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SINGLE SERVER INTERDEPENDENT QUEUEING MODEL WITH CONTROLLABLE ARRIVAL RATES AND RENEGING

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ABSTRACT

This investigation deals with single server markovian queueing model with controllable arrival rates with discouragement factor reneging, in which it is assumed that the arrival and service processes are interdependent. The stationary state solutions of the model are analysed. The expressions for system characteristics as average number of customers in the system and average waiting time are determined. The numerical illustrations are also considered to validate the analytical results and to illustrate the effect of the parameters on several performance characteristics.

KEY WORDS

Interdependent, Markovian queue, Controllable, Reneging, Stationary state.

AMS Mathematics Subject Classification: 60K25, 68M20.

1. INTRODUCTION

In the last few years, research in queueing theory has been successfully applied to predict the performance measures of various service systems dealing with many congestion situations of day to day life. The queueing models based on performance measures are playing an important role in analyzing the practical situations of various fields in modeling of manufacturing, production process, data voice transmissions and computer communication systems. These queueing models provide the basic framework for efficient design and analysis of many practical situations. In real life, many queueing situations arise in which there may be a tendency for customers to be discouraged by a long queue. As a result, customer may decide to depart after joining the queue without getting service due to impatience. Abou-El-Ata and Hariri (1992) considered the multiple servers queueing system with balking and reneging. Zhang et al (2005) analysed the finite capacity queueing model with discouragement factors balking and reneging and server vacations. The performance analysis for multi server system with balking and reneging was discussed by Yue et al (2006). Mitchell and Paulson (1979) presented a single server queueing model with interdependent arrival and service processes. Bhat (1969) has expressed the interdependent queueing model in which arrival and service are correlated. Conolly and Choo (1979) have provided means for exact calculation of the waiting time for a generalized correlated queueing model with exponential demand and service. The busy period of a correlated queue with exponential demand and service was analysed by Borst and Cambe (1992) and Lenganis (1987). Srinivasa Rao et al (2000) have obtained the performance measures in $M/M/1$ interdependent queueing model with controllable arrival rates.

In this paper, a single server queueing system with infinite capacity is considered. We assume that arrival and service processes of the system are correlated and follows a bivariate poisson process. In addition to this interdependence, it is also considered that discouragement factor also affects the smooth functioning of the system, because customers may decide to leave the queue, after some time of joining the queue due to impatience. In section 2, the model description with mathematical formulation of the model are considered. The solution method of steady state equation is described in section 3. Various performance measures are defined in terms of queue size distribution in section 4. The special cases are defined in section 5. Section 6 consists the mathematical illustration to validate the analytical results. Section 7 gives the discussion with the importance of the model in real life situations.

2. MODEL DESCRIPTION

We consider the infinite capacity single server queueing model in which arrival and service processes are correlated. It is assumed that time spent among all state transitions are exponentially distributed. The notations used to formulate the model mathematically, as follows:

$\lambda_0(\lambda_1)$: The faster (slow) arrival rate of customers;
μ : The mean service rate of customers;
ε : The mean dependence rate;
α : The reneging parameter;
$R(r)$: The prescribed integers for operating strategy of the system with controllable arrival rates; $(0 \le r \le R)$
$P_n(0)$: The probability that there are n customers in the system with faster rate of arrivals (λ_0).
$P_n(1)$: The probability that there are n customers in the system with slower rate of arrivals (λ_1).

The state dependent service rate of customers is defined as

$$\mu_n = (\mu - \varepsilon) + \alpha(n-1)$$

- The reneging parameter (α) is introduced in state dependent service rate of customers.

- The mean dependence rate ‘Covariance between the arrival and service processes’ (ε) is used.
- The arrival process and the service process of system are correlated and follow a bivariate poisson process.
- The faster arrival rate of customers (λ_0) decreases to slower arrival rate (λ_1), when the system size increases to R.
- The slower arrival rate of customers (λ_1) increases to faster arrival rate (λ_0), when system size decreases to r $(0 \le r < R)$. This process is repeated.
- The probability that there is no arrival and no service completion during a small interval of time Δt, when the system is in faster rate of arrivals is
  $1-\left\{\left(\lambda_0-\varepsilon\right)+(\mu-\varepsilon)+\alpha(n-1)\right\}\Delta t+o(\Delta t)$
- The probability that there is no arrival and no service completion during a small interval of time Δt, when the system is in slower rate of arrivals is
  $1-\left\{\left(\lambda_1-\varepsilon\right)+(\mu-\varepsilon)+\alpha(n-1)\right\}\Delta t+o(\Delta t)$
- The probability that there is one arrival and no service completion during a small interval of time Δt , when the system is in faster rate of arrivals is
  $\left(\lambda_0-\varepsilon\right)\ \Delta t+o(\Delta t)$
- The probability that there is one arrival and no service completion during a small interval of time Δt, when the system is in slower rate of arrivals is
  $\left(\lambda_1-\varepsilon\right)\ \Delta t+o(\Delta t)$
- The probability that there is no arrival and one service completion during a small interval of time Δt, when the system is either faster or in slower rate of arrivals is
  $\left\{(\mu-\varepsilon)+\alpha(n-1)\right\}\ \Delta t+o(\Delta t)$
- The probability that there is one arrival and one service completion during a small interval of time Δt, when the system is either faster or in slower rate of arrivals, is $o(\Delta t)$.

