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AN INFORMATION APPROACH TO OPTIMAL SPACINGS FOR ESTIMATION IN LOCATION-SCALE MODELS

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ABSTRACT

In this paper, we consider the problem of simultaneous estimation of location and scale parameters based on a fixed number, say *k*, of order statistics from a random sample of size *n*. An asymptotic expression for the information matrix is obtained based on the spacing $0<u_1<u_2<\cdots<u_k<1$, where the ranks of the selected order statistics are given by $n_i = [nu_i]+1$ for i =1, 2, ..., *k*. A spacing will be considered optimal if it maximizes the smaller of the eigenvalues of the information matrix. Optimal spacings are found for the normal, logistic, and exponential distributions.

1. INTRODUCTION

The problem of optimal selection of order statistics for use in the estimation of parameters has a relatively long history. For the case of the estimation of a scalar, the optimality criterion is naturally to minimize the variance of the resulting estimator. Examples of this technique for the small sample case include Ali, Umbach, and Saleh (1982), Beyer, Moore, and Harter (1976), and Umbach, Ali, and Hassanein (1981). A small collection of references concerning asymptotic results include Ali and Umbach (1989), Chan and Cheng (1972), Kubat and Epstein (1980), Ogawa (1951), and Saleh, Ali, and Umbach (1983).

The situation where multiple parameters are to be estimated is much richer. In this case, one is presented with a choice as to the optimality criteria to be used. See Ali and Umbach (1998) for a discussion of a variety of methods that one might apply to the location-scale problem. They do not consider, however, the problem of maximizing the amount of information in the order statistics to be selected.

Arnold, Balakrishnan, and Nagaraja (1992) and Park (1996) discuss the problem of computing the amount of information in a collection of order statistics. The computational complexities for the small sample case make virtually any optimality criterion based on exact results intractable. In this work, we will develop an optimality criterion based on an asymptotic description of the amount of information in order statistics. For the location-scale problem, this presents a 2×2 information matrix. In this work, we propose a maximin procedure that chooses the k order statistics, which maximize the smaller of the two eigenvalues. This procedure is applied to the normal, logistic, and exponential distributions to explicitly obtain the optimal spacings and coefficients of the resulting estimators.

2. ASYMPTOTIC RESULTS

Let $X_1, X_2, ..., X_n$ be a random sample from a continuous location-scale family of distributions with distribution function

$$F_{\lambda,\delta}(x) = F((x-\lambda)/\delta)$$

and density function

$$f_{\lambda,\delta}(x) = f((x-\lambda)/\delta)/\delta$$

for $-\infty < \lambda < \infty$ and $0 < \delta < \infty$. Let X_{in} represent the i^{th} order statistic of the sample for i =1, 2, ..., n. For the spacing $0 < u_1 < u_2 < ... < u_k < 1$, let $n_i = [nu_i] + 1$ for i=1, 2, ..., k, where $[\cdot]$ represents the greatest integer function. Let the vector of *k* selected order statistics be

$$X = (X_{n_1:n}, X_{n_2:n}, \ldots, X_{n_k:n})'.$$

If $F^{-1}(u_i)$ is a point of continuity of *f* for i =1, 2, ..., k, then by Mosteller (1946), we have that X is asymptotically k-variate normal with mean vector

$$(\lambda + \delta F^{-1}(u_1), \lambda + \delta F^{-1}(u_2), ..., \lambda + \delta F^{-1}(u_k))'$$

and covariance matrix given by $(\delta^2/n)W$, where the $(i, j)^{th}$ and $(j, i)^{th}$ entries of W are given by

$$\frac{u_i(1-u_j)}{f(F^{-1}(u_i))f(F^{-1}(u_j))} \quad \text{with } i \le j.$$

Using this result, Ogawa (1962a) shows with the information matrix for the selected order statistics I_X given by

$$\begin{pmatrix} E\left[\left\{\frac{\partial f_{n_1,n_2,\ldots,n_k}(x_1,x_2,\ldots,x_k)}{\partial\lambda}\right\}^2\right] & E\left[-\frac{\partial^2 f_{n_1,n_2,\ldots,n_k}(x_1,x_2,\ldots,x_k)}{\partial\lambda\,\partial\delta}\right] \\ E\left[-\frac{\partial^2 f_{n_1,n_2,\ldots,n_k}(x_x,x_2,\ldots,x_k)}{\partial\lambda\,\partial\delta}\right] & E\left[\left\{\frac{\partial f_{n_1,n_2,\ldots,n_k}(x_1,x_2,\ldots,x_k)}{\partial\delta}\right\}^2\right] \end{pmatrix}$$

