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BAYESIAN MULTIVARIATE NORMAL ANALYSIS UNDER BALANCED LOSS FUNCTION

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ABSTRACT

This paper considers the Bayesian analysis of the multivariate normal distribution under balanced loss function. The Bayes estimators of the mean vector θ and the covariance matrix Σ are obtained. The admissibility of $c\overline{\mathbf{X}}+\mathbf{d}$ for the mean vector is also studied.

Keywords

Admissibility, Balanced loss function, Bayes estimator, Inadmissibility, Inverted Wishart prior, Multivariate normal distribution.

1. Introduction

Let $\mathbf{X}=(X_1,\ldots,X_p)'$ be a random vector with the pdf

$$f(\mathbf{x}|\theta,\Sigma)=\frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}\exp[-\frac{1}{2}(\mathbf{x}-\theta)'\Sigma^{-1}(\mathbf{x}-\theta)], \tag{1.1}$$

where $\theta=(\theta_1,\ldots,\theta_p)$ and Σ is a positive definite symmetric matrix ($\Sigma>0$). We use the notation $\mathbf{X}\sim N_p(\theta,\Sigma)$. Let $\mathbf{X}_\alpha=(X_{1\alpha},\ldots,X_{p\alpha})'$, $\alpha=1,\ldots,N$, be a random sample of size N from (1.1). We would like to estimate the mean vector θ under the balanced loss function (BLF), given by

$$L(\hat{\theta},\theta)=\frac{\omega}{N}\sum_{i=1}^{N}(\mathbf{X}_i-E_{\hat{\theta}}(\mathbf{X}))'Q(\mathbf{X}_i-E_{\hat{\theta}}(\mathbf{X}))+(1-\omega)(\theta-\hat{\theta})'Q(\theta-\hat{\theta})$$

$$=\frac{\omega}{N}\sum_{i=1}^{N}(\mathbf{X}_i-\hat{\theta})'Q(\mathbf{X}_i-\hat{\theta})+(1-\omega)(\theta-\hat{\theta})'Q(\theta-\hat{\theta}) \tag{1.2}$$

where $0<\omega<1$, $\hat{\theta}=(\hat{\theta}_1,\ldots,\hat{\theta}_p)$ is an estimate of θ, and Q is an arbitrary positive definite matrix of dimension $p\times p$. The BLF introduced by Zellner (1994), is formulated to reflect two criteria, namely goodness of fit and precision of estimation. Use of a goodness of fit criterion leads to an estimate which gives good fit and is, an unbiased estimator however, it may not be as precise as an estimator which is biased. Thus there is a need to provide a framework which combines goodness of fit and precision of estimation formally. As mentioned above, first term of the r.h.s. of (1.2) reflects goodness of fit while the second term reflects the precision of estimation.

For estimation under the BLF, for some standard distributions, see Zellner (1994), Rodrigues and Zellner (1995), Chung and Kim (1997), Chung, Kim and Song (1998), Dey, Ghosh and Strawderman (1999).

Note that, if Q=I where I denotes the identity matrix, the loss function (1.2) reduces to

$$L(\hat{\theta},\theta)=\sum_{i=1}^{p}\left(\sum_{j=1}^{N}\frac{\omega}{N}(X_{ij}-\hat{\theta}_i)^2+(1-\omega)(\hat{\theta}_i-\theta_i)^2\right), \tag{1.3}$$

Where X_{ij}'s, for i=1,…,p and j=1,…N are distributed as normal with mean θ_i and variance σ_i^2.