3. STEADY STATE EQUATIONS AND SOLUTION

In this section, we consider the steady state equations and obtain the solutions by using the generating function technique for this model.

$$-\left(\lambda_0-\varepsilon\right)P_0(0)+\left(\mu-\varepsilon\right)P_1(0)=0;\quad n=0 \tag{3.1}$$

$$-\left\{\left(\mu+\lambda_0-2\varepsilon\right)+\alpha\left(n-1\right)\right\}P_n(0)+\left\{\left(\mu-\varepsilon\right)+\alpha n\right\}P_{n+1}(0)+\left(\lambda_0-\varepsilon\right)P_{n-1}(0)=0;\quad 1\le n<r \tag{3.2}$$

$$-\left\{\left(\mu+\lambda_0-2\varepsilon\right)+\alpha\left(r-1\right)\right\}P_r(0)+\left\{\left(\mu-\varepsilon\right)+\alpha r\right\}P_{r+1}(0)+\left(\lambda_0-\varepsilon\right)P_{r-1}(0)+\left\{\left(\mu-\varepsilon\right)+\alpha r\right\}P_{r+1}(1)=0;\quad n=r \tag{3.3}$$

$$-\left\{\left(\mu+\lambda_0-2\varepsilon\right)+\alpha\left(n-1\right)\right\}P_n(0)+\left\{\left(\mu-\varepsilon\right)+\alpha n\right\}P_{n+1}(0)+\left(\lambda_0-\varepsilon\right)P_{n+1}(0)=0;$$
$$n=r+1,\ldots,R-2 \tag{3.4}$$

$$-\left\{\left(\mu+\lambda_0-2\varepsilon\right)+\alpha\left(R-2\right)\right\}P_{R-1}(0)+(\lambda_0-\varepsilon)P_{R-2}(0)=0;\quad n=R-1 \tag{3.5}$$

$$-\left\{\left(\mu+\lambda_1-2\varepsilon\right)+\alpha r\right\}P_{r+1}(1)+\left\{\left(\mu-\varepsilon\right)+\alpha(r+1)\right\}P_{r+2}(1)=0;\quad n=r+1 \tag{3.6}$$

$$-\left\{\left(\mu+\lambda_1-2\varepsilon\right)+\alpha\left(n-1\right)\right\}P_n(1)+\left\{\left(\mu-\varepsilon\right)+\alpha n\right\}P_{n+1}(1)+\left(\lambda_1-\varepsilon\right)P_{n-1}(1)=0;$$
$$n=r+2,\ldots,R-1 \tag{3.7}$$

$$-\left\{\left(\mu+\lambda_1-2\varepsilon\right)+\alpha\left(R-1\right)\right\}P_R(1)+\left\{\left(\mu-\varepsilon\right)+\alpha R\right\}P_{R+1}(1)+(\lambda_1-\varepsilon)P_{R-1}(1)$$
$$+(\lambda_0-\varepsilon)P_{R-1}(0)=0 \tag{3.8}$$

$$-\left\{\left(\mu+\lambda_1-2\varepsilon\right)+\alpha\left(n-1\right)\right\}P_n(1)+\left\{\left(\mu-\varepsilon\right)+\alpha n\right\}P_{n+1}(1)+\left(\lambda_1-\varepsilon\right)P_{n-1}(1)=0;\; n\geq R+1 \tag{3.9}$$

By using the generating function technique, the steady state solution of equations (3.1)-(3.9) is obtained as

$$G(z)=\left[\begin{array}{l}\left\{\begin{array}{l}\left(\left(\mu-\varepsilon\right)+\alpha\left(-1\right)\right)(1-z)P_0(0)\\ +(\lambda_0-\varepsilon)z^{R+1}P_{R-1}(0)-\left(\left(\mu-\varepsilon\right)+\alpha r\right)z^{r+1}P_{r+1}(1)\end{array}\right\}\\ \left\{(\lambda_0-\varepsilon)z^2-(\mu+\lambda_0-2\varepsilon)z+(\mu-\varepsilon)\right\}^{-1}\end{array}\right]+$$

$$\left[\begin{array}{l}\left\{\left(\left(\mu-\varepsilon\right)+\alpha r\right)z^{r+1}P_{r+1}(1)+(\lambda_0-\varepsilon)z^{R+1}P_{R-1}(0)\right\}\\ \left\{(\lambda_1-\varepsilon)z^2-(\mu+\lambda_1-2\varepsilon)z+(\mu-\varepsilon)\right\}^{-1}\end{array}\right];\quad 0<\varepsilon<\lambda_0(\lambda_1)<\mu,\left|z\right|<1.$$