where $f_{n_1,n_2,...,n_k}$ represents the density function of X, that I_X/n converges to

$$I_{u_1,u_2,...,u_k} = \frac{1}{\delta^2}\begin{pmatrix} K_1 & K_3 \\ K_3 & K_2 \end{pmatrix},$$

where

$$K_1 = \sum_{i=1}^{k+1} \frac{(f_i - f_{i-1})^2}{u_i - u_{i-1}}$$
$$K_2 = \sum_{i=1}^{k+1} \frac{(f_i F^{-1}(u_i) - f_{i-1} F^{-1}(u_{i-1}))^2}{u_i - u_{i-1}}$$
$$K_3 = \sum_{i=1}^{k+1} \frac{(f_i - f_{i-1})(f_i F^{-1}(u_i) - f_{i-1} F^{-1}(u_{i-1}))}{u_i - u_{i-1}}$$
$$\Delta = K_1 K_2 - K_3{}^2$$
$$f_i = \begin{cases} f(F^{-1}(u_i)) & \text{for } i = 1,2,\ldots,k \\ 0 & \text{for } i = 0, k+1 \end{cases}$$
$$u_0 = 0$$
$$u_{k+1} = 1$$

The main optimality criterion that has been developed based on $I_{u_1,u_2,\ldots,u_k}$ is to choose the spacings to maximize its determinant, Δ. This is typically arrived at by noting that *n* times the covariance matrix of the ABLUE, the Asymptotically Best Linear Unbiased Estimator, for (λ, δ) converges to

$$I^{-1}_{u_1,u_2,\ldots,u_k} = \frac{\delta^2}{\Delta} \begin{pmatrix} K_2 & -K_3 \\ -K_3 & K_1 \end{pmatrix}.$$

Thus, an asymptotic expression for the generalized variance of the ABLUE of (λ, δ) is given by $\det(I^{-1}_{u_1,u_2,\ldots,u_k}) = \delta^4 / \Delta$, leading to the maximization of Δ, or equivalently to maximization of the determinant of the information matrix.

A small sampling of works that discuss choosing the spacing to maximize Δ include Kulldorf (1963) for the normal distribution, Koutrouvelis (1981) for the Pareto distribution, and Hassanein (1974) for the logistic distribution. Cane (1974) shows that the spacings that maximize Δ for the Cauchy distribution are uniformly spaced over (0,1).

A more conservative approach than maximization of Δ would be to choose the spacing which maximizes the smaller eigenvalue of $I_{u_1,u_2,\ldots,u_k}$. This is especially important if one is considering estimating linear functions of λ and δ. Now, the smaller eigenvalue of $I_{u_1,u_2,\ldots,u_k}$ is given by

$$v_1 = \frac{1}{2}\left\{K_1 + K_2 - \sqrt{(K_1 - K_2)^2 + 4K_3{}^2}\right\},$$

which can alternatively be expressed as

$$v_1 = \frac{1}{2}\left\{K_1 + K_2 - \sqrt{(K_1 + K_2)^2 - 4\Delta}\right\}.$$

Maximization of v_1 is carried out for the normal, logistic, and exponential distributions in subsequent sections.

Note that the ABLUE for (λ, δ) based on the spacing $u_1, u_2, \ldots, u_k$ is given by

$$\hat{\lambda} = \sum_{i=1}^{k} a_i X_{n_i:n}, \quad \hat{\delta} = \sum_{i=1}^{k} b_i X_{n_i:n}$$

where for $i = 1, 2, \ldots, k$, we have

$$a_i = \frac{f_i K_2}{\Delta}\left[\frac{f_i - f_{i-1}}{u_i - u_{i-1}} - \frac{f_{i+1} - f_i}{u_{i+1} - u_i}\right] - \frac{f_i K_3}{\Delta}\left[\frac{f_i F^{-1}(u_i) - f_{i-1}F^{-1}(u_{i-1})}{u_i - u_{i-1}} - \frac{f_{i+1}F^{-1}(u_{i+1}) - f_i F^{-1}(u_i)}{u_{i+1} - u_i}\right],$$

$$b_i = \frac{f_i K_1}{\Delta}\left[\frac{f_i F^{-1}(u_i) - f_{i-1}F^{-1}(u_{i-1})}{u_i - u_{i-1}} - \frac{f_{i+1}F^{-1}(u_{i+1}) - f_i F^{-1}(u_i)}{u_{i+1} - u_i}\right] - \frac{f_i K_3}{\Delta}\left[\frac{f_i - f_{i-1}}{u_i - u_{i-1}} - \frac{f_{i+1} - f_i}{u_{i+1} - u_i}\right].$$

