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MODEL-BASED VARIANCE OF MODIFIED MURTHY ESTIMATOR

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ABSTRACT

In this paper expected model-based variance of the modified Murthy (1957) estimator given by Shahbaz (2004) has been obtained by using the super population model. Empirical study has also been carried out to see the performance of the modified Murthy (1957) estimator under super–population model.

KEY WORDS

Unequal Probability Sampling, Model based variance.

1. INTRODUCTION

Horvitz and Thompson (1952) were the first who developed the theoretical ground and provide following estimator of population total:

$$y'_{HT} = \sum_{i\in S} \frac{y_i}{\pi_i} \tag{1.1}$$

The designed based variance is

$$Var\left(y'_{HT}\right) = \sum_{i=1}^{N} \frac{\left(1-\pi_i\right)}{\pi_i} Y_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}} \frac{\left(\pi_{ij} - \pi_i \pi_j\right)}{\pi_i \pi_j} Y_i Y_j \tag{1.2}$$

Sen (1953) and, independently, Yates and Grundy (1953) obtained the following model based variance of the Horvitz–Thompson (1952) estimator:

$$Var\left(y_{HT}^{/}\right)=\sum_{\substack{i=1\\ j>i}}^{N}\sum_{j=1}\left(\pi_i\pi_j-\pi_{ij}\right)\left(\frac{Y_i}{\pi_i}-\frac{Y_j}{\pi_j}\right)^2 \tag{1.3}$$

Raj (1956) has given following ordered estimator of population total. For n=2:

$$t_{mean} = \frac{1}{2}\left[\frac{y_i}{p_i}\left(1+p_i\right)+\frac{y_i}{p_i}\left(1-p_j\right)\right] \tag{1.4}$$

The variance of (1.4) is

$$Var\left(t_{mean}\right)=\frac{1}{8}\sum_{\substack{i=1\\ j\neq i}}^{N}\sum_{j=1}P_iP_j\left(2-P_i-P_j\right)\left(\frac{Y_i}{P_i}-\frac{Y_j}{P_j}\right)^2 \tag{1.5}$$

Murthy (1957) symmetrize the Raj (1956) estimator to obtain the following unbiased estimator of population total:

$$t_{symm}=\frac{1}{2-p_i-p_j}\left[\frac{y_i}{p_i}\left(1-p_j\right)+\frac{y_j}{p_j}\left(1-p_i\right)\right] \tag{1.6}$$

The variance of (1.5) is

$$Var\left(t_{symm}\right)=\frac{1}{2}\sum_{\substack{i=1\\ j\neq i}}^{N}\sum_{j=1}\frac{P_iP_j\left(1-P_i-P_j\right)}{2-P_i-P_j}\cdot\left(\frac{Y_i}{P_i}-\frac{Y_j}{P_j}\right)^2 \tag{1.7}$$

Shahbaz (2004) obtained the following modified Murthy estimator by using the Durbin (1953) selection procedure in the general Murthy (1957) estimator:

$$t_{MM(D)}=\frac{1}{2}\left[\frac{y_i}{p_i}+\frac{y_j}{p_j}\right] \tag{1.8}$$

The design based variance of (1.8) obtained by Shahbaz (2004) is:

$$Var\left(t_{MM(D)}\right)=\frac{1}{2}\sum_{\substack{i=1\\ j\neq i}}^{N}\sum_{j=1}P_iP_j\left[1-\frac{\left(1-P_i-P_j\right)}{k\left(1-2P_i\right)\left(1-2P_j\right)}\right]\cdot\left\{\frac{Y_i}{P_i}-\frac{Y_j}{P_j}\right\}^2 \tag{1.9}$$

Under the linear stochastic model:

$$\left.\begin{array}{ll} & y_i = \beta z_i + \varepsilon_i \\ \textit{with} & E\left(\varepsilon_i\right) = 0 \\ & E\left(\varepsilon_i, \varepsilon_j\right) = \begin{cases} \sigma_i^2 & i = j \\ 0 & \textit{otherwise} \end{cases} \\ \textit{where} & \sigma_i^2 = \sigma^2 z_i^{2\gamma} \qquad \tfrac{1}{2} \le \gamma \le 1 \end{array}\right\} \tag{1.10}$$

Bayless and Rao (1970) have shown that the expected variance of $y_{HT}^{/}$ is:

$$E_M\left[Var\left(y_{HT}^{/}\right)\right] = \sigma^2\left(\frac{Z}{n}\right)^{2\gamma} \sum_{I=1}^{N}\left(1-\pi_I\right)\pi_I^{2\gamma-1} \tag{1.11}$$

