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FITTED MINIMAX DESIGNS FOR ESTIMATING THE SLOPE OF A THIRD-ORDER POLYNOMIAL MODEL IN A HYPERCUBIC REGION

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ABSTRACT

The criterion of minimizing the variance of the estimated slope maximized over all points in the region of interest is considered for third-order polynomial regression models over hypercubic region. Optimal designs derived are then investigated as functions of the number of factors. Regression models are fitted to obtain the functional relationships which are useful for determining the optimal design for a given number of factors, especially when the number of factors is large.

KEY WORDS

Hypercubic regions; minimax designs; optimal designs; third-order models.

INTRODUCTION

Mukerjee and Huda (1985) considered variance of the estimated slope averaged over all directions and introduced minimization of the variance maximized over all points in the factor space as a design criterion. They derived the optimal second- and third-order designs for regression over hyperspherical regions.

The minimax designs are much more difficult to obtain when the experimental region is a hypercube. Huda and Shafiq (1992) derived the minimax designs for second-order models over hypercubes. Huda and Al-Shiha (1998) worked on the corresponding problem for third-order models over hypercubes.

In this paper, we provide fitted models for the relationship between the minimax designs and the number of factors.

PRELIMINARIES

Assume that the response y depends upon k factors $x_1, x_2, \ldots, x_k$ through a smooth function relationship $y = \eta(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)^t$. Let y_i be the be the observed response taken at the point $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)^t$. It is assumed that $y_i = \eta(\boldsymbol{x}_i) + e_i$ where the e_i's are uncorrelated random errors with zero mean and constant variance σ^2, taken to be unity without loss of generality.

Let $\eta(\boldsymbol{x})$ be a linear parametric function given by $\eta(\boldsymbol{x}) = f^{\,t}(\boldsymbol{x})\,\mathbf{q}$ where $f^{\,t}(\boldsymbol{x})$ contains p linearly independent functions of $\boldsymbol{x}$ and $\mathbf{q}$ is the corresponding column vector of unknown parameters. A design ξ is a probability measure on the experimental region χ. If N observations are taken according to ξ then N cov($\hat{\mathbf{e}}$)= $M^{-1}(\xi)$, where $\hat{\mathbf{e}}$ is least squares estimator of $\mathbf{q}$ and $M(\xi) = \int_{\boldsymbol{c}} f(\boldsymbol{x})\, f^{\,t}(\boldsymbol{x})\, \xi(\mathrm{d}\boldsymbol{x})$ is the information matrix of ξ. The estimated response at $\boldsymbol{x}$ is $\hat{y}(\boldsymbol{x}) = f^{\,t}(\boldsymbol{x})\hat{\mathbf{e}}$. The column vector of estimated slopes along the factor axes at $\boldsymbol{x}$ is given by $\mathrm{d}\hat{y}/\mathrm{d}\boldsymbol{x} = (\partial \hat{y}/\partial x_1, \ldots, \partial \hat{y}/\partial x_k)^t = H\,\hat{\mathbf{e}}$ where H is a $k \times p$ matrix with i-th row given by $\partial f^t(\boldsymbol{x})/\partial x_i$. Thus N cov$(\mathrm{d}\hat{y}/\mathrm{d}\boldsymbol{x}) = HM^{-1}(\xi)H^t$. Under the minimax criterion the objective is $\min_{\xi} \max_{\boldsymbol{x} \in \boldsymbol{c}}\, tr(H\,M^{-1}(\xi)H^t)$.

In what follows we consider the third-order model, that is, the case where $f^t(\boldsymbol{x})$ contains all the ${}_{k+3}C_3$ powers and products of powers up to degree 3 in the x_i's. Also, the design region χ is taken to be a hypercube which without loss of generality is assumed to be $[-1,1]^k$.

Since the model is permutation invariant and the design region is permutation invariant as well as symmetric, we shall consider only symmetric and permutation invariant designs. For such a design ξ, $\xi(\boldsymbol{x})=\xi(-\boldsymbol{x})$ and $\xi(\boldsymbol{x})=\xi(\boldsymbol{z})$ for any permutation $\boldsymbol{z}$ of $\boldsymbol{x}$.

It follows that for a symmetric permutation invariant third-order design ξ,

$$\begin{aligned} V &= tr(HM^{-1}(\xi)H^{t}) \\ &= ka + [4\{\boldsymbol{a}_4 + (k-2)\boldsymbol{a}_{22} - (k-1)\boldsymbol{a}_2^2\}/(\boldsymbol{a}_4 - \boldsymbol{a}_{22})\{\boldsymbol{a}_4 + (k-1)\boldsymbol{a}_{22} - \\ &\quad k\boldsymbol{a}_2^2\} + (k-1)/\boldsymbol{a}_{22} + 6b + 2(k-1)c]\, \boldsymbol{r}_x^2 + \{9d + (k-1)(g+h)\}\, \boldsymbol{r}_x^4 + \\ &\quad [(k-2)/\boldsymbol{a}_{222} - 2\{9d - 6f + (k-5)/g - 3h\}] \sum_{i>j} x_i^2 x_j^2 \qquad (1) \end{aligned}$$

where $\boldsymbol{r}_x^2 = \boldsymbol{x}^t\boldsymbol{x} = \sum_{i=1}^{k} x_i^2$, $\boldsymbol{a}_2 = \int_\chi x_i^2 \xi(dx)$, $\boldsymbol{a}_4 = \int_\chi x_i^4 \xi(dx)$,

$\boldsymbol{a}_6 = \int_\chi x_i^6 \xi(dx)$, $\boldsymbol{a}_{22} = \int_\chi x_i^2 x_j^2 \xi(dx)$, $\boldsymbol{a}_{24} = \int_\chi x_i^4 x_j^2 \xi(dx)$, and