3. NORMAL DISTRIBUTION

There is not much simplification of K_1, K_2 and K_3 that can be done for the normal distribution whose mean is the location parameter λ and whose standard deviation is the scale parameter δ. Thus, there is little that one can do with the maximization problem analytically. However, the numerical capabilities of *Mathematica* were used with the "Continuous Distributions" package to maximize v_1 over $0<u_1<u_2<\cdots<u_k<1$. The results are reported in Table 1 for $k = 2, 3, \ldots, 8$. It should be noted that the resulting spacings are all symmetric. Since the normal distribution is also symmetric, the value of K_3 is zero for these spacings. Thus,

$$v_1 = \frac{1}{2}\left\{K_1 + K_2 - \sqrt{(K_1 - K_2)^2 + 4K_3^{\ 2}}\right\} = \min\{K_1, K_2\},$$

and v_2 the larger eigenvalue, is given by

$$v_2 = \frac{1}{2}\left\{K_1 + K_2 + \sqrt{(K_1 - K_2)^2 + 4K_3^{\ 2}}\right\} = \max\{K_1, K_2\}.$$

These results form an interesting comparison with the results Ogawa (1962b). The maximization of v_1 corresponds to the maximization of K_1 as carried out by Ogawa for k = 3, 4, ...,10. However, for k = 2 maximization of v_1 produces a point where $K_1 = K_2$, which is not a point that maximizes either K_1 or K_2 Nonetheless, we do have $v_1 = K_1$ and $v_2 = K_2$

for this case as well. It should be noted that the values in Table 1 differ slightly than those of Ogawa for k = 3, 4, ...,8. This should be attributed to rounding error in Ogawa's computations.

4. LOGISTIC DISTRIBUTION

The standardized distribution function of the logistic distribution is given by

$$F(x) = \frac{1}{1+e^{-x}}\cdot$$

Unlike the normal problem, the spacing that maximizes v_1 can be determined analytically in this case. In fact, the uniform spacing, i.e. $u_i = i/(k+1)$ maximizes v_1 over all symmetric spacings under the assumption that the optimal spacing is uniform. This is a weak assumption given the preponderance of numerical evidence generated to date and the fact that, as shown in Ogawa (1998), a symmetric spacing maximizes $\det(I_{u_1,u_2,\ldots,u_k}) = K_1K_2 - K_3{}^2 = \Delta$ for all infinitely differentiable, symmetric distributions.

To see that the uniform spacing maximizes v_1 note that if $K_3 = 0$ then $v_1 = \min\{K_1, K_2\}$. Gupta and Gnandesikan (1966) have shown that the uniform spacing maximizes K_1 By direct calculation with the uniform spacing, one easily establishes that $K_2 > K_1$. Thus, $\min\{K_1, K_2\}$ is maximized when K_1 is maximized, that is with the uniform spacing. The pertinent values for estimation of λ and δ for the uniform spacing are contained in Table 2. In this case we always have $v_1 = K_1$ and $v_2 = K_2$, as well.

5. EXPONENTIAL DISTRIBUTION

The standardized distribution function of the exponential distribution is given by

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Table 1. The optimal spacings, coefficients, and other constants for the normal distribution.