On collecting the coefficients of appropriate power of z in $G(z)$, the system size distribution can be obtained as

$$P_n(0)=(\beta_r)^n P_0(0);\quad n=0,1,\ldots,r$$
$$=(\beta_r)^n\left[(\beta_r)^n-(\beta_r)^R\right]MP_0(0);\; n=r,r+1,\ldots,R-1.$$

$$P_n(1)=\left[1-\left(\Delta_r\right)^{n-r}\right]\left(\frac{1}{H_r}\right)^R P_{r+1}(1);\quad n=r+1,\ldots R$$
$$=\left[1-\left(\delta_r\right)^{R-r}\right]\left[\frac{(\lambda_1-\varepsilon)^{n-r}(C_r)^{R-n}}{(B_r)^{R-r}}\right]P_{r+1}(1);\quad n\geq R$$

where

$$M = \left[(\beta_r)^r - (\beta_r)^R\right]^{-1}$$

$$P_{r+1}(1) = (\beta_r)^{R+r}\left(\frac{1}{G_r}\right)MP_0(0)$$

The probability that system is empty can be obtained as

$$P_0(0) = \left[G_r - (R-r)(\beta_r)^{R+r}(\gamma_r)^{R+r}M\right]^{-1}$$

where

$$A_r = (\mu-\varepsilon)+\alpha(r-1);\;\; B_r = (\mu-\lambda_1)+\alpha r;\;\; C_r = (\mu-\varepsilon)+\alpha r;\;\; D_r = (\mu-\lambda_0)+\alpha(r-1)\,.$$

$$G_r = \frac{A_r}{D_r};\;\; H_r = \frac{B_r}{C_r};\;\; \beta_r = \left(\frac{\lambda_0-\varepsilon}{A_r}\right);\;\; \gamma_r = \left(\frac{\lambda_0-\lambda_1}{B_r}\right);\;\; \delta_r = \left(\frac{\lambda_1-\varepsilon}{C_r}\right);\;\; \Delta_r = \left(\frac{\lambda_1-\varepsilon}{B_r}\right).$$

4. PERFORMANCE MEASURES

In this section, we discuss the various performance measures in terms of queue size distribution.

1. System probability in state (i)

$$P(i) = \sum_{n=0}^{\infty} P_n(i);\;\; i = 0,1.$$

$$P(0) = \left[G_r - (R-r)(\beta_r)^{R+r}M\right]P_0(0);$$

$$P(1) = (R-r)\left(\frac{1}{H_r}\right)P_{r+1}(1)$$

2. Probability that the units in the system lies between r and R

$$P(r \le n \le R) = (\beta_r)^r\left[G_n - (\beta_n)^R M\right]\left[(R-r)(\gamma_n) + \left(\frac{D_n}{B_n}\right)\sum_{n=1}^{R-r}(\delta_n)^n\right]P_0(0)$$

3. Average number of customers in the system

$$L_s = L_{s_0} + L_{s_1}$$

with

$$L_{s_0} = \sum_{n=0}^{r} nP_n(0) + \sum_{n=r+1}^{R-1} nP_n(0)$$

$$L_{s_1} = \sum_{n=r+1}^{R-1} nP_n(1) + \sum_{n=R}^{\infty} nP_n(1)$$

On putting the values of $P_n(0)$ and $P_n(1)$ in above expressions, we get

$$L_{s_0} = \left[(\lambda_0 - \varepsilon)\left(\frac{G_r}{D_r}\right) - \left[R - (\beta_r)r\right]\left[(\beta_r)^{R+r} MG_r\right] - \frac{1}{2}(R+r)(R-r-1)(\beta_r)^{R+r} M \right] P_0(0)$$

$$L_{s_1} = \left(\frac{P_{r+1}(1)}{H_r}\right)\left[\frac{1}{2}(R+r)(R-r-1) + (R-\delta_r r)\left(\frac{1}{H_r}\right)\right]$$

4. Expected waiting of customer in the system

$$W_s = \frac{L_s}{\bar{\lambda}}$$

where

$\bar{\lambda}$ (actual mean arrival rate of system) = $\lambda_0 P(0) + \lambda_1 P(1)$

