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ON SUM OF SOME HYPER-GEOMETRIC SERIES FUNCTIONS-I

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ABSTRACT

In this paper sum of some hyper-geometric series functions have been derived, using properties of discrete probability functions.

INTRODUCTION

Jones (1987), Lepage (1978) and Roohi (2002) have worked on the negative moments and inverse factorial moments of some discrete probability functions. In Statistics, some discrete probability functions have been expressed in terms of hyper geometric series functions. (See Bardwell and Crow, 1964). Using properties of the discrete probability functions, simple solutions of sums of hyper geometric series functions have been found.

Theorem 1:

Let ${}_2F_1(a,\ b;\ c;\ \theta)$ be defined as:

$$ {}_2F_1(a,\ b;\ c;\ \theta) = 1 + \frac{ab}{c}\theta + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{\theta^2}{2} + \ldots, c \neq 0 \tag{1}$$

then if a or b is negative, ${}_2F_1$ will be defined as terminating series, then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_2F_1(s,\ -n;\ s+1;\ -\theta) = {}_2F_1[1,\ -n;\ k+1;\ -\theta], \tag{2}$$

$$\theta > 0,\quad n, k = 1, 2, 3, \ldots.$$

Proof: Suppose X is a binomial random variable with parameters n and θ,

$$f(x) = \binom{n}{x} \frac{\theta^x}{(1+\theta)^n}, \qquad x = 0, 1, 2, \ldots, n,\ \theta > 0 \tag{3}$$

It is known that

$$\prod_{s=1}^{k} \frac{1}{x+s} = \sum_{s=1}^{k} \frac{(-1)^{s+1}}{(k-s)!\,(s-1)!} \frac{1}{x+s}, \quad x \geq 0,\ k = 1, 2, 3, \cdots \tag{4}$$

(See Rainville 1960)

By definition of $E[g(x)] = \sum g(x)\, f(x),$ we have

$$E\left(\prod_{s=1}^{k} \frac{1}{X+s}\right) = \sum_{s=1}^{k} \frac{(-1)^{s+1}}{(k-s)!\ (s-1)!} . E\left(\frac{1}{X+s}\right)$$

Now $$E\left(\frac{1}{X+s}\right) = \sum_{x=0}^{x} \frac{1}{x+s} \binom{n}{x} p^x q^{n-x} = \frac{1}{s(1+\theta)^n}\ {}_2F_1[s, -n; s+1; -\theta]$$

Thus $$E\left[\prod\left(\frac{1}{X+s}\right)\right] = \sum_{s=1}^{k} \frac{(-1)^{s+1}}{(k-s)!\ (s-1)!} . \frac{1}{s(1+\theta)^n}\ {}_2F_1[s, -n; s+1; -\theta]$$

$$= \frac{1}{(1+\theta)^n k!} \sum_{s=1}^{k} (-1)^{s+1} \binom{k}{s}\ {}_2F_1(s, -n; s+1; -\theta) \tag{5}$$

Also

$$E\left(\prod_{s=1}^{k} \frac{1}{X+s}\right) = \sum_{x=0}^{n} \binom{n}{x} \frac{\theta^x}{(1+\theta)^n} \prod_{s=1}^{k}\left(\frac{1}{x+s}\right),$$

Expanding the summation, we have

$$= \frac{1}{(1+\theta)^n\, k!}\left[1 + \frac{n}{k+1}\theta + \frac{n(n-1)}{2!} \frac{1\,.\,2}{(k+1)(k+2)}\, \theta^2 + \cdots \right.$$
$$\left. + \frac{n!}{(k+1)\cdots(k+n)} \theta^n \right]$$

$$= \frac{1}{(1+\theta)^n\, k!}\left[1 + \frac{-n}{k+1}(-\theta) + \frac{(-n)(-n+1)1.2}{(k+1)(k+2)}\frac{1}{2!}(-\theta)^2 + \cdots \right.$$

$$\left. + \frac{(-n)(-n+1)\cdots(-1)1.2\ \ldots\ n}{(k+1)\cdots(k+n)\ n!}(-\theta)^n\right]$$

$$= \frac{1}{(1+\theta)^n\, k!}\ {}_2F_1[1, -n;\ k+1;\ -\theta] \tag{6}$$

Thus, we get the result from (4) and (6),

Theorem 2:

Let ${}_2F_1(a,\ b;\ c;\ \theta)$ be defined as in (1), then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\ {}_2F_1(r,\ s;\ s+1;\ q) = {}_2F_1(1,\ r;\ k+1;\ q) \tag{7}$$

r = 1, 2, …. k = 1, 2, ….., 0 < q < 1.

holds.

Proof:
Suppose X has a negative binomial function

$$f(x) = \binom{x+r-1}{x}p^r q^x, \tag{8}$$

r = 1, 2, …. k = 1, 2, ….., 0 < q < 1 and p = 1 – q

If X has the probability function (8), then

$$E\left(\frac{1}{X+s}\right) = \sum_{x=0}^{\infty}\frac{1}{x+s}\binom{x+r-1}{x}p^r q^x$$

$$= \frac{p^r}{s}\left[1 + \frac{s.r}{s+1}q + \frac{r(r+1)s(s+1)}{(s+1)(s+2)}.\frac{q^2}{2!} + \cdots\right]$$

$$= \frac{p^r}{s}\ {}_2F_1[r,\ s;\ s+1;\ q] \tag{9}$$

By definition of expected of a function, and $\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)$ defined as in (3), we have

$$E\left(\prod_{s=1}^{k}\frac{1}{X+s}\right) = \sum_{s=0}^{k}\frac{(-1)^{s+1}}{(k-s)!\,(s-1)!}\,E\left(\frac{1}{X+s}\right) \tag{10}$$

Substituting (9) in (10), we have

$$= \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!\,(s-1)!}\,\frac{p^{r}}{s}\;{}_2F_1\,(r,\ s;\ s+1;\ q).$$

$$= \frac{p^{r}}{k!}\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_2F_1\,(r,\ s;\ s+1;\ q). \tag{11}$$

Also by definition of expectation $(X+s)^{-1}$ when f(x) is defined as (8), we have

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right] = \sum_{x=0}^{\infty}\binom{x+r-1}{x}p^{r}q^{x}\ \prod_{s=1}^{k}\left(\frac{1}{x+s}\right)$$

$$= \frac{p^{r}}{k!}\left[1+\frac{r.1}{k+1}q+\frac{r(r+1)\,1.2}{(k+1)(k+2)}\frac{q^{2}}{2!}+\cdots\right]$$

$$= \frac{p^{r}}{k!}\;{}_2F_1[1,\ r;\ k+1;\ q] \tag{12}$$

We get the result from (11) and (12)

Corollary 1: If r = 1, then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_2F_1\ (1,\ s;\ s+1;\ q) = {}_2F_1\ (1,\ 1;\ k+1;\ q) \tag{13}$$

Proof:

This follows immediately from (6) by putting r = 1.

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