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DISTRIBUTION OF THE RATIO OF GENERALIZED UNIFORM VARIATES

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ABSTRACT

In this paper the distribution and moments of the ratio of independent generalized uniform variates are considered. An application of the distribution is also given.

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1. INTRODUCTION

A generalized uniform distribution is given by

$$f(x;\alpha,\theta) = \frac{\alpha+1}{\theta^{\alpha+1}} x^{\alpha}, \quad 0 < x < \theta, \tag{1.1}$$

where $\alpha>-1$ is the shape parameter and $\theta>0$ is the threshold parameter. The generalized uniform distribution is a uniform distribution if $\alpha=0$ and is a standard power function distribution if $\theta=1$. The density function (1.1) is decreasing if $-1<\alpha<0$, constant if $\alpha=0$, and decreasing if $\alpha>0$.

Proctor (1987) introduced the four parameter generalized uniform distribution, which is a counterpart to Burr type XII distribution. Tiwari et al. (1996) studied Bayes estimation of parameters of a Pareto distribution using the generalized uniform distribution. Lee (2000) studied the MLE and the UMVUE of parameters of the generalized uniform distribution. Dixit, Ali, and Woo (2002) studied estimation of parameters of the generalized uniform distribution in the presence of outliers.

The distribution of the ratio of independent gamma variates with shape parameters equal to 1 was studied by Bowman and Shenton (1998). Ali and Woo (2002) have studied the distribution of the ratio of standard power variates. The problem arises in from a model of ionic current fluctuations in biological membranes.

Here the distribution of ratio V=X/(X+Y) of two independent generalized uniform random variables X and Y each with two parameters are considered. Moments of this distribution are also obtained.

2. DISTRIBUTION OF THE RATIO

Let the independent random variables X and Y have generalized uniform distributions with parameters (α_x,θ_x) and (α_y,θ_y), respectively. Let

$$V = \frac{X}{X+Y}$$

U = X+Y.

Then the joint pdf of U and V is

$$f_{U,V}(u,v) = fx(uv)f_Y(u(1-v)u$$

where $0<uv<\theta_x$, $0<u(1-v)<\theta_y$, $0<v<1$, and f_X and f_Y are the pdf's of X and Y, respectively.

Fact 1: The pdf of V can be obtained as follows:

$$f\left(v;\alpha_x,\alpha_y,\frac{\theta_y}{\theta_x}\right)=\begin{cases}\dfrac{(\alpha_x+1)(\alpha_y+1)}{\alpha_x+\alpha_y+2}\left(\dfrac{\theta_y}{\theta_x}\right)^{\alpha_x+1} v^{\alpha_x}(1-v)^{-\alpha_x-1}, \text{ if } 0<\text{y}<\dfrac{1}{1+\dfrac{\theta_y}{\theta_x}} \\[2ex] \dfrac{(\alpha_x+1)(\alpha_y+1)}{\alpha_x+\alpha_y+2}\left(\dfrac{\theta_x}{\theta_y}\right)^{\alpha_y+1} v^{-\alpha_y-2}(1-v)^{\alpha_y}, \text{ if } \dfrac{1}{1+\dfrac{\theta_y}{\theta_x}}\le v<1.\end{cases} \quad (2.1)$$

The cdf of V is given by

$$F\left(v;\alpha_x,\alpha_y,\frac{\theta_y}{\theta_x}\right)=\begin{cases}\dfrac{\alpha_y+1}{\alpha_x+\alpha_y+2}\left(\dfrac{\theta_y}{\theta_x}\right)^{\alpha_x+1}\left(\dfrac{v}{1-v}\right)^{\alpha_x+1}, & \text{if } 0<v<\dfrac{1}{1+\dfrac{\theta_y}{\theta_x}}\\[2ex] 1-\dfrac{\alpha_x+1}{\alpha_x+\alpha_y+2}\left(\dfrac{\theta_x}{\theta_y}\right)^{\alpha_y+1}\left(\dfrac{1-v}{v}\right)^{\alpha_y+1}, & \text{if } \dfrac{1}{1+\dfrac{\theta_y}{\theta_x}}\le v<1.\end{cases} \quad (2.2)$$

Here if $\theta_x=\theta_y=1$, then the generalized uniform random variable becomes standard power random variable. Ali and Woo (2002) have studied the distribution and the moments of the ratio V in this situation. When $\alpha_x=\alpha_y=0$, the generalized uniform becomes a uniform distribution. In this paper we also consider the ratio V in this special situation.

