SOME STRONG DEVIATION THEOREMS FOR STOCHASTIC PROCESS INDEXED BY A TREE

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ABSTRACT
In this paper we introduce a concept of asymptotic log-likelihood ratio as a measure of deviation of the joint distribution from the product of their margins, we give the generalized strong law of large numbers (strong law expressed by inequalities, i.e. strong deviation theorems) for stochastic process indexed by a tree. As corollaries, we obtain some known results.

KEYWORDS
Asymptotic log-likelihood ratio; strong deviation theorem; tree; Laplace transform.


1. INTRODUCTION
A tree is a graph \( G = (T, E) \) which is connected and contains no circuits. Given any two vertices \( \sigma, t (\sigma \neq t \in T) \), let \( \overline{\sigma t} \) be the unique path connecting \( \sigma \) and \( t \). Define the graph distance \( d(\sigma, t) \) to be the number of edges contained in the path \( \overline{\sigma t} \). Let \( T \) be an arbitrary infinite tree that is partially finite (i.e. it has infinite vertices, and each vertex connects with finite vertices) and has a root \( o \). The set of all vertices with distance \( n \) from the root \( o \) is called the \( n \)-th generation of \( T \), which is denoted by \( L_n \). We say that \( L_n \) is the set of all vertices on level \( n \). We denote by \( T^{(n)} \) the subtree of tree \( T \) containing the vertices from level 0 (the root \( o \)) to level \( n \). Let \( t(t \neq o) \) be a vertex of tree \( T \). Predecessor of the vertex \( t \) is another vertex which is nearest from \( t \) on the unique path from root \( o \) to \( t \). \( X^A = \{X_t, t \in A\} \) is a stochastic process indexed by a set \( A \), and denoted by \(|A|\) the number of vertices of \( A \), \( x^A \) is the realization of \( X^A \).

There have been some works on limit theorems for tree-indexed stochastic process. Benjamini and Peres (1994) have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye (1990) have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Pemantle (1992) has proved a mixing property and a weak law of large numbers for a PPG-invariant and ergodic random field on a homogeneous tree. Ye and Berger (1996, 1998), by using Pemantle's result and a combinatorial approach, have studied the
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Shannon-McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu (2000) have studied a strong law of large numbers for the frequency of occurrence of states for Markov chains field on a Bethe tree (a particular case of tree-indexed Markov chains field and PPG-invariant random field). Yang (2003) has studied the strong law of large numbers for frequency of occurrence of state and Shannon-McMillan theorem for homogeneous Markov chains indexed by a homogeneous tree. Yang and Ye (2007) have studied the strong law of large numbers and Shannon-McMillan theorem for nonhomogeneous Markov chains indexed by a homogeneous tree. Huang and Yang (2008) have studied the strong law of large numbers and Shannon-McMillan theorem for Markov chains indexed by an infinite tree with uniformly bounded degree. Recently, Shi and Yang (2009) have also studied some limit properties of random transition probability for second-order nonhomogeneous Markov chains indexed by a tree. Peng, Yang and Wang (2009) have studied a class of strong deviation theorems for the random fields relative to homogeneous Markov chains indexed by a homogeneous tree. Shi and Yang (2010) have studied the strong law of large numbers and Shannon-McMillan for the $m$th-order nonhomogeneous Markov chains indexed by a $m$ rooted Cayley tree.

Recently, Liu and Yang (1996, 2000) introduced and used the concept of asymptotic logarithmic likelihood ratio, which worked as a measure of the deviation between a multivariate density function and the product of their one-dimensional margins, to prove some strong deviation theorems (strong law expressed by inequalities) of arbitrarily dependent random variables. A technique that partially solves the strong deviation theorems for continuous random variables was established by using the Laplace transformation method. Li, Chen and Zhang (2009) have studied a class of random deviation theorems for a sequence of random variables by using the approach of Laplace transform. Wang and Yang (2011) have also studied some limit theorems for delayed sums of dependent sequence.

Throughout the present paper, let $\{X_t, t \in T\}$ be a collection of continuous nonnegative integrable random variables defined on the underlying probability space $(\Omega, \mathcal{F}, P)$. For any finite set $B(B \subset T)$, if $\{X_t, t \in B\}$ is a collection of nonnegative integrable independence random variables, we say that $\{X_t, t \in T\}$ is a collection of nonnegative integrable independence random variables.