$\boldsymbol{a}_{222} = \int_\chi x_i^2 x_j^2 x_l^2 \xi(dx)$ $(i, j, l = 1, \ldots, k;\ i \neq j,\ i \neq j \neq l)$

with

$$\begin{aligned} a &= A[\boldsymbol{a}_6\{\boldsymbol{a}_{24} + (k-2)\boldsymbol{a}_{222}\} - (k-1)\boldsymbol{a}_{24}^2], \\ b &= -A[\boldsymbol{a}_4\{\boldsymbol{a}_{24} + (k-2)\boldsymbol{a}_{222}\} - (k-1)\boldsymbol{a}_{22}\,\boldsymbol{a}_{24}], \\ c &= -A[\boldsymbol{a}_6\boldsymbol{a}_{22} - \boldsymbol{a}_4\,\boldsymbol{a}_{24}], \\ d &= A[\boldsymbol{a}_2\{\boldsymbol{a}_{24} + (k-2)\boldsymbol{a}_{222}\} - (k-1)\boldsymbol{a}_{22}^2], \\ f &= -A[\boldsymbol{a}_2\boldsymbol{a}_{24} - \boldsymbol{a}_{22}\,\boldsymbol{a}_4], \\ g &= (\boldsymbol{a}_{24} - \boldsymbol{a}_{222})^{-1}, \\ h &= -A[(\boldsymbol{a}_2\boldsymbol{a}_6 - \boldsymbol{a}_4^2)\boldsymbol{a}_{222} - \{\boldsymbol{a}_2\boldsymbol{a}_{24}^2 - 2\boldsymbol{a}_{22}\,\boldsymbol{a}_4\,\boldsymbol{a}_{24} + \boldsymbol{a}_6\,\boldsymbol{a}_{22}^2\}]/(\boldsymbol{a}_{24} - \boldsymbol{a}_{222}), \\ A &= [(\boldsymbol{a}_2\boldsymbol{a}_6 - \boldsymbol{a}_4^2)\{\boldsymbol{a}_{24} + (k-2)\boldsymbol{a}_{222}\} - (k-1)\{\boldsymbol{a}_2\,\boldsymbol{a}_{24}^2 - \boldsymbol{a}_{22}\boldsymbol{a}_4\boldsymbol{a}_{24} + \\ &\quad \boldsymbol{a}_6\boldsymbol{a}_{22}^2\}]^{-1}. \end{aligned}$$

The above expression for V shows that a symmetric, permutation invariant third-order design is slope rotatable over all directions if and only if the last term in (1), that is, the coefficient of $\sum_{i>j} x_i^2 x_j^2$ vanishes for the design. This condition is clearly satisfied if the design is rotatable (Box and Hunter, 1957).

Note that V is a positive definite quadratic form in $\mathbf{z} = (x_1^2, \ldots, x_k^2)^t$ and goes to infinity as $r_x \to \infty$. Hence for $\boldsymbol{x} \in [-1,1]^k$ the maximum of V occurs at $\boldsymbol{x} = \mathbf{0}$ or at the boundary of $[-1,1]^k$. Indeed for all sensible designs, that is designs not close to singularity, the maximum V occurs at $\boldsymbol{x}=(\pm 1, \ldots, \pm 1)$. Denoting the maximum value of V by V_m, the problem is therefore reduced to minimization of

$$k^{-1}V_m = a + 4\{\boldsymbol{a}_4 + (k-2)\boldsymbol{a}_{22} - (k-1)\boldsymbol{a}_2^{\,2}\}/(\boldsymbol{a}_4 - \boldsymbol{a}_{22})\{\boldsymbol{a}_4 + (k-1)\boldsymbol{a}_{22} - k\boldsymbol{a}_2^{\,2}\} + (k-1)/\boldsymbol{a}_{22} + 6b + 2(k-1)c + k\{9d + (k-1)(g+h)\} + [(k-2)/\boldsymbol{a}_{222} - 2\{9d - 6f + (k-5)g - 3h\}](k-1)/2 \qquad (2)$$

with respect to the design ξ.

Huda and Al-Shiha (1998) considered minimization of (2) for two special classes of designs. First, they considered the four-level factorials which are the designs putting equal mass at each support point of a 4-level factorial $[-\sqrt{u}, -\sqrt{t}, \sqrt{t}, \sqrt{u}]^k$ (or a suitably chosen subset of it) where $0 < t < u \le 1$, say. These are the simplest third-order designs for regression over hypercubes. It can be shown that under our minimax criterion, we only need to consider factorials with $u = 1$, the highest possible value. Therefore, the objective function in (2) depends only on one variable t and we may write (2) as $k^{-1}V_m(t)$ which we have to minimize with respect to t. It is very difficult to find the value of t that minimizes $k^{-1}V_m(t)$ algebraically. Huda and Al-Shiha (1998) have numerically obtained the optimal four-level factorial designs for k=2 to k=10 which are reproduced in the Table 1.

Table 1: Minimax four-level factorials given by
Huda and Al-Shiha (1998)

k	2	3	4	5	6
t	0.195	0.181	0.173	0.169	0.169
$k^{-1}V_m$	158.078	208.212	262.749	322.089	386.398
k	7	8	9	10	
t	0.170	0.174	0.179	0.185	
$k^{-1}V_m$	455.710	529.065	609.065	692.852	

Secondly, Huda and Al-Shiha (1998) considered the class of product designs which is found to be quite useful. A design ξ is a product design if $\xi(\mathbf{x})=\prod_{i=1}^{k} \xi_i(x_i)$ where each ξ_i is a design on [-1, 1]. It is symmetric if every ξ_i is symmetric and it is permutation invariant if $\xi_i=\mu$ (i=1, …, k); i.e., $\xi(\mathbf{x})=\prod_{i=1}^{k}\mu(x_i)$. Lim and Studden (1988) found that the D-optimal designs among the symmetric product designs must be permutation invariant. Their numerical studies showed that the D-optimal designs within this class are quite efficient in comparison with the overall D-optimal designs. In view of their findings it is of interest to obtain the minimax designs within this class.