Using formula 3.194(1) in Gradshteyn and Ryzhik (1965) and the density (2.1), we obtain the moments of the ratio V = X/(X+Y) as follows.

Fact 2: For a positive integer k,

$$E(V^k)=\frac{(\alpha_x+1)(\alpha_y+1)}{\alpha_x+\alpha_y+2(k+\alpha_x+1)}\left(\frac{\theta_x}{\theta_y}\right)^k H\left(k,k+\alpha_x+1;k+\alpha_x+2;-\frac{\theta_x}{\theta_y}\right) + \frac{\alpha_x+1}{\alpha_x+\alpha_y+2}H\left(k,\alpha_y+1;\alpha_y+2;-\frac{\theta_y}{\theta_x}\right) \quad (2.3)$$

where H(a,b; c; z) is the hypergeometric function in Gradshteyn and Ryzhik (1965).

Let $m_k=E(V^k)$. Then from formula 3.194(1) in Gradshteyn and Ryzhik (1965) and the density in (2.1), we can obtain the following recursion formula.

Fact 3:

$$m_{k-1}-m_k=\frac{(\alpha_x+1)(\alpha_y+1)}{(\alpha_x+\alpha_y+2)(k+\alpha_x+1)}\left(\frac{\theta_x}{\theta_y}\right)^k H\left(k,k+\alpha_x+1;k+\alpha_x+2;-\frac{\theta_x}{\theta_y}\right) + \frac{(\alpha_x+1)(\alpha_y+1)}{(\alpha_x+\alpha_y+2)(\alpha_y+2)}\frac{\theta_y}{\theta_x}H\left(k,\alpha_y+1;\alpha_y+3;-\frac{\theta_y}{\theta_x}\right),\ k=1,2,3,...$$

Since the right hand side of Fact 3 is positive, we have the inequality

$0<m_k<m_{k-1}\le 1,\ \forall k=1,2,3\ldots$

From Fact 2, especially if X and Y are two independent uniform random variables over $(0,\theta_x)$ and $(0,\theta_y)$, respectively, then from (2.3)

$$E(V^k)=\frac{1}{2(k+1)}\left(\frac{\theta_x}{\theta_y}\right)^k H\left(k,k+1;k+2;-\frac{\theta_x}{\theta_y}\right)+\frac{1}{2}H\left(k,1;2;-\frac{\theta_y}{\theta_x}\right) \tag{2.4}$$

If X and Y are two independent identical uniform random variables over (0,θ), then (2.3) becomes

$$E(V^k)=\frac{1}{2(k+1)}H\left(k,k+1;k+2;-1\right)+\frac{1}{2}H\left(k,1;2;-1\right) \tag{2.5}$$

To evaluate the numerical values of the moments using the formulas 15.1.3, 15.1.8, and 15.2.14 in Abramowitz and Stegun (1970) and formulas 9.137(11) and 9.137(12) in Gradshteyn and Ryzhik (1965), we obtain the following recursive formulas.

Fact 4:

1) H(1,1; 2; -z) = $\frac{1}{z}\ln(1+z)$.

2) H(k+1, 1;2;z) = $\frac{1}{k}\left[(1-z)^{-k}+(k-1)H(k,1;2;z)\right]$, for k=1,2,3,…

3) H(1,α+1, α+2;z) = $\frac{\alpha+1}{\alpha z}(H(1,\alpha;\alpha+1;z)-1)$, for α>0. (Recursion formula)

4) H(k+1,α+1, α+2;z) = $\frac{\alpha+1}{kz}\left[(1-z)^{-k}-H(k,\alpha;\alpha+1;z)\right]$,
   for k=1,2,3,...(recursion formula).