In this paper, our main purpose is similar to Li, Chen and Zhang (2009). Our aim is to extend its results to a tree, and give a random bound of

$$\liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} (X_t - m_t), \quad \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} (X_t - m_t),$$

where $m_t$ is the expectation of $X_t$. We first introduce a concept of asymptotic loglikelihood ratio, and prove some strong deviation theorems and strong law of large numbers for stochastic process indexed by a tree. The crucial part of proof is the construction of a asymptotic log-likelihood ratio depending on a parameter by using the Laplace transform and the application of the Borel-Cantelli Lemma. This paper is organized as follows. Section 1 gives some foundational concepts and key lemmas.
Section 2 introduces the strong deviation theorems for stochastic process indexed by a tree. Finally, section 3 is devoted to proofs of the main results.

Assume that for any $n(n \geq 0)$, $X^{(n)}$ has a joint distribution density function $f_n\left(x^{(n)}\right)$, where $x_i \geq 0$, $t \in T^{(n)}$. Let $f_t(x_t)$ stand for the density of the random variable $X_t\left(t \in T^{(n)}\right)$, that is, $f_t(x_t)$ is the marginal density of $f_n\left(x^{(n)}\right)$. Let

$$\pi_n\left(x^{(n)}\right) = \prod_{t \in T^{(n)}} f_t(x_t)$$

be the reference product density function. Let

$$r_n(\omega) = \ln \left[ \frac{f_n\left(X^{(n)}\right)}{\pi_n\left(X^{(n)}\right)} \right]$$

$$= \ln \left[ \frac{f_n\left(X^{(n)}\right)}{\prod_{t \in T^{(n)}} f_t\left(X_t\right)} \right],$$

where $\omega$ is a sample point. Let

$$r(\omega) = \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \ln r_n(\omega),$$

with $\ln 0 = -\infty$. $r(\omega)$ is called asymptotic log-likelihood ratio.

**Remark 1:**

Although $r(\omega)$ is not a proper metric between probabilities, we nevertheless consider it as a measure of "discrimination" between the dependence (their joint distribution) and independence (the product of their margins). Based on the above discussion of the asymptotic likelihood ratio, it is natural for us to deem $r(\omega)$ as a measure how far (the random deviation) of $\{X_t, t \in T\}$ is from being independent and how dependent they are. The closer $r(\omega)$ approaches to 0, the smaller the deviation is. In particular, if $\{X_t, t \in T\}$ is a collection of independence random variables, i.e. $f_n\left(x^{(n)}\right) = \prod_{t \in T^{(n)}} f_t\left(x_t\right)$, we have $r(\omega) = 0$.

In order to prove our main results, we first propose a concept, which will play a fundamental role in the proofs of our results.

**Definition 1:**

A collection of random variables $\{X_t, t \in T\}$ is called stochastically dominated by a random variable $X$ (we write $\{X_t, t \in T\} \prec X$) if there exists a constant $D > 0$ such that

$$\sup_{t \geq 0} P(X_t > t) \leq DP(X > t) \text{ for all } t > 0.$$
Definition 2:
Let \( \{X_t, t \in T\} \) be a collection of random variables, and \( f_t(x_t), t \in T^{(n)} \) be the marginal density functions of \( f_n \left( x^{(n)} \right) (n \geq 0) \). Let Laplace transform and tail probability Laplace transform as follows:

\[
\tilde{f}_t(s) = \int_0^{+\infty} e^{-sx_t} f_t(x_t) dx_t ,
\]

and

\[
Q_t(s) = \int_0^{+\infty} e^{-sx} P(X_t > x) dx .
\]

Before giving the main results which correspond to the strong deviation for stochastic process indexed by a tree, we begin with the following lemmas.

Lemma 1 (Wang and Yang, 2012).
Let \( \tilde{f}_t(s), Q_t(s) \) be defined by (4) and (5), respectively, then

\[
Q_t(s) = \frac{1 - \tilde{f}_t(s)}{s} ,
\]

and

\[
Q_t(0) = E X_t = m_t, \ t \in T^{(n)}, n \geq 0 .
\]

Proof:
For all \( t \in T \),

\[
Q_t(s) = \int_0^{+\infty} e^{-sx} P(X_t > x) dx
\]

\[
= \int_0^{+\infty} e^{-sx} \int_x^{+\infty} f_t(y) dy dx
\]

\[
= \int_0^{+\infty} \int_0^y e^{-sx} f_t(y) dx dy \quad (\text{Fubini's Theorem})
\]

\[
= \int_0^{+\infty} \frac{1 - e^{-sy}}{s} f_t(y) dy
\]

\[
= \frac{1 - \tilde{f}_t(s)}{s}
\]

and

\[
Q_t(0) = \int_0^{+\infty} P(X_t > x) dx = E X_t .
\]

Thus we complete the proof.