For a permutation invariant symmetric product design ξ with $\xi(\mathbf{x})=\prod_{i=1}^{k}\mu(x_i)$, the non zero elements of the information matrix are given by $\boldsymbol{a}_2 = \int_{-1}^{1} x^2\mu(dx)$, $\boldsymbol{a}_4 = \int_{-1}^{1} x^4\mu(dx)$, $\boldsymbol{a}_6 = \int_{-1}^{1} x^6\mu(dx)$, $\boldsymbol{a}_{22} = \boldsymbol{a}_2^2$, $\boldsymbol{a}_{24} = \boldsymbol{a}_2\,\boldsymbol{a}_4$, and $\boldsymbol{a}_{222} = \boldsymbol{a}_2^3$. For such a design ξ, we have

$a = A[\boldsymbol{a}_2\,\boldsymbol{a}_6\{\boldsymbol{a}_4+(k-2)\boldsymbol{a}_2^2\}-(k-1)\,\boldsymbol{a}_2^2\boldsymbol{a}_4^2]$,

$b = -A\boldsymbol{a}_2\,\boldsymbol{a}_4(\boldsymbol{a}_4-\boldsymbol{a}_2^2)$,

$c = -A\boldsymbol{a}_2\,(\boldsymbol{a}_2\boldsymbol{a}_6-\boldsymbol{a}_4^2)$,

$d = A\,\boldsymbol{a}_2^2\,(\boldsymbol{a}_4-\boldsymbol{a}_2^2)$,

$f = 0$,

$g = \{\boldsymbol{a}_2\,(\boldsymbol{a}_4-\boldsymbol{a}_2^2)\}^{-1}$,

$h = 0$,

where

$A = \{\boldsymbol{a}_2\,(\boldsymbol{a}_4-\boldsymbol{a}_2^2)\,(\boldsymbol{a}_2\,\boldsymbol{a}_6-\boldsymbol{a}_4^2)\}^{-1}$.

Consequently, the objective function (2) reduces to

$$k^{-1}V_m = (k-1)/\boldsymbol{a}_2^2 + (k-1)(k-2)/\,2\boldsymbol{a}_2^3 + \{(k-1)\boldsymbol{a}_2 - 2(k-3)\}/(\boldsymbol{a}_4-\boldsymbol{a}_2^2) + (\boldsymbol{a}_6-6\boldsymbol{a}_4+9\boldsymbol{a}_2)/(\boldsymbol{a}_2\,\boldsymbol{a}_6-\boldsymbol{a}_4^2) + 5(k-1)/\boldsymbol{a}_2\,(\boldsymbol{a}_4-\boldsymbol{a}_2^2) \qquad (3)$$

which, for fixed $\boldsymbol{a}_2$ and $\boldsymbol{a}_4$, is strictly decreasing in $\boldsymbol{a}_6$. Hence in order to minimize (3) with respect to design ξ, we only consider a ξ with μ putting masses, say $w/2$ at each of ± 1 and $(1-w)/2$ at each of $\pm\sqrt{t}$ giving $\boldsymbol{a}_2$=

$w+(1-w)t$, $\alpha_4 = w+(1-w)t^2$, $\alpha_6 = w+(1-w)t^3$, $\alpha_2\alpha_6-\alpha_4^2 = w(1-w)\,t(1-t)^2$, and $\alpha_4-\alpha_2^2 = w(1-w)(1-t)^2$. Thus (3) may be written as a function $k^{-1}V_m(w, t)$ which we wish to minimize with respect to w and t subject to $w, t \in [0,1]$. Although, this minimization is extremely difficult to be done analytically, the results may be obtained quite easily numerically. The minimax symmetric product designs obtained numerically for k=2 to k=10 are reproduced from Huda and Al-Shiha (1998) in Table 2.

Table 2: Minimax symmetric product designs given by Huda and Al-Shiha (1998)

k	2	3	4	5	6
w	0.496	0.539	0.570	0.593	0.612
t	0.195	0.184	0.175	0.168	0.163
$k^{-1}V_m$	158.068	206.603	256.009	306.829	359.285
k	7	8	9	10	
w	0.628	0.642	0.654	0.665	
t	0.158	0.153	0.149	0.146	
$k^{-1}V_m$	413.481	469.464	527.259	586.868	

The four-level factorial designs with equal masses at all support points are in fact special cases of symmetric product designs. Thus a factorial design based on $[-1, -\sqrt{t}, \sqrt{t}, 1]^k$ is in fact a symmetric product design with μ having w=1/2. The minimax product designs are therefore better than the minimax factorials as the results of Table 1 and Table 2 clearly demonstrate.

In sections to follow we consider fitting simple models for predicting minimax optimal designs as functions of k, the number of factors in the response surface.

FITTED MODELS FOR FOUR-LEVEL FACTORIALS

Huda and Al-Shiha (1998) have obtained, partly numerically, the optimal four-level factorial designs for k=2 to k=10. In this paper, we have similarly found the optimal designs for k=2 to k=100. The results are presented in Table 3 and Table 4. Figure 1 and Figure 2 represent the graphs of the optimal value of t as a function of k. Based on these figures, we have fitted the optimal value of t as a function of k using the following models:

$$t=\beta_0+\sum_{i=1}^{5}\beta_i z^i+e \quad \text{(for } k = 11, \ldots, 100) \qquad (4)$$

$$t=\beta_0+\sum_{i=1}^{9}\beta_i z^i+e \quad \text{(for } k = 2, \ldots, 100) \qquad (5)$$

where $z = \ln(k)$. The fitted optimal values $\hat{t}$ are given in Table 3 and Table 4. The analysis of variance results and the least square estimates of the models parameters are given in Table 5 and Table 6. For both models, the value of R-square equals to 1 indicating a perfect fit. Moreover, all residual analyses showed perfect fit of the models to the data. Therefore, to find the optimal design among the 4-level factorials, we need only to know the number of factors k included in the design and then substitute it in the simple model given in equation (4) or (5).

FITTED MODELS FOR PERMUTATION INVARIANT SYMMETRIC PRODUCT DESIGNS

Huda and Al-Shiha (1998) have obtained, partly numerically, the optimal permutation invariant symmetric product designs for k=2 to k=10. In this paper, we have analogously found the optimal designs for k=2 to k=100. These optimal designs are presented in Table 7.

