Pak J. Statist.
2003 Vol. 19(2) pp 273 – 280

INVERSE ASCENDING FACTORIAL MOMENTS OF THE HYPER-POISSON PROBABILITY DISTRIBUTION

by

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ABSTRACT

Inverse ascending factorial moments of the hyper-Poisson distribution have been derived in terms of hypergeometric series function. A recurrence relation for the negative moments and inverse ascending factorial moments is also derived. The Poisson distribution as a special case of the hyper-Poisson distribution has also been dealt with.

KEYWORDS

Negative moments, inverse ascending factorial moments, hypergeometric series function, hyper-Poisson distribution.

1. INTRODUCTION

Negative moments have been under study for quite some time. Many authors [Grab and Savage (1954), Mendenhall and Lehman (1960), Govindarajulu (1962,1963), Tiku (1964), Stancu (1968), Chao and Strawderman (1972), Lepage (1978), Cressie et al (1981), Cressie and Borkent (1986) and Jones (1986, 1987) Roohi (2002)], have worked on the negative moments of discrete distributions, mainly binomial, Poisson, geometric and negative binomial distributions. Ahmad and Sheikh (1983) used Chao and Strawderman (1972) technique to obtain the first negative moment of the hyper-Poisson distribution and stated the conditions under which this moment is identical to that of the Poisson distribution.

In this paper, we derive inverse ascending factorial moments of the hyper-Poisson distribution in terms of hyper-geometric series function. The expressions are simple and easy to compute. A recurrence relation of negative ascending factorial moments is also derived so that higher moments are easily calculated. Similar results have been given for the Poisson distribution.

2. INVERSE ASCENDING FACTORIAL MOMENTS

Theorem 1:

Suppose the random variable X follows a hyper-Poisson distribution with parameters θ and λ. Then the inverse ascending factorial moment of X is:

$$m'_{-[k]} = E\left[\prod_{i=1}^{k}(x+i)\right]^{-1} = C_{q,l}\,(k!)^{-1}\;{}_2F_2\left[1,1;\,l,\,k+1;\,q\right],\ \text{k=1,2,}\ldots$$

where $$C_{q,l} = \left\{{}_1F_1\left[1;\,l;q\right]\right\}^{-1} \tag{1}$$

Proof:

Since $\left[(\mathrm{x}+1)(\mathrm{x}+2)\cdots(\mathrm{x}+\mathrm{k})\right]^{-1}$

$$= \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!}\frac{1}{x+s},$$

$$\mu'_{-[\mathrm{k}]} = \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}}(\mathrm{X}+\mathrm{i})\right]^{-1}$$

$$= \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!}E\left(\frac{1}{X+s}\right) \qquad \text{(see Jones 1987)}$$

$$= \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!}\,.C_{q,l}\;s^{-1}\;{}_2F_2\left[1,s;\,l,\,s+1;\,q\right]$$

(see Ahmad and Sheikh 1983)

$$= C_{q,l}\,(k!)^{-1}\;{}_2F_2\left[1,1;\,l,\,k+1;\,q\right] \qquad \text{(see Ahmad and Roohi 2002)}$$

Corollary:

For λ=1 we get the inverse ascending factorial moment of the Poisson distribution,

$$m'_{-[k]} = \sum_{s=1}^{k} \frac{(-1)^{s+1}}{(s-1)!\,(k-s)!} \;.\; e^{-q}\; s^{-1} \;\; {}_2F_1\left[s,;\; s+1;\; q\right]$$

$$= e^{-q}\left[k!\right]^{-1} \; {}_1F_1\left[1;\; k+1; q\right] \qquad \text{k=1,2,….} \qquad \text{(see Ahmad and Roohi 2002)}$$

3. RECURRENCE RELATION OF NEGATIVE MOMENTS AND INVERSE ASCENDING FACTORIAL MOMENTS OF THE HYPER-POISSON DISTRIBUTION

Theorem 2: Suppose the random variable X has a Hyper-Poisson Distribution with parameters $q \; and \;\; l$. Then

$$E(X + A)^{-1} = \frac{1}{q}\left[1 + \frac{1-l}{A-1} C_{q,l} + (l - A)\, E(X + A - 1)^{-1}\right] \quad \text{for A > 1} \qquad (2)$$

Proof: We have

$$E(X + A)^{-1} = A^{-1} C_{q,l} \;\; {}_2F_2\left[1,\; A;\; l,\; A+1; q\right]$$