For estimating θ when Σ=I, Chung and Kim (1997) showed that $\overline{\mathbf{X}}=(\overline{\mathbf{X}}_1,\ldots,\overline{\mathbf{X}}_p)$ where $\overline{\mathbf{X}}_i=\frac{1}{N}\sum_{\alpha=1}^{N}X_{i\alpha}$ is inadmissible when $p\geq 3$ under the loss function (1.3). They showed that the estimator $\delta(\overline{\mathbf{X}})=(\delta_1(\overline{\mathbf{X}}),\ldots,\delta_p(\overline{\mathbf{X}}))'$ with

$$\delta_i(\overline{\mathbf{X}})=\left(1-\frac{(1-\omega)(p-2)}{n\sum_{i=1}^{p}\overline{\mathbf{X}}_i^2}\right)\overline{\mathbf{X}}_i, \qquad i=1,\ldots,p$$

has uniformly smaller risk than $\overline{\mathbf{X}}$, for all θ. James and Stein (1961), Baranchik (1970), Strawderman (1971), Efron and Morris (1973) and Stein (1981) studied the problem of estimating multivariate normal mean vector under quadratic loss function.

In this paper, we obtain the Bayes estimators of θ and Σ under the loss (1.2) and study the admissibility and inadmissibility of the estimators of the form $c\overline{\mathbf{X}}+\mathbf{d}$ with $c\in\Re$ and $d\in\Re^p$ for the mean vector.

2. BAYES ESTIMATORS OF THE MEAN VECTOR

Suppose $\mathbf{X}_1,\ldots,\mathbf{X}_N$ is a sample of $p\times 1$ mutually independent random vectors that are identically distributed as $N_p(\theta,\Sigma)$ and let $\underline{\mathbf{X}}=(\mathbf{X}_1,\ldots,\mathbf{X}_N)$. To determine the Bayes estimator of θ under the loss (1.2), it is enough to find a value of $\hat{\theta}$ which minimizes

$$E[L(\hat{\theta},\theta)|\underline{\mathbf{X}}]=\frac{\omega}{N}\sum_{i=1}^{N}(\mathbf{X}_i-\hat{\theta})'Q(\mathbf{X}_i-\hat{\theta})+(1-\omega)E[(\theta-\hat{\theta})'Q(\theta-\hat{\theta})|\underline{\mathbf{X}}]$$

or

$$E[L(\hat{\theta},\theta)|\underline{\mathbf{X}}]=\frac{\omega}{N}\sum_{i=1}^{N}(\mathbf{X}_i-\hat{\theta})'Q(\mathbf{X}_i-\hat{\theta})$$

$$+(1-\omega)[\hat{\theta}'Q\hat{\theta}-2\hat{\theta}'QE(\theta|\underline{\mathbf{X}})+E(\theta'Q\theta|\underline{\mathbf{X}})].$$

Differentiating with respect to $\hat{\theta}$ gives

$$\frac{\partial E[L(\hat{\theta},\theta)\,|\,\underline{\mathbf{X}}]}{\partial\hat{\theta}}=\frac{-2\omega}{N}\sum_{i=1}^{N}Q(\mathbf{X}_i-\hat{\theta})+(1-\omega)[2Q\hat{\theta}-2QE(\theta\,|\,\underline{\mathbf{X}})]$$

$$=-2\omega Q(\overline{\mathbf{X}}-\hat{\theta})+2(1-\omega)Q[\hat{\theta}-E(\theta\,|\,\underline{\mathbf{X}})].$$

Setting the derivative equal to zero and solving gives

$$\hat{\theta}_{Bayes}=\omega\overline{\mathbf{X}}+(1-\omega)E(\theta\,|\,\underline{\mathbf{X}}). \qquad (2.1)$$

Note that since

$$\frac{\partial^2 E[L(\hat{\theta},\theta\,|\,\underline{\mathbf{X}})\,|}{\partial\hat{\theta}'\partial\hat{\theta}}=2\omega Q+2(1-\omega)Q=2Q$$

a positive definite matrix, $\hat{\theta}_{Bayes}$ actually corresponds to a minimum value.