On putting the values of $P(0)$ and $P(1)$ in above expression, we get

$$\bar{\lambda} = \lambda_0 \left[G_r - (R-r)(\beta_r)^{R+r} M \right] P_0(0) + \lambda_1 (R-r)\left(\frac{1}{H_r}\right) P_{r+1}(1)$$

5. SPECIAL CASES

1. When $\varepsilon = 0$, $\alpha = 0$, the present model reduces to the single server poisson queueing model with controllable arrival rates without interdependence and without reneging.
2. When $\lambda_0 = \lambda = \lambda_1$, $\varepsilon = \alpha \neq 0$, this model reduces to $M/M/1$ interdependence model with reneging.
3. When $\lambda_0 = \lambda_1$, $\varepsilon = 0$, $\alpha = 0$, this model reduces to single server poisson queueing model without interdependence and without reneging.
4. When $\alpha = 0$, It reduces to $M/M/1$ interdependent queueing model with controllable arrival rates.
5. When $\alpha \neq 0$, $\varepsilon = 0$, It reduces to $M/M/1$ model with reneging.

6. NUMERICAL ILLUSTRATION

In this section, we have computed and tabulated probabilities $P_0(0)$, $P(0)$ and $P(1)$, for various values of different parameters $\lambda_0, \lambda_1, \mu, \varepsilon, r, R, \alpha$ in Table-1. The average number of customers in the system (L_s) and average waiting time (W_s) are tabulated in Table-2. It is observed that

- ❖ $P(0)$ is decreasing when arrival rates λ_0, λ_1 are increasing and other parameters remain constant. It means, when arrival rate of customers increase, the system probability transform from state 0 to 1 is increasing.
- ❖ $P(0)$ is increasing function of service rate of customers (μ), when other parameters remain fixed.
- ❖ $P(0)$ is also increasing function of mean dependence rate (ε) for fixed values of other parameters.

- When reneging parameter (α) increases the probability of state 0 increases.
- When mean dependence rate (ε) increases, the average number of customers in the system (L_s) and average waiting time in the system (W_s) decrease for other fixed parameters.
- When service rate (μ) increases, the average number of customers in the system (L_s) and average waiting time in the system (W_s) decrease for other fixed parameters.
- When arrival rates (λ_0, λ_1) increase, the average number of customers in the system (L_s) and average waiting time in the system (W_s) increase for other fixed parameters.
- When reneging parameter (α) increases, the average number of customers in the system (L_s) and average waiting time in the system (W_s) decrease for other fixed parameters.

7. DISCUSSION

In this paper, we have obtained the steady state solutions of the model by taking controllable arrival rate of customers. The expected number of customers in the system and average waiting times are calculated analytically and verified by numerical illustrations. Such models are much useful in data voice transmission, production process, computer, communication systems etc., where the arrival and service processes are to be made interdependent and controllable in order to have optimal operating policies. Such systems and many others including the multiprocessor systems, which are designed for highly efficient working and systems consisting of a network of processors with various system configurations and requirements in distributions and service sectors. These models help us in study of real life situations in which discouragement factors affect the smooth functioning of system.

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Table-1

Computations of probabilities $P_0(0)$, $P(0)$ and $P(1)$

r	R	λ_0	λ_1	μ	ε	α	$P_0(0)$	$P(0)$	$P(1)$
8	11	9	2	11	.5	.6	.4217	.9963	.0017
8	12	9	2	11	.5	.6	.4218	.9970	.0013
8	12	11	5	12	.5	.8	.3860	.9947	.0027
8	12	10	4	12	.5	.8	.4444	.9982	.0010
9	12	9	2	11	.5	.6	.4443	.9986	.0007
9	12	9	2	12	.5	.6	.4785	.9993	.0003
9	12	9	2	12	.5	.7	.5029	.9996	.0002
9	12	9	2	12	.6	.7	.5059	.9997	.0002

Table-2
Computations of average number of customers in the system (L_s) and average waiting time (W_s)

r	R	λ_0	λ_1	μ	ε	α	L_s	$\overline{\lambda}$	W_s
8	11	9	2	11	.5	.6	1.3217	8.9701	.1473
8	12	9	2	11	.5	.6	1.3386	8.9756	.1491
8	12	11	5	12	.5	.8	1.5566	10.9552	.1421
8	12	10	4	12	.5	.8	1.2381	9.9860	.1240
9	12	9	2	11	.5	.6	1.2406	8.9888	.1380
9	12	9	2	12	.5	.6	1.0860	8.9943	.1207
9	12	9	2	12	.5	.7	0.9863	8.9968	.1096
9	12	9	2	12	.6	.7	0.9745	8.9967	.1083

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