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TWO SERIES CONSTRUCTIONS OF NEAREST NEIGHBOR BALANCED *BIBRC*S

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ABSTRACT

Two series constructions of balanced incomplete block designs with nested rows and columns are given. One series of designs is balanced for nearest neighbors in rows, in columns, and in diagonals, and the second series is balanced for neighbors in rows and columns combined and in diagonals.

KEY WORDS

Block design with nested rows and columns; finite field; method of differences; neighbor balance.

1. INTRODUCTION

Suppose that *bpq* plots are laid out in $p \times q$ arrays of *pq* experimental units or plots each, and *v* treatments are to be assigned to these plots so that each plot receives exactly one treatment. Each such arrangement of treatments is said to be a balanced incomplete block design with nested rows and columns, *BIBRC*(*v, b, p, q*), if the following conditions are satisfied:

(*i*) each block is binary (i.e., every treatment occurs at most once in each block)
(*ii*) every treatment occurs in $r = bpq/v$ blocks
(*iii*) every pair of treatments (*i, j*) satisfies

$$p\lambda_{r(i,j)} + q\lambda_{c(i,j)} - \lambda_{b(i,j)} = r(p-1)(q-1)/(v-1) = \lambda$$

where $\lambda_{r(i,j)}, \lambda_{c(i,j)}$ and $\lambda_{b(i,j)}$ denote the number of times treatment pair (*i, j*) appears, respectively, in *bp* rows, in *bq* columns, and in *b* blocks.
A *BIBRC* is said to be neighbor balanced if it satisfies the following condition:

(*iv*) Every pair of treatments occur in neighboring plots n_r times in rows, n_c times in columns and n_d times in diagonals.

In some situations (e.g., square plots in blocks having equal number of rows and columns), balance of neighbor counts in rows and columns combined and in diagonals may be sufficient for achieving high statistical efficiency. Following Kiefer and Wynn's (1981) two-step procedure, one may argue that a neighbor balanced *BIBRC* would be highly efficient within the class of binary block designs under the fixed effects (row, column, and block) additive model with long range error covariance structures such as three parameter conditional auto normal process (see Gill and Shukla, 1985). We do not, however, address the statistical efficiency issue of the proposed designs in this paper. Although a host of recent papers dealt with the construction of *BIBRC*s (e.g., Singh and Dey, 1979; Agrawal and Prasad, 1982, 1983; Ipinyomi and John, 1985; Cheng, 1986; Sreenath, 1989; Street, 1981; Uddin and Morgan, 1990, 1991; Uddin, 1990, 1992), very little is known on the construction of *BIBRC*s satisfying neighbor property (*iv*) mentioned above. In this paper, we offer two series constructions of *BIBRC*s that are also neighbor balanced.

2. CONSTRUCTIONS OF NEIGHBOR BALANCED *BIBRC*S

We shall let $F_v(x)$ denote the finite field of order *v* with primitive root *x*. For the purpose of our constructions, the *v* treatments will be denoted by *v* elements of $F_v(x) = \{0, x^0, x^1, ..., x^{v-2}\}$. To avoid trivial constructions, we concentrate on designs with fewer than $\frac{v-1}{2}$ initial blocks. Our constructions are based on the method of differences. Under this method, one needs to construct a set of initial blocks that satisfy a *balanced difference* condition which is satisfied if the symmetric pairwise differences arising from elements involved with each factor of interest is multiple copies of the nonzero elements of $F_v(x)$. The full design is then generated by adding field elements to these initial blocks; *t* initial blocks thus give *tv* blocks of the design.

Series 1. *Let* $v = 8m + 1$ be odd prime power. The initial blocks

$$A_i = \begin{pmatrix} x^0 & x^m & x^{2m} \\ x^{7m} & 0 & x^{3m} \\ x^{6m} & x^{5m} & x^{4m} \end{pmatrix} x^{i-1}, i = 1, 2, ..., 2m,$$

when developed over $F_v(x)$, *give a BIBRC*(v; $b = 2mv$, 3, 3) *which is balanced for neighbors in rows and columns combined and balanced for neighbors in diagonals.*

Proof: First we shall show that $A_1, A_2, ..., A_{2m}$ are initial blocks of *BIBRC*(*v*, 2*mv*, 3, 3). Consider the initial block A_1. Each row and each column of the initial block A_1 has three pairs of elements of $F_v(x)$. Thus each row results in six symmetric differences. Similarly each column results in six symmetric differences. The 36 symmetric differences arising from the elements of these three rows and three columns together are found to be two copies of

$$\pm\left\{1-x^m, 1-x^{2m}, x^m\left(1-x^m\right)\right\},$$
$$\pm\left\{1-x^m, 1-x^{2m}, x^m\left(1-x^m\right)\right\}x^{2m}, \pm\left\{x^m, x^{3m}\right\}$$

and one copy of $\pm\left\{2x^m, 2x^{3m}\right\}$. Multiplying these 36 differences successively by $x^0, x^1, ..., x^{2m-1}$, we obtain the differences from all 2*m* initial blocks. The resulting 72*m* differences are all nonzero elements of $F_v(x)$ each exactly nine times. In a similar fashion, we see that the 144*m* (72 from each initial block) symmetric differences arising from the elements of 2*m* initial blocks are each nonzero element of $F_v(x)$ exactly 18 times. Thus the 2*mv* blocks obtained by developing 2*m* initial blocks over $F_v(x)$ is a *BIBRC*(*v*, *b* = 2*mv*, 3,3).