Remark (2.1): Note that $\hat{\theta}_{Bayes}$ is the value of $\hat{\theta}$ that minimizes posterior risk, it is an average of $\overline{\mathbf{X}}$, which minimizes the first term on the r.h.s. of (1.2) and $E(\theta|\overline{\mathbf{X}})$, the posterior mean of θ which minimizes the posterior expectation of the second term on the r.h.s. of (1.2). Accordingly, the Bayes estimator of θ can be expressed as linear combination of sample mean and posterior mean.

3. ADMISSIBILITY OF c$\overline{\mathbf{X}}$+d FOR THE MEAN VECTOR

In this section, the condition of admissibility of the linear estimator $c\overline{\mathbf{X}}+\mathbf{d}$ with $c\in\Re$ and $d\in\Re^p$ is considered for the mean vector. For later use, we give the risk function of the linear estimator $c\overline{\mathbf{X}}+\mathbf{d}$, in Proposition (3.1).

Proposition 3.1. The risk function of the estimator $c\overline{\mathbf{X}}+\mathbf{d}$, relative to the BLF (1.2) is

$$R(\theta,c\overline{\mathbf{X}}+\mathbf{d})=[(c-1)\theta+\mathbf{d}]'Q[(c-1)\theta+\mathbf{d}]+[(c-\omega)^2+\omega(N-\omega)]\frac{tr(\Sigma Q)}{N}. \qquad (3.1)$$

Proof: To see this, note that

$$\mathbf{X}_i-c\overline{\mathbf{X}}-\mathbf{d}\sim N_p((1-c)\theta-\mathbf{d},\frac{c^2-2c+N}{N}\Sigma)$$

and

$$c\overline{\mathbf{X}}+\mathbf{d}-\theta\sim N_p((c-1)\theta+\mathbf{d},\frac{c^2}{N}\Sigma).$$

Also, it is easy to show that for any random vector **X** where $\mathbf{X}\sim N_p(\theta,\Sigma)$, we have

$$E(\mathbf{X}'A\mathbf{X})=\theta'A\theta+tr(\Sigma A)$$

where A is a p×p matrix.

Hence

$$R(\theta, c\overline{\mathbf{X}}+\mathbf{d}) = \frac{\omega}{N}\sum_{i=1}^{N} E[(\mathbf{X}_i - c\overline{\mathbf{X}}-\mathbf{d})'Q(\mathbf{X}_i - c\overline{\mathbf{X}}-\mathbf{d})]$$

$$+(1-\omega)E[(c\overline{\mathbf{X}}+d-\theta)'Q(c\overline{\mathbf{X}}+\mathbf{d}-\theta)]$$

$$= \omega\{[(1-c)\theta-\mathbf{d}]'Q[(1-c)\theta-\mathbf{d}]+tr(\frac{c^2-2c+N}{N}\sum Q)\}$$

$$+(1-\omega)\{[(c-1)\theta+\mathbf{d}]'Q[(c-1)\theta+\mathbf{d}]+tr(\frac{c^2}{N}\sum Q)\}.$$

Now, since for any scalar k, tr(kA)=ktr(A), the proof is completed.

Theorem 3.2: The estimator $c\overline{\mathbf{X}}+\mathbf{d}$ with $c\in\Re$ and $\mathbf{d}\in\Re^p$ is an inadmissible estimator of θ if one of the following conditions holds:

(i) $c > 1$, $\mathbf{d}\in\Re^p$

(ii) $c < \omega$, $\mathbf{d}\in\Re^p$

(iii) c=1 and $\mathbf{d}\neq\mathbf{0}$ (**0** is an $p\times 1$ vector with all elements equal to zero).

Proof: (i) If $c > 1$, then $(c-\omega)^2 > (1-\omega)^2$ and hence from (3.1)

$$R(\theta, c\overline{\mathbf{X}}+\mathbf{d}) \geq [(c-\omega)^2+\omega(N-\omega)]\frac{tr(\sum Q)}{N}$$

$$> [(1-\omega)^2+\omega(N-\omega)]\frac{tr(\sum Q)}{N}$$

$$= R(\theta, \overline{X}).$$

Thus, $c\overline{\mathbf{X}}+\mathbf{d}$ is dominated by $\overline{\mathbf{X}}$.