(see Ahmad and Sheikh 1983)

Using the identity (see Rainville 1960)

$${}_2F_2\left[\alpha_1,\, \alpha_2;\, \beta_1,\, \beta_2;\, x\right] = {}_2F_2\left[\alpha_1 - 1,\, \alpha_2;\, \beta_1,\, \beta_2;\, x\right] + x\left\{\frac{\alpha_2 - \beta_1}{\beta_1(\beta_2 - \beta_1)}\right.$$

$$\left. {}_2F_2\left[a_1,\, a_2;\; b_1 + 1,\; b_2; x\right] + \frac{a_2 - b_2}{b_2(b_1 - b_2)} \; {}_2F_2\left[a_1,\, a_2;\; b_1,\, b_2 + 1; x\right]\right\} \qquad (3)$$

For $\alpha_1 = \text{A},\ \alpha_2 = 1,\ \beta_1 = \text{A},\ \beta_2 = \lambda$ and x = θ, we get:

$${}_2\text{F}_2\left[1,\ \text{A};\ \lambda,\ \text{A};\ \theta\right] = {}_2\text{F}_2\left[1,\ \text{A} - 1;\ \lambda,\ \text{A};\ \theta\right] + \theta\left\{\frac{1 - \text{A}}{\text{A}(\lambda - \text{A})} \; {}_2\text{F}_2\left[1,\ \text{A};\ \lambda,\ \text{A} + 1;\ \theta\right]\right.$$

$$\left. + \frac{1 - \lambda}{\lambda(\text{A} - \lambda)} \; {}_2\text{F}_2\left[1,\ \text{A};\ \lambda + 1,\ \text{A};\ \theta\right]\right\}$$

We know that

$${}_2F_2[1,\ A;\ l,\ A;q] = {}_1F_1(1;\ l;q)$$

and
$${}_2F_2[1,\ A;\ l+1,\ A;q] = {}_1F_1(1;\ l+1;q)$$

Then

$${}_1F_1[1;\ l;q] = \frac{A-1}{C_{q,l}}E(X+A-1)^{-1} + \frac{q(1-A)}{(l-A)}\cdot\frac{1}{C_{q,l}}E(X+A)^{-1} + \frac{q(1-l)}{l(A-l)}\,{}_1F_1(1;\ l+1;q)$$

Hence

$$E(X+A)^{-1} = \frac{C_{q,l}}{1-A}\left\{\frac{l-A}{q}\,{}_1F_1[1;\ l;q] + \frac{1-l}{l}\,{}_1F_1[1;\ l+1;q]\right\}$$

$$+\frac{\lambda-A}{\theta}E(X+A-1)^{-1}$$

We know that

$${}_1F_1[1;\ \lambda+1;\ \theta] = \frac{\lambda}{\theta}\{{}_1F_1[1;\ \lambda;\ \theta]-1\} \qquad (5)$$

Substituting (5) in (4) we get:

$$E(X+A)^{-1} = \frac{C_{\theta,\lambda}}{1-A}\left\{\frac{1-A}{\theta}\,{}_1F_1[1;\ \lambda;\ \theta] - \frac{1-\lambda}{\theta}\right\} + \frac{\lambda-A}{\theta}E(X+A-1)^{-1}$$

Hence the result.

Corollary:

For $\lambda = 1$, we get the recurrence relation for the negative moment of the Poisson distribution.

$$E(X+A)^{-1} = \frac{1}{\theta}\left\{1-(A-1)E(X+A-1)^{-1}\right\},\ A>1$$

This result was also obtained by Chao and Strawderman (1972) by integrating the probability generating function.

Theorem 3:

Suppose that the random variable X has a Hyper-Poisson distribution with parameters θ and λ, and $\mu'_{[k]}$ is the kth inverse ascending factorial moment of X. Then the relation

$$k^2\theta\mu'_{-[k+1]} = (k+\theta)(1-\lambda+k)\mu'_{-[k]} - (k+1-\lambda)\mu'_{-[k-1]}$$

$$+\frac{q(l-1)^2}{k!}C_{q,l}\ {}_2F_2[1,1;\ l+1,\ k+1;q]$$

holds for k > 1.