Rao and Bayless (1969) has obtained following model based variance of the Raj(1956) estimator under model

$$E_M\left[Var\left(t_{mean}\right)\right] = \frac{\sigma^2}{4}\left(\frac{Z}{2}\right)^{2\gamma} \sum_{\substack{I=1 \\ J \neq I}}^{N}\sum_{J=1} \pi_I^{2\gamma-1}\pi_J\left(4-\pi_I-\pi_J\right) \tag{1.12}$$

The expected variance of Murthy’s variance under model obtained by Rao and Bayless (1969), is:

$$E_M\left[Var\left(t_{symm}\right)\right] = \sigma^2\left(\frac{Z}{2}\right)^{2\gamma} \sum_{\substack{I=1 \\ J \neq I}}^{N}\sum_{J=1} \pi_I^{2\gamma-1}\pi_J\left(\frac{2-\pi_I-\pi_J}{4-\pi_I-\pi_J}\right) \tag{1.13}$$

Rao and Bayless (1969) and Bayless and Rao (1970) have conducted extensive empirical study for the expected variances given above.

2. THE EXPECTED VARIANCE

In this section the expected variance of (1.9) under the model (1.10) has been obtained. For this write $\pi_I = 2P_I$, in equation (1.9):

$$Var\left(t_{MM(D)}\right) = \frac{1}{2}\sum_{\substack{I=1 \\ J \neq I}}^{N}\sum_{J=1} \pi_I\pi_J\left[1-\frac{2-\pi_I-\pi_J}{2k\left(1-\pi_I\right)\left(1-\pi_J\right)}\right]\left(\frac{Y_I}{\pi_I}-\frac{Y_J}{\pi_I}\right)^2 \tag{2.1}$$

Now the expected variance is:

$$E_M\left[Var\left(t_{MM(D)}\right)\right] = E_M\left[\frac{1}{2}\sum_{\substack{I=1 \\ J \neq I}}^{N}\sum_{J=1} \pi_I\pi_J\left\{1-\frac{2-\pi_I-\pi_J}{2k\left(1-\pi_I\right)\left(1-\pi_J\right)}\right\}\left(\frac{Y_I}{\pi_I}-\frac{Y_J}{\pi_I}\right)^2\right]$$

Substituting the values of Y_I and Y_J from (1.10) we have:

$$E_M\left[Var\left(t_{MM(D)}\right)\right]=E_M\left[\frac{1}{2}\sum_{\substack{I=1\\J\neq I}}^{N}\sum_{J=1}\pi_I\pi_J\left\{1-\frac{2-\pi_I-\pi_J}{2k\left(1-\pi_I\right)\left(1-\pi_J\right)}\right\}\left(\frac{\varepsilon_I}{\pi_I}-\frac{\varepsilon_J}{\pi_I}\right)^2\right]$$

Opening the square and applying model expectation we have:

$$E_M\left[Var\left(t_{MM(D)}\right)\right]=\frac{1}{2}\sum_{\substack{I=1\\J\neq I}}^{N}\sum_{J=1}\pi_I\pi_J\left\{1-\frac{2-\pi_I-\pi_J}{2k\left(1-\pi_I\right)\left(1-\pi_J\right)}\right\}\left(\frac{\sigma_I^2}{\pi_I^2}+\frac{\sigma_J^2}{\pi_J^2}\right)$$

$$=\frac{\sigma^2}{2}\left(\frac{Z}{2}\right)^{2\gamma}\sum_{\substack{I=1\\J\neq I}}^{N}\sum_{J=1}\pi_I\pi_J\left\{1-\frac{2-\pi_I-\pi_J}{2k\left(1-\pi_I\right)\left(1-\pi_J\right)}\right\}\left(\pi_I^{2\gamma-2}+\pi_J^{2\gamma-2}\right)$$

$$=\sigma^2\left(\frac{Z}{2}\right)^{2\gamma}\sum_{\substack{I=1\\J\neq I}}^{N}\sum_{J=1}\pi_I^{2\gamma-1}\pi_J\left\{1-\frac{2-\pi_I-\pi_J}{2k\left(1-\pi_I\right)\left(1-\pi_J\right)}\right\}$$

$$=\sigma^2\left(\frac{Z}{2}\right)^{2\gamma}\left[\sum_{I=1}^{N}\pi_I^{2\gamma-1}\left(2-\pi_I\right)-\sum_{I=1}^{N}\pi_I^{2\gamma-1}\right]$$

$$E_M\left[Var\left(t_{MM(D)}\right)\right]=\sigma^2\left(\frac{Z}{2}\right)^{2\gamma}\sum_{I=1}^{N}\pi_I^{2\gamma-1}\left(1-\pi_I\right) \tag{2.2}$$

Now comparing (2.2) with (1.11) we can see that the modified Murthy (1957) estimator given by Shahbaz (2004) has same model based variance as the model based variance of Horvitz and Thompson (1952) estimator under exact probability proportional to size selection procedures.