(ii) If $c < \omega$, then $(c-1)^2 > (\omega-1)^2$ and hence

$$R(\theta, c\overline{\mathbf{X}}+\mathbf{d}) = [(c-1)\theta+\mathbf{d}]'Q[(c-1)\theta+\mathbf{d}]+[(c-\omega)^2+\omega(N-\omega)]\frac{tr(\sum Q)}{N}$$

$$= (c-1)^2[\theta+\frac{\mathbf{d}}{c-1}]'Q[\theta+\frac{\mathbf{d}}{c-1}]+[(c-\omega)^2+\omega(N-\omega)]\frac{tr(\sum Q)}{N}$$

$$> (\omega-1)^2[\theta+\frac{\mathbf{d}}{c-1}]'Q[\theta+\frac{\mathbf{d}}{c-1}]+\omega(N-\omega)\frac{tr(\sum Q)}{N}$$

$$= [(\omega-1)\theta+\frac{\mathbf{d}(\omega-1)}{c-1}]'Q[(\omega-1)\theta+\frac{\mathbf{d}(\omega-1)}{c-1}]+\omega(N-\omega)\frac{tr(\sum Q)}{N}$$

$$= R(\theta, \omega\overline{\mathbf{X}}+\frac{\mathbf{d}(\omega-1)}{c-1}).$$

Thus in this case, $c\overline{\mathbf{X}}+\mathbf{d}$ is dominated by $\omega\overline{\mathbf{X}}+\frac{(\omega-1)\mathbf{d}}{c-1}$.

(iii) If c=1 and $\mathbf{d}\neq\mathbf{0}$, then

$$R(\theta, \overline{\mathbf{X}}+\mathbf{d}) = \mathbf{d}'Q\mathbf{d} + [(1-\omega)^2 + \omega(N-\omega)]\frac{tr(\Sigma Q)}{N}$$

$$> [(1-\omega)^2 + \omega(N-\omega)]\frac{tr(\Sigma Q)}{N} \qquad \text{(since Q>0)}$$

$$= R(\theta, \omega\overline{\mathbf{X}}).$$

Thus c $\overline{\mathbf{X}}$ +**d** is dominated by $\overline{\mathbf{X}}$ when condition (iii) holds.

Theorem 3.3 The estimator c$\overline{\mathbf{X}}$+**d** is admissible for the mean vector under the BLF (1.2), whenever $\omega < c < 1$ and $d \in \Re^p$.

Proof: It is often the case to choose a hierarchical model as an appropriate conjugate family of priors (Berger, 1985, P. 288). Suppose that conditional on Σ, θ has a multivariate normal distribution with mean ν and covariance matrix $(\frac{1}{\beta})\Sigma$, where $\nu \in \Re^p$ and $\beta>0$, i.e., $\theta|\Sigma \sim N_p(\nu, \frac{\Sigma}{\beta})$. Furthermore assume that Σ has a pdf h(Σ). The likelihood function in (1.1) is now combined with the joint prior density for θ and Σ to obtain the conditioned density of θ given $\underline{\mathbf{x}}$, Σ as

$$q(\theta \mid \underline{\mathbf{x}}, \Sigma) \propto \exp[\frac{-1}{2}\{\beta(\theta-\nu)'\Sigma^{-1}(\theta-\nu) + N(\theta-\overline{\mathbf{x}})'\Sigma^{-1}(\theta-\overline{\mathbf{x}})\}]. \qquad (3.2)$$