Proof: The formulas (1) and (2) come from the formulas 15.1.3, 15.1.8 and 15.2.14 in Abramowitz and Steguan (1970) and the formulas (3) and (4) come from 9.137(11) and 9.137(12) in Gradshteyn and Ryzhik (1965) and the formulas 15.1.8 in Abramowitz and Stegaun (1970).

Let X and Y be two independent uniform random variables over $(0,\theta_x)$ and $(0,\theta_y)$, respectively. Then from the moment (2.4) and Fact 2, we can obtain the following first and second moments of the ratio V=X/(X+Y).

Fact 5:

$$E(V)=\frac{1}{2}\left[1-\frac{\theta_x}{\theta_y}\ln\left(1+\frac{\theta_x}{\theta_y}\right)+\frac{\theta_x}{\theta_y}\ln\left(1+\frac{\theta_y}{\theta_x}\right)\right], \text{ and}$$

$$E(V^2)=1-\frac{\theta_y}{\theta_x}\ln\left(1+\frac{\theta_x}{\theta_y}\right)-\frac{1}{2}.\frac{\theta_x}{\theta_y}\left(1+\frac{\theta_x}{\theta_y}\right)^{-1}+\frac{1}{2}\left(1+\frac{\theta_y}{\theta_x}\right)^{-1}. \tag{2.6}$$

Especially, if X and Y are two independent identical uniform random variables over $(0,\theta)$, then the first and second moments of V=X/(X+Y) becomes E(V)=0.5, and Var(V)=0.75-ln 2.

When α_x and α_y are not integers, from Fact 2, Fact 3, and formulas (3) and (4) in Fact 4 and formula 15.3.1 in Abramowitz and Stegun (1970), we can obtain the following numerical expectations and variances of ratio V as in Table 1.

Table 1:
Numerical means and variances (in parentheses) of the ratio V=X/(X+Y) when X~Uniform (α_x,θ_x) and Y~Uniform (α_x,θ_x).

(α_x,α_y)	$\theta_x=\theta_y=1$	$\theta_x=0.5,\ \theta_y=0.25$
(-0.25, -0.5)	0.57876 (0.02403)	0.68222 (0.01993)
(1,1)	0.50003 (0.05172)	0.65283 (0.04283)
(0.25, 0.5)	0.47532 (0.04437)	0.61891 (0.03931)
(2.5, 3.25)	0.48818 (0.05870)	0.65134 (0.04860)

3. APPLICATION

In reliability studies, the problem of testing stress-strength reliability, viz. P(X<Y), where X and Y denote respectively the stress and strength, is of great importance. If X and Y be independent random variables with X~Uniform (α_x,θ_x) and Y~Uniform (α_y,θ_y), where α_x and α_y denote the shape parameters and θ_x and θ_y the threshold parameters, then

R = P(X<Y) = P(V < 0.5),

where $V=\dfrac{X}{X+Y}$, having the distribution given by (2.1).

Hence, from (2.3), we have

$$R=\begin{cases}\dfrac{\alpha_y+1}{\alpha_x+\alpha_y+2}\rho^{\alpha_x+1}, & \text{if } 0<\rho<1\\[2ex] 1-\dfrac{\alpha_x+1}{\alpha_x+\alpha_y+2}\rho^{-(\alpha_y+1)}, & \text{if } \rho\le 1,\ \rho\equiv\dfrac{\theta_y}{\theta_x}\end{cases}$$

Let α_x and α_y be known.