To show that the design is neighbor balanced, consider the neighbor differences in rows, columns, and diagonals. Neighbor differences arising from within-row and within column elements of A_1 are $\pm\left\{x^0, x^m, ..., x^{7m}\right\}\left(1-x^m\right)$, $\pm\left\{x^{3m}, x^{3m}, x^m, x^m\right\}$. Multiplying these elements successively by $x^0, x^1, ..., x^{2m-1}$, we obtain each nonzero element of $F_v(x)$ exactly six times. Thus the design is neighbor balanced in rows and columns combined with $n_r + n_c = 6$. In a similar fashion, the symmetric diagonal neighbor differences obtained from A_1 are $\pm\left\{x^m, x^{3m}, x^{5m}, x^{7m}\right\}\left(1-x^{2m}\right)$, $\pm\left\{x^{2m}, x^{2m}, x^0, x^0\right\}$. Multiplying these elements by $x^0, x^1, ..., x^{2m-1}$, we obtain each nonzero element of $F_v(x)$ exactly four times and hence the diagonal neighbor balance with $n_d = 4$, completing the proof.

Example 1.

The two initial blocks of a *BIBRC*(*v* = 9, *b* = 18, *p* = 3, *q* = 3) having neighbor properties described in Series 1 above are

$$\begin{pmatrix} (0,1) & (1,0) & (2,1) \\ (1,1) & (0,0) & (2,2) \\ (1,2) & (2,0) & (0,2) \end{pmatrix} \text{ and } \begin{pmatrix} (1,0) & (2,1) & (2,2) \\ (0,1) & (0,0) & (0,2) \\ (1,1) & (1,2) & (2,0) \end{pmatrix}.$$

Here the elements of the finite field $F_{3^2}(x)$ are 0 = (0, 0), x^0 = (0,1), x^1 = (1,0), x^2 = (2, 1), x^3 = (2,2), x^4 = (0,2), x^5 = (2, 0), x^6 = (1,2), x^7 = (1,1). The full design with 18 blocks are then generated by adding elements of $F_{3^2}(x)$, mod(3,3), to these two initial blocks.

For the purpose of our next construction, v-1 treatments are denoted by the elements of $F_{v-1}(x)$ and the v-th treatment is denoted by the symbol ∞. Also, we use the convention that $\infty + y = \infty$ for all $y \in F_{v-1}(x)$.

Series 2. Let v-1 = 4m+3 be a prime power. Then the initial blocks $B_1, \ldots, B_{m+1}$, when developed over $GF_{v-1}(x)$, produce a *BIBRC*(v, (m + 1)(v - 1), v/2, 2) which is balanced for neighbors in rows, in columns, and in diagonals.

$$B_1 = \begin{pmatrix} 0 & \infty \\ x^{v-4} & x^{v-3} \\ x^0 & x^1 \\ x^2 & x^3 \\ \vdots & \vdots \\ x^{v-6} & x^{v-5} \end{pmatrix}, \qquad B_{i+1} = \begin{pmatrix} x^0 & x^1 \\ 0 & \infty \\ x^2 & x^3 \\ x^4 & x^5 \\ \vdots & \vdots \\ x^{v-4} & x^{v-3} \end{pmatrix} x^{4(i-1)}, i = 1, 2, \ldots, m.$$

Proof: That the design of the above series is neighbor balanced in rows, in columns, and in diagonals follows from Theorem 3.2 of Morgan and Uddin (1999). Since the blocks are complete blocks and q = 2 with row neighbor balance, we only need to show that the column component design is a balanced incomplete block design. This last property holds from the fact that $(0, x^0, x^2, \ldots, x^{v-3})$ and $(\infty, x^1, x^3, \ldots x^{v-2})$ are initial blocks of a balanced incomplete block design (see exercise 23 in Raghavarao, 1971, and compare).

Example 2.

A neighbor balanced *BIBRC*(v = 12, b = 33, p = 6, q = 2) can be generated using the following three initial blocks:

$$\begin{pmatrix} 0 & \infty \\ 3 & 6 \\ 1 & 2 \\ 4 & 8 \\ 5 & 10 \\ 9 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & \infty \\ 4 & 8 \\ 5 & 10 \\ 9 & 7 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 10 \\ 0 & \infty \\ 9 & 7 \\ 3 & 6 \\ 1 & 2 \\ 4 & 8 \end{pmatrix}.$$

The full design with 33 blocks are generated by adding elements of the finite field $F_{11}(x)$ to these initial blocks with the convention that $\infty + y = \infty$ for all $y \in F_{11}(x)$.

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