	k=2	*K*=3	*k=4*	*k=5*	*k*=6	*k*=7	*k*=8
u_1	0.2321	0.1631	0.1067	0.0740	0.0536	0.0402	0.0311
a_1	0.5000	0.2953	0.1918	0.1329	0.0964	0.0725	0.0561
b_1	-0.6831	-0.5094	-0.3502	-0.2552	-0.1937	-0.1515	-0.1214
u_2	0.7679	0.5000	0.3511	0.2550	0.1910	0.1469	0.1155
a_2	0.5000	0.4094	0.3082	0.2326	0.1787	0.1400	0.1117
b_2	0.6831	0.0000	-0.1681	-0.1985	-0.1905	-0.1721	-0.1522
u_3		0.8369	0.6489	0.5000	0.3896	0.3083	0.2479
a_3		0.2953	0.3082	0.2691	0.2249	0.1863	0.1546
b_3		0.5094	0.1681	0.0000	-0.0766	-0.1088	-0.1195
u_4			0.8933	0.7450	0.6104	0.5000	0.4122
a_4			0.1918	0.2326	0.2249	0.2026	0.1776
b_4			0.3502	0.1985	0.0766	0.0000	-0.0447
u_5				0.9260	0.8090	0.6917	0.5878
a_5				0.1329	0.1787	0.1863	0.1776
b_5				0.2552	0.1905	0.1088	0.0447
u_6					0.9464	0.8531	0.7521
a_6					0.0964	0.1400	0.1546
b_6					0.1937	0.1721	0.1195
u_7						0.9598	0.8845
a_7						0.0724	0.1117
b_7						0.1515	0.1522
u_8							0.9689
a_8							0.0561
b_8							0.1214
K_1	0.8026	0.8825	0.9201	0.9420	0.9560	0.9655	0.9721
K_2	0.8026	1.0647	1.3142	1.4767	1.5879	1.6672	1.7256

Thus, the location parameter is the left endpoint of the support of the distribution in this case.

The maximization problem for the exponential distribution is more easily attacked through the quantiles than through the spacings.

Table 2. The optimal spacings, coefficients, and other constants for the logistic distribution.

	k=2	*k*=3	*k*=4	*k*=5	*k*=6	*k*=7	*k*=8
u_1	1/3	1/4	1/5	1/6	1/7	1/8	1/9
a_1	0.5000	0.3000	0.2000	0.1429	0.1072	0.0833	0.0667
b_1	-0.7213	-0.4551	-0.3312	-0.2592	-0.2121	-0.1790	-0.1545
u_2	2/3	1/2	2/5	1/3	2/7	1/4	2/9
a_2	0.5000	0.4000	0.3000	0.2286	0.1786	0.1429	0.1167
b_2	0.7213	0	-0.1006	-0.1196	-0.1181	-0.1109	-0.1025
u_3		3/4	3/5	1/2	3/7	3/8	1/3
a_3		0.3000	0.3000	0.2571	0.2143	0.1786	0.1500
b_3		0.4551	0.1006	0	-0.0410	-0.0584	-0.0652
u_4			4/5	2/3	4/7	1/2	4/9
a_4			0.2000	0.2286	0.2143	0.1905	0.1667
b_4			0.3312	0.1196	0.0410	0	-0.0224
u_5				5/6	5/7	5/8	5/9
a_5				0.1429	0.1786	0.1786	0.1667
b_5				0.2592	0.1181	0.0584	0.0224
u_6					6/7	3/4	2/3
a_6					0.1072	0.1429	0.1500
b_6					0.2121	0.1109	0.0652
u_7						7/8	7/9
a_7						0.0833	0.1167
b_7						0.1790	0.1025
u_8							8/9
a_8							0.0667
b_8							0.1545
K_1	0.2963	0.3125	0.3200	0.3241	0.3265	0.3281	0.3292
K_2	0.4271	0.6789	0.8364	0.9423	1.0177	1.0739	1.1173

Reparameterizing with $x_i = F^{-1}(u_i) = -\ln(1-u_i)$ for *i* = 1, 2, ..., *k*, we find that

$$K_1 = \frac{1}{e^{x_1} - 1}$$

$$K_2 = \sum_{i=1}^{k} \frac{(x_i - x_{i-1})^2}{e^{x_i} - e^{x_{i-1}}}, \quad x_0 = F^{-1}(0) = 0$$

$$K_3 = \frac{x_1}{e^{x_1} - 1}.$$

Maximization of v_1 is to be carried out over $0<x_1<x_2<\cdots<x_k<\infty$.

Further reparameterization with $t_i = x_{i+1} - x_1$ for i = 1, 2, ..., k-1, yields

$$K_2 = \frac{x_1^{\ 2}}{e^{x_1} - 1} + \frac{K_2^*}{e^{x_1}},$$

where

$$K_2^* = \sum_{i=1}^{k-1} \frac{(t_i - t_{i-1})^2}{e^{t_i} - e^{t_{i-1}}}, \quad t_0 = 0.$$

With these substitutions, we find that

$$2v_1 = \frac{1 + x_1^{\ 2}}{e^{x_1} - 1} + \frac{K_2^*}{e^{x_1}} - \sqrt{\left(\frac{1 - x_1^2}{e^{x_1} - 1} - \frac{K_2^*}{e^{x_1}}\right)^2 + \frac{4x_1^{\ 2}}{(e^{x_1} - 1)^2}},$$

which is to maximized over $0 < x_1 < \infty, 0 < t_1 < t_2 < \cdots < t_{k-1} < \infty$.