Remark 2:
We denote the tail probability Laplace transform of \( X \) by
\[ Q(s) = \int_0^{+\infty} e^{-sx} P(X > x) dx. \] (8)

It is not difficult to see that if there exists a real number \( s_0 > 0 \), such that \( Q(-s_0) < \infty \) and \( \{X_t, t \in T\} \prec X \), then \( \tilde{f}_i(s) < \infty \), \( Q_i(s) < \infty \), \( EX_i < \infty, s \in [-s_0, s_0] \) and \( \tilde{f}_i(s), Q_i(s) \), \( t \in T \) are all continuous on \( [-s_0, s_0] \).

**Lemma 2:**

Let \( f_n\left(X_{T(n)}\right) \) be the joint distribution of \( X_{T(n)} \), and \( g_n\left(X_{T(n)}\right) \) be another probability density function on \( (\Omega, \mathcal{F}, P) \). Let

\[
t_n(\omega) = \frac{g_n\left(X_{T(n)}\right)}{f_n\left(X_{T(n)}\right)},
\]

then

\[
\limsup_{n \to \infty} \frac{1}{|T(n)|} \ln t_n(\omega) \leq 0 \quad \text{a.s.} \tag{10}
\]

**Proof:**

It is easy to see that \( Et_n \leq 1 \). Hence for all \( \varepsilon \geq 0 \), we have by Markov's inequality

\[
\sum_{n=0}^{\infty} P\left(|T(n)|^{-1} \ln t_n > \varepsilon \right) \leq \sum_{n=0}^{\infty} \exp\left(-|T(n)| \varepsilon \right) < \infty. \tag{11}
\]

Since \( \varepsilon > 0 \) is arbitrary, by the Borel-Cantelli Lemma, (10) follows.

2. MAIN RESULT

In this section, we present the main results of the paper.

**Theorem 1:**

Let \( T \) be a tree and \( \{X_t, t \in T\} \) be a collection of nonnegative integrable random variables on the probability space \( (\Omega, \mathcal{F}, P) \), \( r(\omega), \tilde{f}_i(s), Q_i(s) \) be given as above. Let

\[
D = \{\omega: r(\omega) < +\infty\}, P(D) = 1, \tag{12}
\]

and \( s_0 \) be a positive real numbers. If \( \{X_{t}, t \in T\} \prec X \) and \( EX < \infty \). Then

\[
\liminf_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} (X_t - m_t) \geq \alpha(r(\omega)) \quad \text{a.s.} \tag{13}
\]

where

\[
\alpha(x) = \sup \{\varphi(s, x), 0 < s \leq s_0\}, \quad x \geq 0, \tag{14}
\]
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\[ \varphi(s, x) = \liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left[ Q_t(s) - Q_t(0) \right] - \frac{x}{s^2}, \quad 0 < s \leq s_0, \]  

(15)

and then

\[ \alpha(x) \leq 0, \lim_{x \to 0^+} \alpha(x) = \alpha(0) = 0. \]  

(16)

**Theorem 2:**

Let \( T \) be a tree and \( \{X_t, t \in T\} \) be a collection of nonnegative integrable random variables on the probability space \((\Omega, \mathcal{F}, P), r(o), \tilde{f}_t(s), Q_t(s)\) be given as above. Let

\[ D = \{\omega: r(o) < +\infty\}, P(D) = 1. \]

If \( \{X_t, t \in T\} \prec X \), and there exists a number \( s_0 > 0 \), such that \( Q(-s_0) < \infty \), then

\[ \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} (X_t - m_t) \leq \beta(r(o)) \text{ a.s.} \]  

(17)

where

\[ \beta(x) = \inf \{\psi(s, x), -s_0 \leq s < 0\}, \quad x \geq 0, \]  

(18)

\[ \psi(s, x) = \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left[ Q_t(s) - Q_t(0) \right] - \frac{x}{s}, \quad -s_0 \leq s < 0, \]  

(19)

and then

\[ \beta(x) \geq 0, \lim_{x \to 0^+} \beta(x) = \beta(0) = 0. \]  

(20)

**Corollary 1:**

If \( \{X_t, t \in T\} \) is a collection of independent nonnegative integrable random variables, then

\[ \lim_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} (X_t - m_t) = 0 \text{ a.s.} \]  

(21)

**Proof:**

In this case, \( f_n(x^{(n)}) = \prod_{t \in T^{(n)}} f_t(x_t) \), and \( r(o) = 0 \text{ a.s.} \), hence (21) follows directly from (16) and (20).