It can be easily verified, by adding and subtracting $b^* = \frac{N\overline{\mathbf{x}}+\beta\nu}{N+\beta}$ in the above parentheses, that

$$\beta(\theta-\nu)'\Sigma^{-1}(\theta-\nu) + N(\theta-\overline{\mathbf{x}})'\Sigma^{-1}(\theta-\overline{\mathbf{x}})$$

$$= (N+\beta)(\theta-b^*)'^{-1}\Sigma^{-1}(\theta-b^*) + \frac{N\beta}{N+\beta}(\overline{\mathbf{x}}-\nu)'\Sigma^{-1}(\overline{\mathbf{x}}-\nu).$$

Hence (3.2) reduces to

$$q(\theta \mid \underline{\mathbf{x}}, \Sigma) \propto \exp[\frac{-1}{2}\{(N+\beta)(\theta-b^*)'\Sigma^{-1}(\theta-b^*)\}].$$

Therefore, $\theta \mid \underline{\mathbf{x}}, \Sigma \sim N_p(b^*, \frac{\Sigma}{N+\beta})$. Now, from (2.1) the Bayes estimator of θ for an arbitrary prior distribution of Σ, is

$$\hat{\theta}_{Bayes} = \omega\overline{\mathbf{X}} + (1-\omega)E(\theta \mid \underline{\mathbf{X}})$$

$$= \omega\overline{X} + (1-\omega)E[E(\theta \mid \underline{\mathbf{X}}, \Sigma)]$$

$$= \frac{N+\beta\omega}{N+\beta}\overline{\mathbf{X}} + \frac{\beta(1-\omega)\nu}{N+\beta}. \qquad (3.3)$$

We see that the coefficient $\frac{N+\beta\omega}{N+\beta}$ of $\overline{\mathbf{X}}$, is strictly between ω and 1. Also since the loss (1.2) is strictly convex, (3.3) is the unique Bayes estimator and hence admissible. It follows that c $\overline{\mathbf{X}}$ +**d** is admissible when ω<c<1 and d$\in \Re^p$.

Remark (3.3): It is seen that $\overline{\mathbf{X}}$ is the limit of Bayes estimators (3.3) relative to the normal prior, when $\beta\to 0$, and it is conjectured that it is admissible. The admissibility of $\overline{\mathbf{X}}$ has been studied by Chang and Kim (1997).

4. BAYES ESTIMATORS OF THE COVARIANCE MATRIX

For the estimation of Σ, the quadratic loss function

$$L(\Sigma,\hat{\Sigma}) = \mathrm{tr}(\Sigma-\hat{\Sigma})^2$$

has been considered by various authors (James and Stein, 1961; Haff, 1979). Now, in this case, it is helpful to use a balanced loss function L_1, similar to (1.2), as

$$L_1(\Sigma,\hat{\Sigma}) = \frac{\omega}{N}\sum_{i=1}^{N}\mathrm{tr}[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'-\hat{\Sigma}]^2 + (1-\omega)\mathrm{tr}(\Sigma-\hat{\Sigma})^2. \tag{4.1}$$

Note that $E[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'] = \Sigma$.

Now, to determine the Bayes estimator of Σ, it is enough to find a positive definite symmetric matrix $\hat{\Sigma}$ which minimizes $E[L_1(\Sigma,\hat{\Sigma}) \mid \underline{X}]$. Given a posterior probability density function for Σ, it can be employed to obtain the expectation of the BLF in (4.1) namely:

$$E[L_1(\Sigma,\hat{\Sigma}) \mid \underline{\mathbf{X}}] = \frac{\omega}{N}\sum_{i=1}^{N}\mathrm{tr}[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'-\hat{\Sigma}]^2$$
$$+(1-\omega)E[\mathrm{tr}(\Sigma-\hat{\Sigma})^2 \mid \underline{\mathbf{X}}] \tag{4.2}$$

But

$$E[\mathrm{tr}(\Sigma-\hat{\Sigma})^2 \mid \underline{\mathbf{X}}] = E[\mathrm{tr}(\Sigma-M+M-\hat{\Sigma})^2 \mid \underline{\mathbf{X}}]$$
$$= E[\mathrm{tr}(\Sigma-M)^2 \mid \underline{\mathbf{X}}] + \mathrm{tr}(M-\hat{\Sigma})^2 \tag{4.3}$$