Suppose we want to test H_0 : $R=R_0$. Since R is a monotone increasing function of ρ, H_0 is equivalent to $H_0^*:\rho=\rho_0$, where

$$\rho_0 = \begin{cases} \left(\dfrac{\alpha_x + \alpha_y + 2}{\alpha_y + 1} R_0 \right)^{\frac{1}{\alpha_x+1}}, & \text{if } 0 < R_0 < \dfrac{\alpha_y + 1}{\alpha_x + \alpha_y + 2} \\ \left[\dfrac{\alpha_x + \alpha_y + 2}{\alpha_x + 1} (1 - R_0) \right]^{-\frac{1}{\alpha_y+1}}, & \text{if } R_0 \geq \dfrac{\alpha_y + 1}{\alpha_x + \alpha_y + 2}. \end{cases}$$

Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, X_n$ denote respectively random samples of sizes m and n drawn from the distributions of X and Y. Let $X_{(1)}, X_{(2)}, \ldots, X_{(m)}$ and $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ denote the corresponding order statistics. Then, $X_{(m)}$ and $Y_{(n)}$ are respectively complete sufficient statistics for θ_x and θ_y, and, from Ali et al. (2005), these are independently distributed with $X_{(m)} \sim Uniform(\alpha_x^*, \theta_x)$ and $Y_{(n)} \sim Uniform(\alpha_y^*, \theta_y)$, where $\alpha_x^* = m(\alpha_x + 1) - 1$, $\alpha_y^* = m(\alpha_y + 1) - 1$. So, from (2.1), $V_{m,n} = X_{(m)}/(X_{(m)} + Y_{(n)})$ is distributed with pdf $f(v; \alpha_x^*, \alpha_y^*, \rho)$.

A test for H_0 based on $V_{m,n}$ can be constructed. We shall try to find the UMP test for one-sided alternatives among the class of tests based on $V_{m,n}$.

Case (i): Alternative hypothesis is $H_1 : R > R_0$, which is equivalent to $H_1^* : \rho > \rho_0$.

We first find the most powerful test (MPT) against $H_1^* : \rho = \rho_1 (> \rho_0)$. The critical region of the MPT is given by

$$W_1 = \{v_{m,n} \mid \lambda(v_{m,n}) > k\},$$

where k is determined from the size condition and

$$\lambda(v_{m,n}) = \frac{f(v_{m,n;}\rho_1)}{f(v_{m,n;}\rho_0)} = \begin{cases} \left(\dfrac{\rho_1}{\rho_0} \right)^{\alpha_x^*+1} & \text{if } 0 < v_{m,n} < \dfrac{1}{1+\rho_1} \\ \dfrac{1}{\rho_x^* + 1} \left(\dfrac{1 - v_{m,n}}{v_{m,n}} \right)^{\alpha_x^* + \alpha_y^* + 2} & \text{if } 0 < v_{m,n} < \dfrac{1}{1+\rho} \\ \left(\dfrac{\rho_0}{\rho_1} \right)^{\alpha_y^*+1} & \text{if } \dfrac{1}{1+\rho_0} < v_{m,n} < 1, \end{cases}$$

which is clearly a non-increasing function of $v_{m,n}$. Hence,

$$\lambda(v_{m,n}) > k \Leftrightarrow V_{m,n} < v_0,$$

where v_0 is such that the size condition is satisfied, i.e.,

$P(V_{m,n}) < v_0 \mid H_0) = \alpha$ (specified).

If $F\left(\frac{1}{1+\rho_0};\alpha_x^*,\alpha_y^*,\rho_0\right)=\frac{\alpha_y^*+1}{\alpha_x^*+\alpha_y^*+2}>\alpha,$

$$\text{then } v_0 \ni \frac{1-v_0}{v_0}=\frac{1}{\rho_0}\left(\frac{\alpha_x^*+\alpha_y^*+2}{\alpha_y^*+1}\right)^{\frac{1}{\alpha_x^*+1}}.$$

If $F\left(\frac{1}{1+\rho_0};\alpha_x^*,\alpha_y^*,\rho_0\right)=\frac{\alpha_y^*+1}{\alpha_x^*+\alpha_y^*+2}<\alpha,$

$$\text{then } v_0 \ni \frac{v_0}{1-v_0}=\rho_0\left[\frac{\alpha_x^*+\alpha_y^*+2}{\alpha_x^*+1}(1-\alpha)\right]^{\frac{1}{\alpha_y^*+1}}.$$

As v_0 is independent of ρ_1, W_1 is the critical region of the UMP test for testing $H_0 : R = R_0$ against $H_1 : R > R_0$.