If the successor of each vertex of the tree \( T \) has only one vertex, then the stochastic process on the tree \( T \) degenerate into a sequence of random variables. Thus we directly obtain the results in Li, Chen and Zhang (2009).
Corollary 2 (Li, Chen and Zhang, 2009).

Let \( \{X_n, n \geq 1\} \) be a sequence of nonnegative integrable random variables on the probability space \((\Omega, F, P)\), \( r(\omega), \tilde{f}_r(s), Q_t(s), q_t(x) \) be given as in Li, Chen and Zhang (2009), and \( \tilde{f}_r(s) \) is defined in \([-s_0, s_0]\). Let

\[
m_k = \int_0^{+\infty} x_k f_r(x_k) dx_k < +\infty, \quad k = 1, 2, \ldots, n, \tag{22}
\]

\[
D = \{\omega : r(\omega) < +\infty\}, P(D) = 1. \tag{23}
\]

If there exists function \( q(x)(x \geq 0) \), such that

\[
q_k(x) \leq q(x), \quad x \geq 0, \quad k = 1, 2, \ldots, n, \tag{24}
\]

\[
\int_0^{+\infty} q(x) dx = m < +\infty. \tag{25}
\]

Then

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - m_k) \geq \alpha \left( r(\omega) \right) \text{ a.s.} \tag{26}
\]

where

\[
\alpha(x) = \sup \{\varphi(s, x), 0 < s \leq s_0\}, \quad x \geq 0, \tag{27}
\]

\[
\varphi(s, x) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ Q_k(s) - Q_k(0) \right] - \frac{x}{s}, \quad 0 < s \leq s_0, \tag{28}
\]

and then

\[
\alpha(x) \leq 0, \lim_{x \to 0^+} \alpha(x) = \alpha(0) = 0. \tag{29}
\]

**Proof:**

In fact, from \( EX < \infty \) and \( \{X_t, t \in T\} \prec X \) in Theorem 1, (22), (24) and (25) holds. Thus Corollary 2 can be proved directly by Theorem 1.

Corollary 3 (Li, Chen and Zhang, 2009)

Under the assumptions of Corollary 2, then

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - m_k) \leq \beta \left( r(\omega) \right) \text{ a.s.} \tag{30}
\]

where

\[
\beta(x) = \inf \{\psi(s, x), -s_0 \leq s < 0\}, \quad x \geq 0, \tag{31}
\]

\[
\psi(s, x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[ Q_k(s) - Q_k(0) \right] - \frac{x}{s}, \quad -s_0 \leq s < 0. \tag{32}
\]
and then
\[ \beta(x) \geq 0, \quad \lim_{x \to 0^+} \beta(x) = \beta(0) = 0. \] (33)

The proof of Corollary 3 is similar to Corollary 2, so we omit it.

**Corollary 4** (Li, Chen and Zhang, 2009)
If \( \{X_n, n \geq 1\} \) is a sequence of independent nonnegative integrable random variables, then
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - m_k) = 0 \quad a.s. \] (34)

### 3. PROOFS OF MAIN THEOREMS

In this section, we give the proofs of Theorem 1, 2.