Figure 3 represents the graph of the optimal value of w as a function of k. Figure 4 represents the graph of the optimal value of t as a function of k. Figure 5 represents the graph of the optimal value of t as a function of the optimal value w. Based on these figures, we have fitted the following models:

$$w = \beta_0 + \sum_{i=1}^{3} \beta_i z^i + e \qquad (6)$$

$$t = \beta_0 + \sum_{i=1}^{3} \beta_i z^i + e \qquad (7)$$

$$t = \beta_0 + \sum_{i=1}^{3} \beta_i w^i + e \qquad (8)$$

where $z = \ln(k)$. Equation (6) is fitted in order to predict the optimal value of w as a function of k. The fitted optimal values of w are given in Table 8. Equation (7) is fitted in order to predict the optimal value of t as a function of k. The fitted optimal values of t are given in Table 9. Equation (8) is fitted in order to predict the optimal value of t as a function of the optimal value of w. The fitted optimal values of t as function of the optimal values of w are given in Table 10.

The analysis of variance results and the least square estimates of the models parameters are given in Table 11, Table 12, and Table 13. For all models, the value of R-square equals to 1 indicating a perfect fit. Moreover, all residual analyses showed perfect fit of the models to the data. Therefore, to find the optimal permutation invariant symmetric product design, we need only to know the number of factors k included in the design and then substitute it in the simple models given in equations (6), (7), and (8).

COMMENTS AND DISCUSSION

Although traditionally the response surface methodology is used in situations involving a small and moderate number of factors, in many fields of applications response surfaces involving very large number of factors arise quite frequently (cf. Aslett et al (1998), Bates et al (1996)). Often, scientific phenomena, particularly in fields like electrical engineering, are now investigated by complicated computer models or codes. A computer experiment is a number of runs of codes with various inputs. Off course, the output of such experiments are deterministic but can be viewed as the realization of a stochastic process thereby allowing statistical design of experiments to be employed to obtain efficient experiments. Often, the codes are computationally expensive to run and hence optimal design theory should be employed to use the runs in the most informative and economic manner (cf. Sacks, Schiller and Welch (1989) and Sacks et al (1989)). In view of the recent developments in computer experiments and other set-ups involving large number of factors, the optimal designs presented in this paper should find many applications.

The results presented allow the experimenter to identify his/her optimal design, under the minimax criterion, as soon as the number of factors involved is decided.

Table 3: Minimax four-level factorials (k=11 to 100)

no. of factors k	Optimal Value t	Fitted Value $\hat{t}$	no. of factors k	Optimal Value t	Fitted Value $\hat{t}$	no. of factors k	Optimal Value t	Fitted Value $\hat{t}$
11	0.1912	0.1913	41	0.3584	0.3584	71	0.4320	0.4320
12	0.1986	0.1985	42	0.3616	0.3617	72	0.4338	0.4338
13	0.2062	0.2061	43	0.3648	0.3649	73	0.4355	0.4356
14	0.2140	0.2140	44	0.3680	0.3681	74	0.4374	0.4373
15	0.2218	0.2218	45	0.3712	0.3712	75	0.4391	0.4391
16	0.2293	0.2295	46	0.3743	0.3742	76	0.4408	0.4408
17	0.2370	0.2371	47	0.3771	0.3771	77	0.4424	0.4424
18	0.2444	0.2444	48	0.3800	0.3800	78	0.4440	0.4441
19	0.2515	0.2515	49	0.3827	0.3828	79	0.4456	0.4457
20	0.2584	0.2583	50	0.3856	0.3855	80	0.4472	0.4473
21	0.2650	0.2650	51	0.3883	0.3882	81	0.4488	0.4489
22	0.2712	0.2713	52	0.3907	0.3908	82	0.4504	0.4505
23	0.2775	0.2775	53	0.3934	0.3934	83	0.4520	0.4520
24	0.2835	0.2834	54	0.3960	0.3959	84	0.4536	0.4535
25	0.2891	0.2891	55	0.3983	0.3983	85	0.4552	0.4550
26	0.2946	0.2946	56	0.4008	0.4007	86	0.4566	0.4565
27	0.2999	0.2999	57	0.4031	0.4031	87	0.4579	0.4580
28	0.3051	0.3050	58	0.4055	0.4054	88	0.4594	0.4594
29	0.3099	0.3100	59	0.4077	0.4077	89	0.4608	0.4608
30	0.3147	0.3147	60	0.4099	0.4099	90	0.4623	0.4622
31	0.3195	0.3193	61	0.4120	0.4121	91	0.4635	0.4636
32	0.3239	0.3238	62	0.4143	0.4142	92	0.4651	0.4650
33	0.3280	0.3281	63	0.4163	0.4164	93	0.4663	0.4663
34	0.3323	0.3323	64	0.4184	0.4184	94	0.4675	0.4676
35	0.3363	0.3364	65	0.4206	0.4205	95	0.4690	0.4690
36	0.3403	0.3403	66	0.4224	0.4225	96	0.4703	0.4703
37	0.3440	0.3441	67	0.4243	0.4244	97	0.4715	0.4715
38	0.3479	0.3478	68	0.4264	0.4264	98	0.4728	0.4728
39	0.3515	0.3515	69	0.4283	0.4283	99	0.4742	0.4741
40	0.3550	0.3550	70	0.4302	0.4301	100	0.4752	0.4753