Where $M = E[\Sigma \mid \underline{\mathbf{X}}]$ is the posterior mean of the Σ. Also, by adding and subtracting M and S, where

$$(N-1)S = \sum_{i=1}^{N}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'$$

in the first term on the r.h.s. of (4.2), we have

$$\frac{\omega}{N}\sum_{i=1}^{N}\mathrm{tr}[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'-\hat{\Sigma}]^2$$
$$=\frac{\omega}{N}\sum_{i=1}^{N}\mathrm{tr}[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'-S]^2$$

$$+\omega \text{tr}(S-M)^2+\omega \text{tr}(M-\hat{\Sigma})^2+2\omega \text{tr}[(S-M)(M-\hat{\Sigma})]. \quad (4.4)$$

From (4.3) and (4.4), (4.2) reduces to

$$E[L_1(\Sigma,\hat{\Sigma})\,|\,\underline{\mathbf{X}}]=\frac{\omega}{N}\sum_{i=1}^{N}\text{tr}[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'-S]^2+\text{tr}(M-\hat{\Sigma})^2$$

$$+\omega \text{tr}(S-M)^2+2\omega \text{tr}[(S-M)(M-\hat{\Sigma})]$$

$$+(1-\omega)E[\text{tr}(\Sigma-M)^2\,|\,\underline{\mathbf{X}}].$$

on completing the square on $\hat{\Sigma}$, we have

$$E[L_1(\Sigma,\hat{\Sigma})\,|\,\underline{\mathbf{X}}]=\text{tr}(\hat{\Sigma}-\hat{\Sigma}_*)^2+\omega(1-\omega)\text{tr}(S-M)^2$$

$$+\frac{\omega}{N}\sum_{i=1}^{N}\text{tr}[\frac{N}{N-1}(\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})'-S]^2$$

$$+(1-\omega)E[\text{tr}(\Sigma-M)^2\,|\,\underline{\mathbf{X}}]. \quad (4.5)$$

where

$$\hat{\Sigma}_*=\omega S+(1-\omega)M. \qquad (\text{say } \hat{\Sigma}_{Bayes}). \quad (4.6)$$

From (4.5), $\hat{\Sigma}_*$ is the value of $\hat{\Sigma}$ that minimizes posterior expected loss and, is thus the Bayes estimator of Σ relative to the BLF in (4.1).

5. EXAMPLES

Let $\mathbf{x}_1,\ldots,\mathbf{x}_N$ be observations from $N_p(\theta,\Sigma)$. Let

$$\overline{\mathbf{x}}=\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \quad \text{and} \quad V=\sum_{i=1}^{N}(\overline{\mathbf{x}}_i-\overline{\mathbf{x}})(\overline{\mathbf{x}}_i-\overline{\mathbf{x}})'.$$

In this section two examples according to diffuse prior and conjugate prior are given as follows. For each case, Bayes estimators will be calculated. Details are given in Press (1972) and Anderson (1984).

Example (5.1). Natural Conjugate Prior: Consider the normal-inverted Wishart distribution as a prior for (θ,Σ), i.e,

$$h(\theta,\Sigma)\propto\frac{1}{|\Sigma|^{\frac{1}{2}}}\exp[\frac{-1}{2}(\theta-\Phi)'\Sigma^{-1}(\theta-\Phi)b]\frac{\exp(\frac{-1}{2}\text{tr}\,\Sigma^{-1}G)}{|\Sigma|^{m/2}} \quad (5.1)$$

Where b>0, m>2p and G is a positive definite matrix.