Note that the partial derivative of the expression above with respect to K_2^* is given by

$$e^{-x_1}\left(1 - \frac{\dfrac{1 - x_1^{\ 2}}{e^{x_1} - 1} - \dfrac{K_2^*}{e^{x_1}}}{\sqrt{\left(\dfrac{1 - x_1^{\ 2}}{e^{x_1} - 1} - \dfrac{K_2^*}{e^{x_1}}\right)^2 + \dfrac{4x_1^{\ 2}}{(e^{x_1} - 1)^2}}}\right).$$

Since

$$\frac{1 - x_1^{\ 2}}{e^{x_1} - 1} - \frac{K_2^*}{e^{x_1}} < \sqrt{\left(\frac{1 - x_1^{\ 2}}{e^{x_1} - 1} - \frac{K_2^*}{e^{x_1}}\right)^2 + \frac{4x_1^{\ 2}}{(e^{x_1} - 1)^2}},$$

we see that v_1 is an increasing function of K_2^* for fixed x_1. The problem of maximizing K_2^* for $0<t_1<t_2<\cdots<t_{k-1}<\infty$ has been solved by Ogawa (1960). In particular, the value of K_2^* is 0.6476 for k=2 and increases monotonically to a limit of 1 as $k\to\infty$.

Substituting the optimal value for K_2^* into the expression for v_1, we now must optimize over $0<x_1<\infty$. It turns out that

$$\lim_{x_1 \to 0^+} v_1 = \infty,$$

which is not surprising since

$$\lim_{x_1 \to 0^+} K_1 = \infty, \ \lim_{x_1 \to 0^+} K_2 = K_2^*, \text{ and } \lim_{x_1 \to 0^+} K_3 = 1.$$

Thus, we will always choose the first order statistic and choose the remaining *k*-1 order statistics according to Ogawa (1960). For a careful treatment of the asymptotics in this case, the reader is referred to Ali, Umbach, and Saleh (1985). We note that the ABLUE for (λ,δ) in this case is

$$(X_{1:n}, \sum_{i=2}^{k} c_{i-1}(X_{n_i:n} - X_{1:n}))$$

where the coefficients $c_1, c_2, ..., c_{k-1}$ are the coefficients for the estimation of the scale parameter based on λ=0 using the *k*-1 order statistics that maximize K_2^* as given in Ogawa (1960). Thus,

$$b_1 = -\sum_{i=1}^{k-1} c_i \text{ and } b_i = c_{i-1} \text{ for } i = 2, 3, \ldots, k.$$

Table 3 contains these results. Please note that in every case, we have $\hat{\lambda}=X_{1:n}$ and thus, the values of a_i are not included, since a_1=1 and a_i=0 for *i*=2, 3, ..., *k*.

Table 3. The optimal spacings, coefficients, and other constants for the exponential distribution.

	k=2	*k*=3	*k*=4	*k*=5	*k*=6	*k*=7	*k*=8
u_1	0	0	0	0	0	0	0
b_1	-0.6275	-0.7022	-0.7518	-0.7872	-0.8183	-0.8342	-0.8507
u_2	0.7968	0.6386	0.5296	0.4514	0.3931	0.3478	0.3121
b_2	0.6275	0.5232	0.4477	0.3907	0.3463	0.3108	0.2819
u_3		0.9266	0.8300	0.7419	0.6670	0.6042	0.5513
b_3		0.1790	0.2266	0.2361	0.2320	0.2228	0.2119
u_4			0.9655	0.9067	0.8434	0.7828	0.7277
b_4			0.0775	0.1195	0.1402	0.1492	0.1519
u_5				0.9810	0.9434	0.8978	0.8506
b_5				0.0409	0.0709	0.0902	0.1017
u_6					0.9885	0.9631	0.9297
b_6					0.0243	0.0456	0.0615
u_7						0.9925	0.9746
b_7						0.0156	0.0311
u_8							0.9948
b_8							0.0107
K_2^*	0.6476	0.8203	0.8910	0.9269	0.9476	0.9606	0.9693

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