Table 4: Minimax four-level factorials (k=2 to 100)

no. of factors k	Optimal Value t	Fitted Value $\hat{t}$	no. of factors k	Optimal Value t	Fitted Value $\hat{t}$	no. of factors k	Optimal Value t	Fitted Value $\hat{t}$
2	0.1953	0.1953	35	0.3363	0.3364	68	0.4264	0.4264
3	0.1809	0.1808	36	0.3403	0.3404	69	0.4283	0.4283
4	0.1731	0.1732	37	0.3440	0.3442	70	0.4302	0.4302
5	0.1695	0.1695	38	0.3479	0.3479	71	0.4320	0.4320
6	0.1687	0.1687	39	0.3515	0.3514	72	0.4338	0.4339
7	0.1703	0.1701	40	0.3550	0.3549	73	0.4355	0.4356
8	0.1737	0.1736	41	0.3584	0.3583	74	0.4374	0.4374
9	0.1785	0.1785	42	0.3616	0.3617	75	0.4391	0.4391
10	0.1845	0.1846	43	0.3648	0.3649	76	0.4408	0.4408
11	0.1912	0.1914	44	0.3680	0.3680	77	0.4424	0.4425
12	0.1986	0.1987	45	0.3712	0.3711	78	0.4440	0.4441
13	0.2062	0.2063	46	0.3743	0.3741	79	0.4456	0.4457
14	0.2140	0.2140	47	0.3771	0.3770	80	0.4472	0.4473
15	0.2218	0.2218	48	0.3800	0.3799	81	0.4488	0.4489
16	0.2293	0.2294	49	0.3827	0.3827	82	0.4504	0.4504
17	0.2370	0.2369	50	0.3856	0.3854	83	0.4520	0.4520
18	0.2444	0.2442	51	0.3883	0.3881	84	0.4536	0.4535
19	0.2515	0.2513	52	0.3907	0.3907	85	0.4552	0.4550
20	0.2584	0.2582	53	0.3934	0.3933	86	0.4566	0.4564
21	0.2650	0.2649	54	0.3960	0.3958	87	0.4579	0.4579
22	0.2712	0.2713	55	0.3983	0.3983	88	0.4594	0.4593
23	0.2775	0.2774	56	0.4008	0.4007	89	0.4608	0.4607
24	0.2835	0.2834	57	0.4031	0.4031	90	0.4623	0.4621
25	0.2891	0.2891	58	0.4055	0.4054	91	0.4635	0.4635
26	0.2946	0.2947	59	0.4077	0.4077	92	0.4651	0.4649
27	0.2999	0.3000	60	0.4099	0.4099	93	0.4663	0.4663
28	0.3051	0.3051	61	0.4120	0.4121	94	0.4675	0.4676
29	0.3099	0.3101	62	0.4143	0.4143	95	0.4690	0.4689
30	0.3147	0.3148	63	0.4163	0.4164	96	0.4703	0.4703
31	0.3195	0.3194	64	0.4184	0.4185	97	0.4715	0.4716
32	0.3239	0.3239	65	0.4206	0.4205	98	0.4728	0.4729
33	0.3280	0.3282	66	0.4224	0.4225	99	0.4742	0.4742
34	0.3323	0.3324	67	0.4243	0.4245	100	0.4752	0.4755

Table 5: ANOVA Table for model (4)

Source	DF	SS	MS	F	P-value
Regression	5	0.54155	0.10831	1.536E+07	0.000
Error	84	0.00000	0.00000		
Total	89	0.54155			
R-Sq = 100.0%		R-Sq(adj) = 100.0%			

The regression equation is $\hat{t} = \hat{\beta}_0 + \sum_{i=1}^{5} \hat{\beta}_i z^i$, where the least squares estimates of the model Parameters are:

$\hat{\beta}_0 = 2.07115$ $\hat{\beta}_1 = -2.6627$ $\hat{\beta}_2 = 1.39448$ $\hat{\beta}_3 = -0.34432$ $\hat{\beta}_4 = 0.042329$
$\hat{\beta}_5 = -0.0020845$

Table 6: ANOVA Table for model (5)

Source	DF	SS	MS	F	P-value
Regression	9	0.875906	0.097323	7.875E+06	0.000
Error	89	0.000001	0.000000		
Total	98	0.875907			
R-Sq = 100.0%		R-Sq(adj) = 100.0%			

The regression equation is $\hat{t} = \hat{\beta}_0 + \sum_{i=1}^{9} \hat{\beta}_i z^i$, where the least squares estimates of the model Parameters are:

$\hat{\beta}_0 = -0.06217$	$\hat{\beta}_1 = 1.4439$	$\hat{\beta}_2 = -3.2268$
$\hat{\beta}_3 = 3.8455$	$\hat{\beta}_4 = -2.7659$	$\hat{\beta}_5 = 1.2470$
$\hat{\beta}_6 = -0.35224$	$\hat{\beta}_7 = 0.060426$	$\hat{\beta}_8 = -0.0057557$
$\hat{\beta}_9 = 0.00023362$		

Table 7: Minimax permutation invariant symmetric product designs (*k*=2 to 100)

k	*w*	*t*	*k*	*w*	*t*	*k*	*w*	*t*
2	0.496	0.195	35	0.782	0.105	68	0.833	0.084
3	0.540	0.184	36	0.784	0.104	69	0.834	0.084
4	0.570	0.175	37	0.786	0.103	70	0.835	0.083
5	0.593	0.168	38	0.789	0.102	71	0.836	0.083
6	0.612	0.163	39	0.791	0.101	72	0.836	0.083
7	0.628	0.158	40	0.793	0.100	73	0.837	0.082
8	0.642	0.153	41	0.795	0.100	74	0.838	0.082
9	0.654	0.149	42	0.797	0.099	75	0.839	0.082
10	0.665	0.146	43	0.799	0.098	76	0.840	0.081
11	0.674	0.143	44	0.800	0.097	77	0.841	0.081
12	0.683	0.140	45	0.802	0.097	78	0.842	0.080
13	0.691	0.137	46	0.804	0.096	79	0.843	0.080
14	0.698	0.135	47	0.806	0.095	80	0.843	0.080
15	0.705	0.132	48	0.807	0.095	81	0.844	0.079
16	0.711	0.130	49	0.809	0.094	82	0.845	0.079
17	0.717	0.128	50	0.810	0.094	83	0.846	0.079
18	0.723	0.126	51	0.812	0.093	84	0.847	0.078
19	0.728	0.124	52	0.813	0.092	85	0.847	0.078
20	0.732	0.123	53	0.815	0.092	86	0.848	0.078
21	0.737	0.121	54	0.816	0.091	87	0.849	0.077
22	0.741	0.120	55	0.817	0.091	88	0.850	0.077
23	0.745	0.118	56	0.819	0.090	89	0.850	0.077
24	0.749	0.117	57	0.820	0.089	90	0.851	0.076
25	0.753	0.115	58	0.821	0.089	91	0.852	0.076
26	0.756	0.114	59	0.823	0.088	92	0.852	0.076
27	0.760	0.113	60	0.824	0.088	93	0.853	0.076
28	0.763	0.112	61	0.825	0.087	94	0.854	0.075
29	0.766	0.111	62	0.826	0.087	95	0.854	0.075
30	0.769	0.109	63	0.827	0.087	96	0.855	0.075
31	0.772	0.108	64	0.828	0.086	97	0.856	0.074
32	0.774	0.107	65	0.829	0.086	98	0.856	0.074
33	0.777	0.106	66	0.830	0.085	99	0.857	0.074
34	0.780	0.105	67	0.831	0.085	100	0.858	0.073

