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RECORDS AND EXCEEDANCES WHEN UNDERLYING DISTRIBUTION CONTAINS ATOMS

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ABSTRACT

Record indicators and exceedance statistics are investigated when the underlying distribution contains atoms.

KEY WORDS

Record indicator, exceedance statistics, atom.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of random variables (r.v.) with a common distribution function (d.f.) F. $M_n = \max(X_1, X_2, \ldots, X_n)$ for $n = 1,2,\ldots$ and

$$x_1 \equiv 1 \text{ and } x_n = I_{\{M_n > M_{n-1}\}} \text{ for } n = 2,3,\ldots$$

that is, x_n's are indicators of upper records.

There are many studies to calculate the success probability $p_n = P\{x_n = 1\}$, i.e. the probability of " X_n is a record " for different aspects of the sequence $X_1, X_2, \ldots$. It is well known in the records literature that, for sequences of independent random variables $X_1, X_2, \ldots$ from the same continuous d.f. F, the record indicators $x_n\ (n \geq 1)$ are independent random variables and $p_n = 1/n$ for $n = 1,2,\ldots$ see, Renyi

(1962). This property also holds for symmetrically dependent random variables. For details on record theory we refer to Ahsanullah (1995) and Nevzorov (2000).

Ballerini (1994) considered the Archimedean copula process (AC process). A sequence $\{X_i\}$ with marginal distribution functions $\{F_i\}$ is said to be an AC process if for any $n = 1,2,...$

$$P\{X_1 < t_1, X_2 < t_2, \dots, X_n < t_n\} = B\left(\sum_{i=1}^{n} A(F_i(t_i))\right),$$

where B is a monotone dependence function such that $B(0) = 1$ and $A = B^{-1}$ is the inverse of the dependence function B. He studied the AC process with $B(s) = \exp\{-s^{1/g}\}$ for $g \geq 1$, and $F_i(x) = (F(x))^{a_i}$ for $i = 1,2,...$, where $F(x)$ is a continuous d.f. and $a_1, a_2, ...$ are positive constants. Under this model he proved that $p_n = a_n^g \Big/ \sum_{i=1}^{n} a_i^g$, $n = 1,2,...$.

Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous distribution with a common d.f. $F_X(.)$. Denote the order statistics of this sample by $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$. Let $Y_1, Y_2, ..., Y_m$ be another sample with an absolutely d.f. $F_Y(.)$. It is well known that when $F_X(.) = F_Y(.)$,

$$P\{Y_k \in (X_{i:n}, X_{j:n})\} = \frac{j-i}{n+1}$$

for $1 \leq i < j \leq n, k \in \{1,2,...,m\}$, i.e. $(X_{i:n}, X_{j:n})$ is a distribution free confidence interval in the class of all absolutely continuous distribution functions.

Define the statistics

$$S_m(X_{r:n}) = \#\{k \leq m : Y_k < X_{r:n}\} \tag{1}$$

which denotes the number of Y's falling below the random threshold $X_{r:n}$. This type of statistics which are called exceedance statistics are important in many practical problems when we are interested in the behavior of sequence of observations in relation to a certain order statistics (minimum, median, quantile, maximum) from a given population. They are also used in many practical applications such as reliability and nature events.

If we take $n = r = 1$ in (1) then we obtain another exceedance statistics

$$S_m(X) = \#\{k \le m : Y_k < X\}$$

which denotes the number of Y's falling below the random threshold X which has a distribution $F_X(.)$.

Wesolowski and Ahsanullah (1998) studied the exact and asymptotic distribution of $S_m(X_{r:n})$ and $S_m(X)$ when underlying distributions are continuous. Bairamov and Kotz (2001) studied the exact distribution of $S_m(X_{r:n})$ when the underlying distribution function contains atoms, i.e. there exists on the real line points of discontinuity $a_1, a_2, ..., a_l$, where $F(a_i - 0) < F(a_i)$. According to their result for $k = 0,1,..., m$,

$$P\{S_m(X_{r:n}) = k\} = \binom{m}{k} \Bigg\{ \frac{1}{B(r, n-r+1)} \sum_{j=0}^{l} \int_{F(a_j)}^{F(a_{j+1}-0)} t^{k+r-1}(1-t)^{m+n-k-r}\, dt$$

$$+ \sum_{j=1}^{l} F^k(a_j - 0)(1 - F(a_j - 0))^{m-k} \left(F_r(a_j) - F_r(a_j - 0)\right) \qquad (2)$$

where $\{a_0, a_1, a_2, ..., a_l, a_{l+1}\}$ is the set of atoms of the distribution F. $a_0 = -\infty, a_{l+1} = \infty$ and $F_X(.) = F_Y(.) = F(.)$. F_r is the d.f. of rth order statistic.

