining a test statistic which is some function of the data measuring the distance between the hypothesis and the data. Here, we are using Kolmogorov-Smirnov and Chi-Squared goodness of fit tests for testing the suitability of various data sets.

3.3.1 Kolmogorov-Smirnov Test

This test may be used to decide if a sample comes from a hypothesized continuous/discrete distribution (see Gibbons, 1971, p.85). In it an empirical cumulative distribution function (ECDF) is computed. Suppose that we have a random sample \( x_1, \ldots, x_n \) from some continuous/discrete distribution with CDF \( F(x) \). The empirical CDF is denoted by

\[
F_n(x) = \frac{\text{Number of observations} \leq x}{n}
\]

**Definition**

The Kolmogorov-Smirnov test statistic \( D \) is based on the largest vertical difference between \( F(x) \) and \( F_n(x) \). It is defined as

\[
D_n = \sup_x |F_n(x) - F(x)|
\]

\( H_0 \) : The data follow the specified distribution.

\( H_A \) : The data do not follow the specified distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level \( (\alpha) \) if the test statistic, \( D \), is greater than the critical value obtained from a table.

3.3.2 Chi-Squared Test

This test is used to determine whether a sample comes from a population with a specific distribution or not. It is applied to grouped data that is why its test statistic depends on how the data is grouped.

Since there is no optimal choice for the number of classes \( (k) \), so there are several formulas which are used to calculate the number of classes which are based on the sample size \( (N) \). For this purpose the Sturges’ empirical formula is frequently used i.e. \( k = 1 + 3.3 \log N \).

Generally the data can be grouped into intervals of equal probability or equal width. Each class should contain at least 5 or more data points, so, in order to satisfy this condition certain adjacent classes sometimes need to be joined together.

**Definition**

The Chi-Squared test statistic is defined as

\[
\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}
\]

where \( o_i \) and \( e_i \) are the observed and expected frequencies for class \( i \) respectively. For testing the
compatibility of the data with theoretical probability function we formulate the following null and alternative hypothesis

\[ H_0 : \text{The data follow the specified distribution.} \]
\[ H_A : \text{The data do not follow the specified distribution.} \]

The hypothesis regarding the distributional form is rejected at the chosen significance level \((\alpha)\) if the test statistic is greater than the critical value defined as \(\chi^2_{1-k-m} \) \((1-\alpha)\).

Where \(k-m\) denotes the degree of freedom with \(m\) the number of parameters to be estimated. Usually a smaller computed value of chi-square indicates a good fit whereas larger value showed a poor fit.

3.4 Applications

Here, we are now presenting some applications and comparisons of the proposed model with the Poisson under real life scenario.

Example 1

The following data set give the number of times that computer break down in each of the 128 consecutive week of operation (see Chakaraborty and Chakravarty (2012)). The empirical failure function is presented in Fig. 3.4.1.

\[
\{4, 0, 0, 0, 3, 2, 0, 0, 6, 7, 6, 2, 1, 11, 6, 1, 2, 1, 1, 2, 0, 2, 2, 1, 0, 12, 8, 4, 5, 0, 5, 4, 1, 0, 8, 2, 5, 2, 1, 12, 8, 9, 10, 17, 2, 3, 4, 8, 1, 2, 5, 2, 2, 2, 3, 1, 2, 0, 2, 1, 6, 3, 6, 11, 10, 4, 3, 0, 2, 4, 2, 1, 5, 3, 3, 2, 5, 3, 4, 1, 3, 6, 4, 4, 5, 2, 10, 4, 1, 5, 6, 9, 7, 3, 1, 3, 0, 2, 2, 1, 4, 2, 13, 0, 2, 1, 1, 0, 3, 16, 22, 5, 1, 2, 4, 7, 8, 6, 11, 3, 0, 4, 7, 8, 4, 4, 5} \]

By using MLE’s, we have fitted the failure functions of discrete inverse Rayleigh and Poisson distributions. Kolmogrov-Smirnov (KS) test for goodness of fit (see Gibbons, 1971, p.85) and AIC are computed to compare their performance. Findings are computed in R computational package.

<table>
<thead>
<tr>
<th>Model</th>
<th>K.S</th>
<th>AIC</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete Inverse Rayleigh (0.0243)</td>
<td>0.1765</td>
<td>715.718</td>
<td>0.9631</td>
</tr>
<tr>
<td>Poisson(2.99993)</td>
<td>0.4706</td>
<td>810.8884</td>
<td>0.04495</td>
</tr>
</tbody>
</table>

From the above table 3, it is evident that discrete inverse Rayleigh distribution provides marginally better fit as compare to Poisson distribution not only in larger p-value but also in least loss of information i.e. smaller AIC.
Example 2: Modeling Probability distribution of Count data

We have also investigated that whether the proposed discrete inverse Rayleigh distribution can compete with the Poisson distribution in modeling real life count data, other than reliability. In this example we fitted the proposed and the Poisson distributions to an over-dispersed data set. The following data set is the distribution of yeast cells in 400 squares of haemacytometer observed by “Student” (1907) (see Roy and Gupta 1999).

Table 4:
Distribution of yeast cells in 400 squares of haemacytometer observed by “Student” (1907) Data is taken from Roy and Gupta (1999)

<table>
<thead>
<tr>
<th>No. of Cells</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>≥5</th>
<th>Total</th>
<th>Chi-square</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>213</td>
<td>128</td>
<td>37</td>
<td>18</td>
<td>3</td>
<td>1</td>
<td>400</td>
<td>calculated</td>
<td></td>
</tr>
<tr>
<td>Expected frequencies dIR(0.5335)</td>
<td>213.4</td>
<td>128.5</td>
<td>31.2</td>
<td>11.6</td>
<td>5.5</td>
<td>9.9</td>
<td>400</td>
<td>2.01</td>
<td>0.3660</td>
</tr>
<tr>
<td>Expected frequencies Poi.(0.6825)</td>
<td>202.1</td>
<td>138.0</td>
<td>47.1</td>
<td>10.7</td>
<td>1.8</td>
<td>0.3</td>
<td>400</td>
<td>10.09</td>
<td>0.0066</td>
</tr>
</tbody>
</table>

The above is an over dispersed data with mean = 0.6825, variance = 0.8137 and index of dispersion = 1.1922. Based on the value of chi-square and p-value it follows that dIR(q) provide a good fit to the data set. It is also worth mentioning that while using the same data set, our proposed model gives the closest fit among all the alternative models studied by Roy and Gupta (1999).
CONCLUDING REMARKS

In this paper, a discrete Inverse Rayleigh distribution is developed by using the discrete concentration approach, which can be used in over dispersed data structure as an alternate to single parameter Poisson distribution. Moreover, this newly proposed model can be applied to model not only the count data but also seems suitable for modeling number of claims in Actuarial science and in discrete life testing data where the hazard function shows non-monotonic behavior.

ACKNOWLEDGEMENT

The authors are thankful to the referees for their suggestions, which led to the improvement of this paper.

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