Table 8: Minimax optimal values of w as a function of k

k	w	$\hat{w}$	k	w	$\hat{w}$	k	w	$\hat{w}$
2	0.496	0.496	35	0.782	0.782	68	0.833	0.833
3	0.540	0.539	36	0.784	0.784	69	0.834	0.834
4	0.570	0.570	37	0.786	0.786	70	0.835	0.835
5	0.593	0.593	38	0.789	0.789	71	0.836	0.836
6	0.612	0.612	39	0.791	0.791	72	0.836	0.837
7	0.628	0.628	40	0.793	0.793	73	0.837	0.837
8	0.642	0.642	41	0.795	0.795	74	0.838	0.838
9	0.654	0.654	42	0.797	0.797	75	0.839	0.839
10	0.665	0.665	43	0.799	0.799	76	0.840	0.840
11	0.674	0.674	44	0.800	0.800	77	0.841	0.841
12	0.683	0.683	45	0.802	0.802	78	0.842	0.842
13	0.691	0.691	46	0.804	0.804	79	0.843	0.843
14	0.698	0.698	47	0.806	0.806	80	0.843	0.844
15	0.705	0.705	48	0.807	0.807	81	0.844	0.844
16	0.711	0.711	49	0.809	0.809	82	0.845	0.845
17	0.717	0.717	50	0.810	0.810	83	0.846	0.846
18	0.723	0.722	51	0.812	0.812	84	0.847	0.847
19	0.728	0.728	52	0.813	0.813	85	0.847	0.847
20	0.732	0.732	53	0.815	0.815	86	0.848	0.848
21	0.737	0.737	54	0.816	0.816	87	0.849	0.849
22	0.741	0.741	55	0.817	0.817	88	0.850	0.850
23	0.745	0.745	56	0.819	0.819	89	0.850	0.850
24	0.749	0.749	57	0.820	0.820	90	0.851	0.851
25	0.753	0.753	58	0.821	0.821	91	0.852	0.852
26	0.756	0.756	59	0.823	0.823	92	0.852	0.852
27	0.760	0.760	60	0.824	0.824	93	0.853	0.853
28	0.763	0.763	61	0.825	0.825	94	0.854	0.854
29	0.766	0.766	62	0.826	0.826	95	0.854	0.854
30	0.769	0.769	63	0.827	0.827	96	0.855	0.855
31	0.772	0.772	64	0.828	0.828	97	0.856	0.856
32	0.774	0.774	65	0.829	0.830	98	0.856	0.856
33	0.777	0.777	66	0.830	0.831	99	0.857	0.857
34	0.780	0.779	67	0.831	0.832	100	0.858	0.857

Table 9: Minimax optimal values of t as a function of k

k	t	$\hat{t}$	k	t	$\hat{t}$	k	t	$\hat{t}$
2	0.195	0.195	35	0.105	0.105	68	0.084	0.084
3	0.184	0.184	36	0.104	0.104	69	0.084	0.084
4	0.175	0.175	37	0.103	0.103	70	0.083	0.083
5	0.168	0.168	38	0.102	0.102	71	0.083	0.083
6	0.163	0.163	39	0.101	0.101	72	0.083	0.083
7	0.158	0.158	40	0.100	0.100	73	0.082	0.082
8	0.153	0.153	41	0.100	0.100	74	0.082	0.082
9	0.149	0.149	42	0.099	0.099	75	0.082	0.081
10	0.146	0.146	43	0.098	0.098	76	0.081	0.081
11	0.143	0.143	44	0.097	0.097	77	0.081	0.081
12	0.140	0.140	45	0.097	0.097	78	0.080	0.080
13	0.137	0.137	46	0.096	0.096	79	0.080	0.080
14	0.135	0.135	47	0.095	0.095	80	0.080	0.080
15	0.132	0.132	48	0.095	0.095	81	0.079	0.079
16	0.130	0.130	49	0.094	0.094	82	0.079	0.079
17	0.128	0.128	50	0.094	0.093	83	0.079	0.079
18	0.126	0.126	51	0.093	0.093	84	0.078	0.078
19	0.124	0.124	52	0.092	0.092	85	0.078	0.078
20	0.123	0.123	53	0.092	0.092	86	0.078	0.078
21	0.121	0.121	54	0.091	0.091	87	0.077	0.077
22	0.120	0.120	55	0.091	0.091	88	0.077	0.077
23	0.118	0.118	56	0.090	0.090	89	0.077	0.077
24	0.117	0.117	57	0.089	0.089	90	0.076	0.076
25	0.115	0.115	58	0.089	0.089	91	0.076	0.076
26	0.114	0.114	59	0.088	0.088	92	0.076	0.076
27	0.113	0.113	60	0.088	0.088	93	0.076	0.076
28	0.112	0.112	61	0.087	0.087	94	0.075	0.075
29	0.111	0.111	62	0.087	0.087	95	0.075	0.075
30	0.109	0.109	63	0.087	0.087	96	0.075	0.075
31	0.108	0.108	64	0.086	0.086	97	0.074	0.074
32	0.107	0.107	65	0.086	0.086	98	0.074	0.074
33	0.106	0.106	66	0.085	0.085	99	0.074	0.074
34	0.105	0.105	67	0.085	0.085	100	0.073	0.074