3. EMPIRICAL STUDY

In this section we have conducted the empirical study of the expected variances of various estimators. For this empirical study we have used ten populations. The expected variances given in section 2 have been calculated for value of the parameter γ ranging from 0.5 to 1.0. The expected variances have been given in Table–1. Looking at the expected variances given in Table–1 we can readily see that for all choices of the constant γ, the Murthy (1957) estimator outperform all other estimators involved in the study expect for $\gamma = 1.0$ in which case the modified Murthy (1957) estimator given by Shahbaz (2004) outperform other estimators involved in the study. Further, we can see that the Raj (1956) estimator has worst performance for all the values of γ as this estimator has largest expected variance in all the populations.

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Table-1

	γ = 0.5				γ = 0.6			
Pop	**HT**	**Raj**	**Murthy**	**New**	**HT**	**Raj**	**Murthy**	**New**
1.	10.00	22.00	9.40	10.00	11.26	25.39	10.77	11.26
2.	9.00	19.72	8.37	9.00	9.90	22.31	9.41	9.90
3.	15.00	31.40	14.41	15.00	16.43	34.88	15.94	16.43
4.	30.00	62.50	28.52	30.00	36.96	78.50	35.63	36.96
5.	50.00	103.14	49.11	50.00	64.93	134.71	64.06	64.93
6.	40.00	82.00	38.85	40.00	48.62	100.70	47.59	48.62
7.	90.00	181.80	88.55	90.00	113.38	230.25	112.03	113.38
8.	200.00	400.20	196.70	200.00	265.07	533.09	261.79	265.07
9.	220.00	439.50	215.87	220.00	295.39	593.52	291.22	295.39
10.	780.00	1552.83	770.66	780.00	1130.99	2259.48	1120.85	1130.99

Table–1 (Continued)

	γ = 0.7				γ = 0.8			
Pop	**HT**	**Raj**	**Murthy**	**New**	**HT**	**Raj**	**Murthy**	**New**
1.	12.80	29.63	12.48	12.80	14.71	34.93	14.62	14.71
2.	10.97	25.46	10.69	10.97	12.23	29.33	12.24	12.23
3.	18.17	39.12	17.82	18.17	20.28	44.30	20.11	20.28
4.	46.36	100.34	45.32	46.36	59.09	130.23	58.56	59.09
5.	84.82	176.97	84.06	84.82	111.43	233.82	110.93	111.43
6.	59.90	125.28	59.06	59.90	74.71	157.69	74.17	74.71
7.	144.58	295.04	143.42	144.58	186.35	381.98	185.53	186.35
8.	356.89	721.16	353.86	356.89	487.22	988.86	484.85	487.22
9.	403.90	815.99	400.00	403.90	561.20	1139.53	558.12	561.20
10.	1673.88	3354.83	1663.54	1673.88	2520.85	5067.09	2511.65	2520.85

Table–1 (Continued)

	γ = 0.9				γ = 1.0			
Pop	**HT**	**Raj**	**Murthy**	**New**	**HT**	**Raj**	**Murthy**	**New**
1.	17.07	41.60	17.30	17.07	20.00	50.00	20.67	20.00
2.	13.72	34.10	14.16	13.72	15.50	40.00	16.52	15.50
3.	22.86	50.64	22.90	22.86	26.00	58.40	26.33	26.00
4.	76.40	171.32	76.72	76.40	100.00	228.00	101.74	100.00
5.	147.20	310.62	147.20	147.20	195.50	414.80	196.35	195.50
6.	94.21	200.57	94.16	94.21	120.00	257.50	120.67	120.00
7.	242.44	499.01	242.18	242.44	318.00	657.00	318.63	318.00
8.	673.21	1371.97	672.25	673.21	940.00	1923.00	941.66	940.00
9.	790.69	1613.04	789.40	790.69	1127.50	2310.00	1129.68	1127.50
10.	3852.36	7763.90	3847.13	3852.36	5960.00	12040.00	5964.22	5960.00