The marginal posterior distribution of θ is

$$h(\theta\,|\,\bar{\mathbf{x}},V) \propto \frac{1}{[1+(\theta-\theta^*)'B^{-1}(\theta-\theta^*)]^{\frac{N+m-p}{2}}}$$

Where

$$\theta^* = \frac{N\bar{\mathbf{x}}+b\Phi}{N+b}, \qquad B = [\frac{V+G}{N+b} + \frac{Nb}{(N+b)^2}(\Phi-\bar{\mathbf{x}})(\Phi-\bar{\mathbf{x}})'].$$

Thus, $(\theta\,|\,\bar{\mathbf{x}},V)$ follows a multivariate student t-distribution. Also the marginal posterior of $\sum$ will be an inverted Wishart distribution with scale matrix V+G+ $\frac{Nb}{N+b}(\bar{\mathbf{x}}-\Phi)(\bar{\mathbf{x}}-\Phi)'$, dimension p, and N+m degrees of freedom, i.e,

$$\textstyle\sum\,|\,\bar{\mathbf{x}},V \sim W^{-1}(V+G+\frac{Nb}{N+b}(\bar{\mathbf{x}}-\Phi)(\bar{\mathbf{x}}-\Phi)', p, N+m).$$

We have

$$E(\theta\,|\,\bar{\mathbf{x}},V) = \frac{N\bar{\mathbf{x}}+b\Phi}{N+b}, \qquad N+m-p>1$$

and

$$E(\textstyle\sum\,|\,\bar{\mathbf{x}},V) = \dfrac{V+G+\frac{Nb}{N+b}(\bar{\mathbf{x}}-\Phi)(\bar{\mathbf{x}}-\Phi)'}{N+m-2p-2}, N+m-2p>2.$$

Hence from (2.1) and (4.6)

$$\hat{\theta}_{Bayes} = \frac{N+b\omega}{N+b}\bar{\mathbf{X}} + \frac{(1-\omega)b\Phi}{N+b},$$

and

$$\hat{\textstyle\sum}_{Bayes} = \omega\frac{V}{N-1} + (1-\omega)\dfrac{V+G+\frac{Nb}{N+b}(\bar{\mathbf{x}}-\Phi)(\bar{\mathbf{x}}-\Phi)'}{N+m-2p-2}$$

Example (5.2). Diffuse prior: Assume

$$h(\theta,\textstyle\sum) \propto \dfrac{1}{|\sum|^{\frac{p+1}{2}}}$$

The marginal posterior distribution of θ is

$$h(\theta\,|\,\bar{\mathbf{x}},V) \propto \frac{1}{[1+N(\theta-\bar{\mathbf{x}})'V^{-1}(\theta-\bar{\mathbf{x}})]^{N/2}}.$$

Thus, $(\theta \mid \bar{\mathbf{x}}, V)$ follows a multivariate student t-distribution. Also

$$\Sigma \mid \bar{\mathbf{x}}, V \sim W^{-1}(V, p, N+p).$$

We have

$$E(\theta \mid \bar{\mathbf{x}}, V) = \bar{\mathbf{x}}, N > 1 \qquad \text{and} \qquad E(\Sigma \mid \bar{\mathbf{x}}, V) = \frac{1}{N-p-2} V, N-p > 2.$$

Hence

$$\hat{\theta}_{Bayes} = \omega \bar{\mathbf{X}} + (1-\omega)\bar{\mathbf{X}} = \bar{\mathbf{X}},$$

and

$$\hat{\Sigma}_{Bayes} = \omega S + (1-\omega)\frac{V}{N-p-2} = \frac{N-\omega p - \omega - 1}{N-p-2} V.$$

Remark (5.1): Note that the diffuse prior $h(\theta, \Sigma) \propto |\Sigma|^{-\frac{p+1}{2}}$ obtain from (5.1) by putting b=0, G=**0** and m=p. Therefore the estimators $\bar{X}$ and $\frac{n-\omega p-\omega-1}{N-p-2} V$ are generalized Bayes estimators for θ and Σ respectively.

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