Proof:

We know that (see equation (1))

$$\mu'_{-[k]} = \frac{C_{\theta,\lambda}}{k!}\ {}_2F_2[1,1;\ \lambda,\ k+1;\ \theta]$$

then

$$\mu'_{-[k+1]} = \frac{C_{\theta,\lambda}}{(k+1)!}\ {}_2F_2[1,1;\ \lambda,\ k+2;\ \theta]$$

Using the identity (see Rainville 1960)

$$(\alpha_1-\beta_1+1)\ {}_2F_2[\alpha_1,\alpha_2;\ \beta_1,\ \beta_2;\ x] = \alpha_1\ {}_2F_2[\alpha_1+1,\ \alpha_2;\ \beta_1-1,\ \beta_2;\ x]$$
$$-(b_1-1)\ {}_2F_2[a_1,\ a_2;\ b_1-1,\ b_2;\ x] \qquad (7)$$

If $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = k+2$, $\beta_2 = \lambda$, $x = \theta$, we have

$$(-k)\ {}_2F_2[1,1;\ l,\ k+2;q] = {}_2F_2[1,2,\ l,\ k+2;q]$$
$$-(k+1)\ {}_2F_2[1,1;\ \lambda,\ k+1;\ \theta] \qquad (8)$$

Using the identity (3) for $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = \lambda$, $\beta_2 = k+1$, $x = \theta$, we get:

$${}_2F_2[1,2;\ l,\ k+1;q] = {}_2F_2[1,1,\ l,k+1;q] + q\left\{\frac{1-l}{(k+1-l)}\ {}_2F_2[1,2,\ l+1,k+1;q]\right.$$
$$\left.-\frac{k}{(k+1)(\lambda-k-1)}\ {}_2F_2[2,1;\ \lambda,\ k+2;\ \theta]\right\}$$

Hence

$$\frac{kq}{(k+1)(k+1-l)}\ {}_2F_2[1,2;\ l,\ k+2;q] = {}_2F_2[1,2;\ l,\ k+1;q] - {}_2F_2[1,1;\ l,\ k+1;q]$$

$$+\frac{q(l-1)}{l(k+1-l)}\ {}_2F_2[1,2;\ l+1,\ k+1;q]$$

$$
\begin{aligned}
{}_2F_2[1,\ 2;\ \lambda,\ k+2;\ \theta] &= \frac{(k+1)(k+1-\lambda)}{k\theta}\ {}_2F_2[1,\ 2;\ \lambda,\ k+1;\ \theta] \\
&\quad - \frac{(k+1)(k+1-l)}{kq}\ {}_2F_2[1,\ 1;\ l,\ k+1;\ q] + \frac{(\lambda-1)(k+1)}{\lambda k}\ {}_2F_2[1,\ 2;\ \lambda+1,\ k+1;\ \theta] \qquad (9)
\end{aligned}
$$

Substituting (9) in (8) we get:

$$
\begin{aligned}
(-k)\ {}_2F_2[1,\ 1;\ \lambda,\ k+2;\ \theta] &= \frac{(k+1)(k+1-\lambda)}{k\theta}\ {}_2F_2[1,\ 2;\ \lambda,\ k+1;\ \theta] \\
&\quad - \frac{(k+1)(k+1-\lambda)}{k\theta}\ {}_2F_2[1,\ 1;\ \lambda,\ k+1;\ \theta] \\
&\quad + \frac{(\lambda-1)(k+1)}{\lambda k}\ {}_2F_2[1,\ 2;\ \lambda+1,\ k+1;\ \theta] - (k+1)\ {}_2F_2[1,\ 1;\ \lambda,\ k+1;\ \theta]
\end{aligned}
$$

Thus

$$
\begin{aligned}
{}_2F_2[1,\ 1;\ \lambda,\ k+2;\ \theta] &= \frac{(k+1)(k+1-\lambda+k\theta)}{k^2\theta}\ {}_2F_2[1,\ 1;\ \lambda,\ k+1;\ \theta] \\
&\quad - \frac{(k+1)(k+1-\lambda)}{k^2\theta}\ {}_2F_2[1,\ 2;\ \lambda,\ k+1;\ \theta] \\
&\quad - \frac{(l-1)(k+1)}{lk^2}\ {}_2F_2[1,\ 2;\ l+1,\ k+1;\ q] \qquad (10)
\end{aligned}
$$

Using the identity (7) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = k+1$, $\beta_2 = \lambda$, $x = \theta$, we have:

$$
{}_2F_2[1,\ 2;\ l,\ k+1;\ q] = k\ {}_2F_2[1,\ 1;\ l,\ k;\ q] - (k-1)\ {}_2F_2[1,\ 1;\ l,\ k+1;\ q] \qquad (11)
$$