**Proof of Theorem 1.**
For arbitrary \( s \in [0, +\infty) \), let
\[ g_t(s, x_i) = e^{-sx_i} \frac{f_t(x_i)}{\tilde{f}_t(s)}. \] (35)

Then
\[ \int_0^{+\infty} g_t(s, x_i)dx_i = 1. \] (36)

Let
\[ q_n(s, x_T^{(n)}) = \prod_{t \in T^{(n)}} g_t(s, x_i) = \prod_{t \in T^{(n)}} \left[ e^{-sx_i} \frac{f_t(x_i)}{\tilde{f}_t(s)} \right] \]
\[ = \frac{1}{\prod_{t \in T^{(n)}} \tilde{f}_t(s)} \exp \left( -s \sum_{t \in T^{(n)}} x_i \right) \cdot \prod_{t \in T^{(n)}} f_t(x_i). \] (37)

Therefore, \( q_n(s, x_T^{(n)}) \) is a \( |T^{(n)}| \) multivariate probability density function. By Lemma 2, there exists \( A(s) \in F, \ P(A(s)) = 1, \) such that
\[ \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \ln \frac{q_n(s, x_T^{(n)})}{f_n(x_T^{(n)})} \leq 0, \quad \omega \in A(s). \] (38)

By (37) and (38), we obtain
\[
\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \left\{ \frac{1}{\prod_{t \in T^{(n)}} f_t(s)} \exp \left( -s \sum_{t \in T^{(n)}} x_t \right) \cdot \prod_{t \in T^{(n)}} f_t(x_t) \right\} \leq 0, \quad \omega \in A(s). \quad (39)
\]

this is

\[
\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \left\{ - \sum_{t \in T^{(n)}} \ln \tilde{f}_t(s) - s \sum_{t \in T^{(n)}} X_t - r_n(\omega) \right\} \leq 0, \quad \omega \in A(s). \quad (40)
\]

By (2) and (40), we have

\[
\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} X_t(-s) \leq \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \ln \tilde{f}_t(s) + r(\omega), \quad \omega \in A(s). \quad (41)
\]

Let \(0 < s \leq s_0\). Then, dividing the two sides of (41) by \(-s\), we obtain

\[
\liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} X_t \geq \liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( - \frac{\ln \tilde{f}_t(s)}{s} \right) - \frac{r(\omega)}{s}, \quad \omega \in A(s). \quad (42)
\]

By (42), the property of the inferior limit

\[
\liminf_{n \to \infty} (a_n - b_n) \geq d \implies \liminf_{n \to \infty} (a_n - c_n) \geq \liminf_{n \to \infty} (b_n - c_n) + d
\]

and the inequality \(\ln x \leq x - 1(x > 0)\), and Lemma 1, for \(\omega \in A(s)\), we have

\[
\liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} (X_t - m_t)
\geq \liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( - \frac{\ln \tilde{f}_t(s)}{s} - m_t \right) - \frac{r(\omega)}{s}
\geq \liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( - \frac{\tilde{f}_t(s) - 1}{s} - m_t \right) - \frac{r(\omega)}{s}
= \liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} [Q_t(s) - Q_t(0)] - \frac{r(\omega)}{s}. \quad (43)
\]

Let \(Q^+\) be the set of rational numbers in the interval \((0, s_0]\) and let \(A^* = \bigcap_{s \in Q^+} A(s)\), then

\[
P(A^*) = 1.
\]

By (43), we have
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\[ \liminf_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} (X_t - m_t) \]
\[ \geq \liminf_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(s) - Q_t(0)] - \frac{r(o)}{s}, \quad o \in A^*, \forall s \in Q^+. \] (44)

Let
\[ g(s) = \liminf_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(s) - Q_t(0)], \quad 0 < s \leq s_0. \] (45)

By (15) and (45), we have
\[ \varphi(s, x) = g(s) - \frac{x}{s}, \quad 0 < s \leq s_0, \quad x \geq 0, \] (46)
\[ \alpha(x) = \sup \left\{ g(s) - \frac{x}{s}, 0 < s \leq s_0 \right\}. \] (47)