Table 10: Minimax optimal values of t as a function of w

k	w	t	$\hat{t}$	k	w	t	$\hat{t}$	k	w	t	$\hat{t}$
2	0.496	0.195	0.195	35	0.782	0.105	0.104	68	0.833	0.084	0.084
3	0.540	0.184	0.183	36	0.784	0.104	0.104	69	0.834	0.084	0.084
4	0.570	0.175	0.175	37	0.786	0.103	0.103	70	0.835	0.083	0.083
5	0.593	0.168	0.168	38	0.789	0.102	0.102	71	0.836	0.083	0.083
6	0.612	0.163	0.162	39	0.791	0.101	0.101	72	0.836	0.083	0.083
7	0.628	0.158	0.158	40	0.793	0.100	0.100	73	0.837	0.082	0.082
8	0.642	0.153	0.153	41	0.795	0.100	0.099	74	0.838	0.082	0.082
9	0.654	0.149	0.149	42	0.797	0.099	0.099	75	0.839	0.082	0.082
10	0.665	0.146	0.146	43	0.799	0.098	0.098	76	0.840	0.081	0.081
11	0.674	0.143	0.143	44	0.800	0.097	0.097	77	0.841	0.081	0.081
12	0.683	0.140	0.140	45	0.802	0.097	0.097	78	0.842	0.080	0.080
13	0.691	0.137	0.137	46	0.804	0.096	0.096	79	0.843	0.080	0.080
14	0.698	0.135	0.135	47	0.806	0.095	0.095	80	0.843	0.080	0.080
15	0.705	0.132	0.132	48	0.807	0.095	0.095	81	0.844	0.079	0.079
16	0.711	0.130	0.130	49	0.809	0.094	0.094	82	0.845	0.079	0.079
17	0.717	0.128	0.128	50	0.810	0.094	0.093	83	0.846	0.079	0.079
18	0.723	0.126	0.126	51	0.812	0.093	0.093	84	0.847	0.078	0.078
19	0.728	0.124	0.124	52	0.813	0.092	0.092	85	0.847	0.078	0.078
20	0.732	0.123	0.123	53	0.815	0.092	0.091	86	0.848	0.078	0.078
21	0.737	0.121	0.121	54	0.816	0.091	0.091	87	0.849	0.077	0.077
22	0.741	0.120	0.120	55	0.817	0.091	0.091	88	0.850	0.077	0.077
23	0.745	0.118	0.118	56	0.819	0.090	0.090	89	0.850	0.077	0.077
24	0.749	0.117	0.117	57	0.820	0.089	0.089	90	0.851	0.076	0.076
25	0.753	0.115	0.115	58	0.821	0.089	0.089	91	0.852	0.076	0.076
26	0.756	0.114	0.114	59	0.823	0.088	0.088	92	0.852	0.076	0.076
27	0.760	0.113	0.113	60	0.824	0.088	0.088	93	0.853	0.076	0.076
28	0.763	0.112	0.112	61	0.825	0.087	0.087	94	0.854	0.075	0.075
29	0.766	0.111	0.111	62	0.826	0.087	0.087	95	0.854	0.075	0.075
30	0.769	0.109	0.109	63	0.827	0.087	0.087	96	0.855	0.075	0.075
31	0.772	0.108	0.108	64	0.828	0.086	0.086	97	0.856	0.074	0.074
32	0.774	0.107	0.108	65	0.829	0.086	0.086	98	0.856	0.074	0.074
33	0.777	0.106	0.106	66	0.830	0.085	0.085	99	0.857	0.074	0.074
34	0.780	0.105	0.105	67	0.831	0.085	0.085	100	0.858	0.073	0.073

Source	DF	SS	MS	F	P-value
Regression	3	0.56237	0.18746	1.669E+06	0.000
Error	95	0.00001	0.00000		
Total	98	0.56238			

R-Sq = 100.0% R-Sq(adj) = 100.0%

The regression equation is $\hat{w} = \beta_0 + \sum_{i=1}^{3} \beta_i z^i + e$, where the least squares estimates of the model Parameters are:

$\hat{\beta}_0 = 0.425405$ $\hat{\beta}_1 = 0.0996379$ $\hat{\beta}_2 = 0.0050600$

$\hat{\beta}_3 = -0.00137463$

Table 12: ANOVA Table for model (7)

Source	DF	SS	MS	F	P-value
Regression	3	0.071916	0.023972	243769.50	0.000
Error	95	0.000009	0.000000		
Total	98	0.071925			

R-Sq = 100.0% R-Sq(adj) = 100.0%

The regression equation is $\hat{t} = \beta_0 + \sum_{i=1}^{3} \beta_i z^i + e$, where the least squares estimates of the model Parameters are:

$\hat{\beta}_0 = 0.211683$ $\hat{\beta}_1 = -0.0209245$ $\hat{\beta}_2 = -0.0047002$

$\hat{\beta}_3 = 0.00059335$

Table 13: ANOVA Table for model (8)

Source	DF	SS	MS	F	P-value
Regression	3	0.071916	0.023972	248990.84	0.000
Error	95	0.000009	0.000000		
Total	98	0.071925			

R-Sq = 100.0% R-Sq(adj) = 100.0%

The regression equation is $\hat{t} = \beta_0 + \sum_{i=1}^{3} \beta_i w^i + e$, where the least squares estimates of the model Parameters are:

$\hat{\beta}_0 = 0.32558$ $\hat{\beta}_1 = -0.30643$ $\hat{\beta}_2 = 0.19087$ $\hat{\beta}_3 = -0.20535$

Figure 1: Minimax four-level factorials (*k*=11 to 100)