Again using (7) for $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = \lambda+1$, $\beta_2 = k+1$, $x = \theta$, we have:

$$
{}_2F_2[2,\ 1;\ k+1,\ \lambda+1;\ \theta] = (\lambda)\ {}_2F_2[1,\ 1;\ k+1,\ \lambda;\ \theta] - (\lambda-1)\ {}_2F_2[1,\ 1;\ \lambda+1,\ k+1;\ \theta] \qquad (12)
$$

Substituting (11) and (12) in (10), we obtain:

$$
\begin{aligned}
{}_2F_2[1,\ 1;\ \lambda,\ k+2;\ \theta] &= \frac{(k+1)(k+1-\lambda+k\theta)}{k^2\theta}\ {}_2F_2[1,\ 1;\ \lambda,\ k+1;\ \theta] \\
&\quad - \frac{(k+1)(k+1-\lambda)}{k^2\theta}\left\{(k)\ {}_2F_2[1,\ 1;\ \lambda,\ k;\ \theta] - (k-1)\ {}_2F_2[1,\ 1;\ \lambda,\ k+1;\ \theta]\right\} \\
&\quad - \frac{(\lambda-1)(k+1)}{k^2\lambda}\ \{(\lambda)\ {}_2F_2[1,\ 1;\ \lambda,\ k+1;\ \theta]
\end{aligned}
$$

$$-(\lambda-1)\,{}_2F_2[1,\,1;\,\lambda+1,\,k+1;\,\theta]\}$$

$$=\frac{(k+1)(k\theta+k^2+k-k\lambda-\theta\lambda+\theta)}{k^2\theta}\,{}_2F_2[1,\,1;\,\lambda,\,k+1;\,\theta]$$

$$-\frac{(k+1)(k+1-\lambda)}{k\theta}\,{}_2F_2[1,\,1;\,\lambda,\,k;\,\theta]$$

$$+\frac{(l-1)^2(k+1)}{k^2 l}\,{}_2F_2[1,\,1;\,l+1,\,k+1;\,q]$$

Hence

$$\mu'_{-[k+1]}=\frac{(k\theta+k^2+k-k\lambda-\theta\lambda+\theta)}{k^2\theta}\mu'_{-[k]}-\frac{k+1-\lambda}{k^2\theta}\mu'_{-[k-1]}$$

$$+\,C_{\theta,\lambda}\,\frac{(\lambda-1)^2}{k^2\lambda\;k!}\,{}_2F_2[1,\,1;\,\lambda+1,\,k+1;\,\theta]$$

$$k^2\theta\,\mu'_{-[k+1]}=(k+\theta)(1-\lambda+k)\mu'_{-[k]}-(k+1-\lambda)\mu'_{-[k-1]}$$

$$+\frac{\theta(\lambda-1)^2}{k!}C_{\theta,\lambda}\;{}_2F_2[1,1;\,\lambda+1,\,k+1;\,\theta], \qquad k>1$$

Corollary:

When λ = 1, in equation (6), the recurrence relation for the inverse ascending factorial moment of Poisson distribution is:

$$k\theta\,\mu'_{-[k+1]}=(k+\theta)\mu'_{-[k]}-\mu'_{-[k-1]}, \quad k>1$$

4. Estimation

Consider the hyper-Poisson distribution with λ known and θ unknown.

$$Var(\hat{q})=\frac{1}{nA_1^2\;{}_1F_1[1;\,l;\,q]}\left[{}_3F_3\,[1,\,1,1;\,2,\,2,\,l;\,q]-\frac{\{{}_2F_2[1,\,1;\,2,\,l;\,q]\}^2}{{}_1F_1[1;\,l;\,q]}\right] \qquad (13)$$

(see Roohi and Ahmad 2003)

for λ = 1 in (13), we get the variance of the negative moment estimator $\hat{\theta}$ of θ, the parameter of a Poisson distribution.

$$Var(\hat{\theta})=\left(\frac{\theta e^{-\theta}+e^{-\theta}-1}{\theta^2}\right)^{-2}\frac{e^{-\theta}}{n}\left[{}_2F_2\,[1,\,1;\,2,\,2;\,\theta]-\frac{\{{}_1F_1[1;\,2;\,\theta]\}^2}{e^{\theta}}\right]$$

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