Obviously \( g(s) \leq 0, \varphi(s, x) \leq 0, \) hence \( \alpha(x) \leq 0. \) Now we are going to show that \( g(s) \) is continuous on interval \([0, s_0]\). If \( 0 \leq s-t < s_0 < \infty, \) by (45), (5) and (8), note that
\[ \{X_t, t \in T\} < X, \]
we have
\[ 0 < g(s-t) - g(s) \]
\[ = \liminf_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(s-t) - Q_t(0)] - \liminf_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(s) - Q_t(0)] \]
\[ = \liminf_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(s-t) - Q_t(0)] + \limsup_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(0) - Q_t(s)] \]
\[ \leq \limsup_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} [Q_t(s-t) - Q_t(s)] \]
\[ = \limsup_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} \left[ \int_0^{+\infty} e^{-(s-t)x} P(X_t > x)dx - \int_0^{+\infty} e^{-sx} P(X_t > x)dx \right] \]
\[ = \limsup_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} \int_0^{+\infty} \left( e^{-(s-t)x} - e^{-sx} \right) P(X_t > x)dx \]
\[ \leq D \limsup_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} \int_0^{+\infty} \left( e^{-(s-t)x} - e^{-sx} \right) P(X > x)dx \]
\[ \leq D \limsup_{n \to \infty} \frac{1}{T(n)} \sum_{t \in T(n)} \left[ \int_0^{+\infty} e^{-(s-t)x} P(X > x)dx - \int_0^{+\infty} e^{-sx} P(X > x)dx \right] \]
\[ \leq D \left[ \int_0^{+\infty} e^{-(s-t)x} P(X > x)dx - \int_0^{+\infty} e^{-sx} P(X > x)dx \right] \]
\[ = D \left[ Q(s-t) - Q(s) \right]. \] (48)
By (48) and $EX < \infty$, we know that $g(s)$ is a continuous function with respect to $s$ on the interval $[0, s_0]$. Now it is easy to see that $\varphi(s, x)$ is also a continuous function with respect to $s$ on the interval $[0, s_0]$. By (47), for each $\omega \in A^* \cap A(0) \cap D$, take $s_n(\omega) \in Q^+, (n = 1, 2, \cdots)$, such that
\[ \lim_{n \to \infty} \varphi(s_n(\omega), r(\omega)) = \alpha(r(\omega)). \quad (49) \]

By (44)-(47), we have
\[ \liminf_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} (X_t - m_t) \geq \varphi(s_n(\omega), r(\omega)), \ \ \omega \in A^* \cap A(0) \cap D, \ t \in T(n). \]

By (49) and (50), we have
\[ \liminf_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} (X_t - m_t) \geq \alpha(r(\omega)) \ \ \omega \in A^* \cap A(0) \cap D. \quad (51) \]

Since $P(A^* \cap A(0) \cap D) = 1$, (13) holds by (51).

Next, we are going to prove (16). When $0 < s \leq s_0$, note that $\{X_t, t \in T\} \prec X$, we have
\[ \frac{1}{|T(n)|} \sum_{t \in T(n)} [Q_t(s) - Q_t(0)] \\
= \frac{1}{|T(n)|} \sum_{t \in T(n)} \left[ \int_0^{+\infty} e^{-sx} P(X_t > x)dx - \int_0^{+\infty} P(X_t > x)dx \right] \\
= \frac{1}{|T(n)|} \sum_{t \in T(n)} \left[ \int_0^{+\infty} (e^{-sx} - 1)P(X_t > x)dx \right] \\
\geq D \frac{1}{|T(n)|} \sum_{t \in T(n)} \left[ \int_0^{+\infty} ((e^{-sx} - 1)P(X > x)dx, \ (e^{-sx} - 1) \leq 0 \right] \\
= D \int_0^{+\infty} e^{-sx} P(X > x)dx - \int_0^{+\infty} P(X > x)dx \\
= D[Q(s) - Q(0)]. \quad (52) \]

For $x > 0$, we have
\[ \alpha(x) \geq \varphi(\sqrt{x}, x) = g(\sqrt{x}) - \frac{x}{\sqrt{x}} \geq Q(\sqrt{x}) - Q(0) - \sqrt{x}. \quad (53) \]

While for $x = 0$, we have
\[ \alpha(0) \geq g\left(\frac{1}{\sqrt{n}}\right) \geq Q\left(\frac{1}{\sqrt{n}}\right) - Q(0), \ n \geq 1. \quad (54) \]
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Noticing that \( \alpha(x) \leq 0, (x \geq 0) \), (16) follows from (53) and (54).

**Proof of Theorem 2.**

Analogous to Theorem 1, for \( s \in [-s_0, 0] \), let

\[
g_t(s, x_t) = e^{-sx_t} f_t(x_t) / \tilde{f}_t(s).
\]

Obviously

\[
q_n(s, x^{T(n)}_t) = \prod_{t \in T^{(n)}} g_t(s, x_t)
\]

\[
= \frac{1}{\prod_{t \in T^{(n)}} f_t(s)} \exp\left(-s \sum_{t \in T^{(n)}} x_t\right) \cdot \prod_{t \in T^{(n)}} f_t(x_t)
\]

is a \( |T^{(n)}| \) multivariate probability density function. By Lemma 2, (41) also holds.