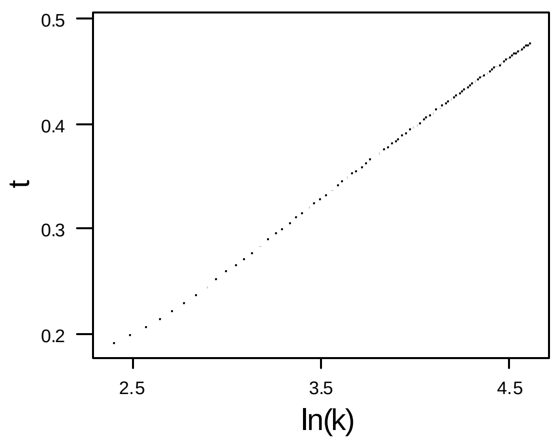


Figure 2: Minimax four-level factorials (*k*=2 to 100)

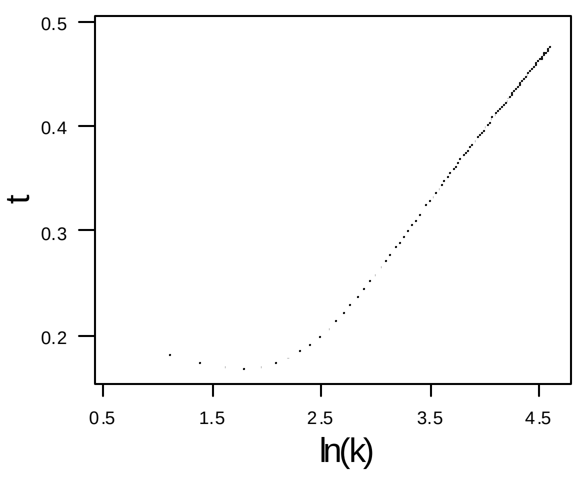


Figure 3: The graph of the optimal value of w as a function of k

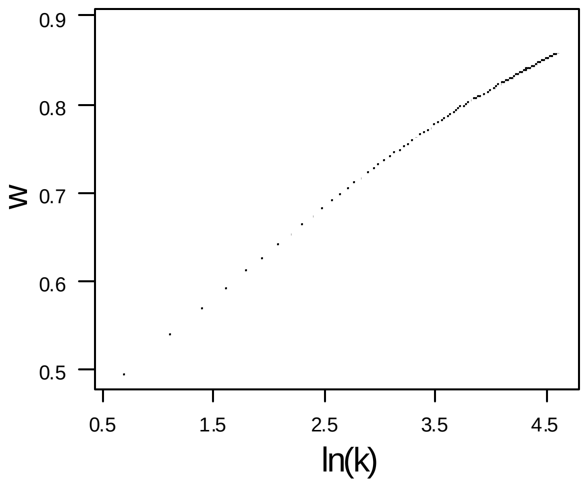


Figure 4: The graph of the optimal value of t as a function of k

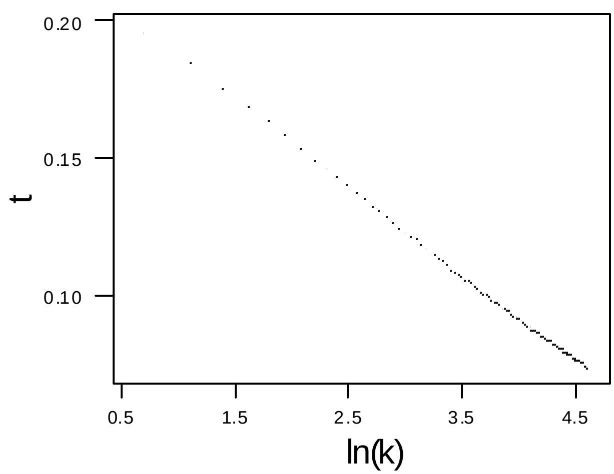
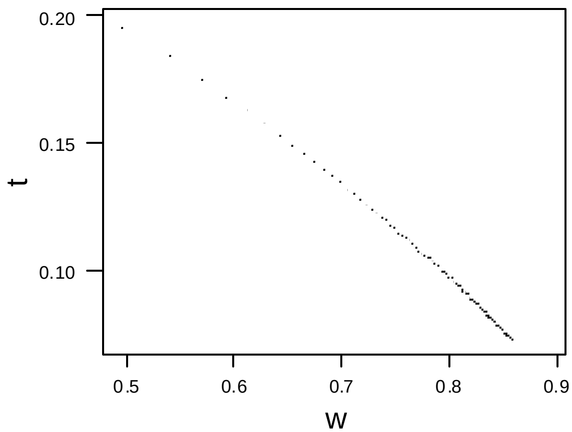


Figure 5: The graph of the optimal value of *t* as a function of *w*



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REFERENCES

1. Aslett, R., Buck, R.J., Duvall, S.G., Sacks, J. and Welch, W.J. (1998). Circuit optimization via sequential computer experiments: design of an output buffer. Appl. Statist., 47, 31-48.
2. Bates, R.A., Buck, R.J., Riccomagno, E. and Wynn, H.P. (1996). Experimental Design and Observation for Large Systems, J. Roy. Statist. Soc. B, 58, 77-94.
3. Box, G. E. P. and Hunter, J. S. (1957). Multifactor experimental designs for exploring response surfaces, Ann. Math. Statist., 28, 195-241.

4. Huda, S. and Al-Shiha, A. A. (1998). Minimax designs for estimating the slope of a third-order response surface in a hypercubic region. Commun. Statist. - Simula., 27(2), 345-356.
5. Huda, S. and Shafiq, M. (1992). Minimax designs for estimating the slope of a second-order response surface in a cubic region, J. Appl. Statist., 19, 501-507.
6. Lim, Y. B. and Studden, W. J. (1988). Efficient D_s-optimal designs for multivariate polynomial regression on the q-cube, Ann. Statist., 16, 1225-1240.
7. Mukerjee, R. and Huda, S. (1985). Minimax second- and third-order designs to estimate the slope of a response surface, Biometrika, 72, 173-178.
8. Sacks, J., Schiller, S.B. and Welch, W.J. (1989). Designs for Computer Experiments, Technometrics, 31, 41-47.
9. Sacks, J., Welch, W.J., Mitchell, T.J. and Wynn, H.P. (1989). Design and Analysis of Computer Experiments, Statistical Science, 4, 409-435.