The power function of the test is given by

$$P_1(\rho)=\begin{cases}\alpha\left(\frac{\rho}{\rho_0}\right)^{\alpha_x^*+1}, & \text{if } 0<v_0<\frac{1}{1+\rho}\\ 1-(1-\alpha)\left(\frac{\rho_0}{\rho}\right)^{\alpha_y^*+1}, & \text{if } \frac{1}{1+\rho}\le v_0<1.\end{cases}$$

Case (ii): Alternative hypothesis is $H_2 : R<R_0$, which is equivalent to $H_2^* : \rho<\rho_0$.

We first find the most powerful test (MPT) for $H_1^* : \rho=\rho_0$ against $H_2^* : \rho_1(<\rho_0)$. The critical region of the MPT is given by

$W_2 = \{v_{m,n} \mid \lambda(v_{m,n}) > k\},$

where k is determined from the size condition and

$$\lambda(v_{m,n}) = \frac{f(v_{m,n};\rho_1)}{f(v_{m,n};\rho_0)} = \begin{cases} \left(\dfrac{\rho_1}{\rho_0}\right)^{\alpha_x^*+1} & \text{if } 0 < \mathrm{v}_{\mathrm{m,n}} < \dfrac{1}{1+\rho_0} \\ \rho_0^{\alpha_y^*+1}\rho_1^{\alpha_x^*+1}\left(\dfrac{v_{m,n}}{1-v_{m,n}}\right)^{\alpha_x^*+\alpha_y^*+2} & \text{if } \dfrac{1}{1+\rho_0} < \mathrm{v}_{\mathrm{m,n}} < \dfrac{1}{1+\rho_1} \\ \left(\dfrac{\rho_0}{\rho_1}\right)^{\alpha_y^*+1} & \text{if } \dfrac{1}{1+\rho_0} < \mathrm{v}_{\mathrm{m,n}} < 1. \end{cases}$$

Clearly $\lambda(\mathrm{v}_{\mathrm{m,n}})$ is such that size condition is satisfied, i.e.,

$P(V_{m,n} > v_0 \mid H_0) = \alpha$ (specified).

If $$F\left(\frac{1}{1+\rho_0};\alpha_x^*,\alpha_y^*,\rho_0\right) = \frac{\alpha_y^*+1}{\alpha_x^*+\alpha_y^*+2} > 1-\alpha,$$

then $$v_0 \ni \frac{v_0}{1-v_0} = \frac{1}{\rho_0}\left(\frac{\alpha_x^*+\alpha_y^*+2}{\alpha_y^*+1}(1-\alpha)\right)^{\frac{1}{\alpha_x^*+1}}.$$

If $$F\left(\frac{1}{1+\rho_0};\alpha_x^*,\alpha_y^*,\rho_0\right) = \frac{\alpha_y^*+1}{\alpha_x^*+\alpha_y^*+2} < 1-\alpha,$$

then $$v_0 \ni \frac{1-v_0}{v_0} = \rho_0\left[\frac{\alpha_x^*+\alpha_y^*+2}{\alpha_x^*+1}\alpha\right]^{\frac{1}{\alpha_y^*+1}}.$$

As v_0 is independent of ρ_1, W_2 is the critical region of the UMP test for testing $H_0 : R=R_0$ against $H_1 : R<R_0$.

The power function of the test is given by

$$P_2(\rho) = \begin{cases} 1-(1-\alpha)\left(\dfrac{\rho}{\rho_0}\right)^{\alpha_x^*+1}, & \text{if } 0 < v_0 < \dfrac{1}{1+\rho} \\ \alpha\left(\dfrac{\rho_0}{\rho}\right)^{\alpha_y^*+1}, & \text{if } \dfrac{1}{1+\rho} \le v_0 < 1. \end{cases}$$

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