Let \(-s_0 \leq s < 0\), dividing the two sides of (41) by \(-s\), we obtain

\[
\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} X_t \leq \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( -\frac{\tilde{f}_t(s)}{s} \right) - \frac{r(\omega)}{s}, \quad \omega \in A(s). \tag{55}
\]

By (55), the property of the superior limit

\[
\limsup_{n \to \infty} \left( a_n - b_n \right) \leq d \Rightarrow \limsup_{n \to \infty} \left( a_n - c_n \right) \leq \limsup_{n \to \infty} \left( b_n - c_n \right) + d
\]

and the inequality \( \ln x \leq x - 1, (x > 0) \), and Lemma 1, for \( \omega \in A(s) \), we have

\[
\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( X_t - m_t \right)
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( -\frac{\ln \tilde{f}_t(s)}{s} - m_t \right) - \frac{r(\omega)}{s}
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left( \frac{\tilde{f}_t(s) - 1}{s} - m_t \right) - \frac{r(\omega)}{s}
\]

\[
= \limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \left[ Q_t(s) - Q_t(0) \right] - \frac{r(\omega)}{s}. \tag{56}
\]

Let \( Q^- \) be the set of rational numbers in the interval \( (-s_0, 0] \) and let \( A_\omega = \bigcap_{s \in Q^-} A(s) \), then

\[ P(A_\omega) = 1. \]

By (56), we have
\[ \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} (X_t - m_t) \leq \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} \left[ Q_t(s) - Q_t(0) \right] - \frac{r(\omega)}{s}, \]
\[ \omega \in A_s, \forall s \in Q^- . \quad (57) \]

Let
\[ h(s) = \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} \left[ Q_t(s) - Q_t(0) \right], \quad -s_0 \leq s < 0. \quad (58) \]
By (19) and (58), we have
\[ \psi(s, x) = h(s) - \frac{x}{s}, \quad -s_0 \leq s < 0, \quad x \geq 0, \]
\[ \beta(x) = \inf \left\{ h(s) - \frac{x}{s}, -s_0 \leq s < 0 \right\}. \quad (60) \]

Obviously \( h(s) \geq 0, \psi(s, x) \geq 0 \), hence \( \alpha(x) \geq 0 \). By imitating (48) and \( Q(-s_0) < \infty \), we can also know that \( h(s) \) is a continuous function with respect to \( s \) on the interval \([-s_0, 0]\), then it is easy to see that \( \psi(s, x) \) is also a continuous function with respects to \( s \) on the interval \([-s_0, 0]\). By (18), for each \( \omega \in A_s \cap A(0) \cap D \), take \( \lambda_n(\omega) \in Q^- \), (\( n = 1, 2, \ldots \)), such that
\[ \lim_{n \to \infty} \psi(\lambda_n(\omega), r(\omega)) = \beta(r(\omega)). \quad (61) \]
By (57)-(60), we have
\[ \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} (X_t - m_t) \leq \psi(s_n(\omega), r(\omega)), \quad \omega \in A_s \cap A(0) \cap D, \quad t \in T(n). \quad (62) \]
By (61) and (62), we have
\[ \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n)} (X_t - m_t) \leq \beta(r(\omega)), \quad \omega \in A_s \cap A(0) \cap D. \quad (63) \]
Since \( P(A_s \cap A(0) \cap D) = 1 \), (17) holds by (63). When \(-s_0 \leq s < 0\), we have like (52)
\[ \frac{1}{|T(n)|} \sum_{t \in T(n)} \left[ Q_t(s) - Q_t(0) \right] \leq Q(s) - Q(0), \quad \left( e^{-sx} - 1 \geq 0, q_s(x) \leq q(x) \right). \quad (64) \]
For \( x > 0 \), we have
\[ \beta(x) \leq \psi(\sqrt{x}, x) = g(\sqrt{x}) - \frac{x}{\sqrt{x}} \leq Q(\sqrt{x}) - Q(0) - \sqrt{x}. \quad (65) \]
While for \( x = 0 \), we have
\[ \beta(0) \leq g\left( \frac{I}{\sqrt{n}} \right) \leq Q\left( \frac{I}{\sqrt{n}} \right) - Q(0), \quad n \geq 1. \]  

(66)

Noticing that \( \beta(x) \geq 0, (x \geq 0) \), (20) follows from (65) and (66).

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