Define the interval $J_{i,q}=\left(X_{i:n},X_{i+q:n}\right)$. Let T be the number of Y's falling in to $J_{i,q}$. When $F_X(.)$ and $F_Y(.)$ are absolutely contnuous distribution functions and $F_X(.)=F_Y(.)$, the distribution of T is

$$P\{T=t\}=\binom{m}{t}\frac{q(q+1)...(q+t-1)(n+1-q)...(n+m-t-q)}{(n+1)(n+2)...(n+m)}$$

$$t=0,1,...,m.$$

Recent interest to the exceedance statistics in the work of Bairamov (1997), Wesolowski and Ahsanullah (1998), Bairamov and Eryýlmaz (2000), Bairamov and Eryýlmaz (2001), Bairamov and Kotz (2001), Eryýlmaz (2002).

In the main section of this paper, we study on record indicators when F contains atoms. We also present the asymptotic distribution of $S_m(X)$ and the exact distribution of statistic T when underlying distribution possess atoms.

2. RECORD INDICATORS

Let X be a random variable defined on probability space $\{\Omega,\Im,P\}$ with d.f. $F(x)=P\{X\le x\}$. Assume that F possess atoms, i.e. there exists on the real line points of discontinuity $a_1,a_2,...,a_l$, where $F(a_i-0)<F(a_i)$. This means F has atoms at the points $a_1,a_2,...,a_l$ and continuous otherwise in R. Let $M=\{a_1,a_2,...,a_l\}$, $(a_1<a_2<...<a_l)$ be the set of atoms of the distribution F.

Lemma 0. (Bairamov and Kotz, 2001) Let $A\in\Im$ and $P\{A|X=x\}$ exists for all $x\in R$. Then

$$P\{A\}=\sum_{k=0}^{l}\int_{a_k}^{a_{k+1}-0}P\{A|X=x\}dF(x)+\sum_{k=1}^{l}P\{A|X=a_k\}P\{X=a_k\}$$

where $a_0 = -\infty, a_{l+1} = \infty$.

Through this paper we will assume that F contains atoms.

Let $X_1, X_2, \ldots$ be a sequence of r.v.'s with d.f. F.

Lemma 1. For $n = 1,2,\ldots$ it is true that,

$$p_n = \sum_{k=0}^{l} \frac{F^n(a_{k+1} - 0) - F^n(a_k)}{n} + \sum_{k=1}^{l} F^{n-1}(a_k - 0)(F(a_k) - F(a_k - 0))$$
(3)

The proof is a direct application of Lemma 0 when $A = \{X_1 < X_n, X_2 < X_n, \ldots, X_{n-1} < X_n\}$.

Let in (3) $F(a_i - 0) = F(a_i)$, $i = 1,2,\ldots,l$, i.e. F is a continuous d.f., then one obtains that $p_n = 1/n$.

It is clear that if we have only one atom, i.e. $M = \{a\}$ then

$$p_n = \frac{1}{n} + \frac{F^n(a-0) - F^n(a)}{n} + F^{n-1}(a-0)(F(a) - F(a-0)).$$

It is not difficult to observe that for only one atom the expected value of the number of records among the sequence of observations $X_1, X_2, \ldots, X_n$ is

$$E\left(\sum_{k=1}^{n} x_k\right) = \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{F^k(a-0) - F^k(a)}{k} + F^{k-1}(a-0)(F(a) - F(a-0))\right).$$

It is stated in the following lemma, the record indicators are not independent if F contains atom.

Lemma 2. The random variables $\{x_n\}$, $n \geq 1$ are dependent.

Proof. Consider the probability

$$P\{x_1 = 1, x_2 = 0, \ldots, x_{n-1} = 0, x_n = 1\}$$

$$= P\{X_1 > X_2, \ldots, X_1 > X_{n-1}, X_1 \leq X_n\}$$

$$= \int_{-\infty}^{a-0} F^{n-2}(x)(1-F(x))dF(x) + F^{n-2}(a-0)(1-F(a-0))(F(a)-F(a-0))$$

$$+ \int_{a}^{\infty} F^{n-2}(x)(1-F(x))dF(x)$$

$$= \int_{0}^{F(a-0)} u^{n-2}(1-u)du + F^{n-2}(a-0)(1-F(a-0))(F(a)-F(a-0))$$

$$+ \int_{F(a)}^{1} u^{n-2}(1-u)du$$

$$= \frac{1}{n(n-1)} + \frac{F^{n-1}(a-0) - F^{n-1}(a)}{n-1} + \frac{F^n(a) - F^n(a-0)}{n}$$

$$+ F^{n-2}(a-0)(1-F(a-0))(F(a)-F(a-0)).$$

Since

$$P\{x_1 = 1\}P\{x_2 = 0\}\ldots P\{x_{n-1} = 0\}P\{x_n = 1\}$$
$$= 1(1-p_2)\ldots(1-p_{n-1})p_n,$$

and hence

$$P\{x_1 = 1, x_2 = 0, \ldots, x_{n-1} = 0, x_n = 1\} \neq 1(1-p_2)\ldots(1-p_{n-1})p_n$$

the lemma is proved.

3. EXCEEDANCES

Let $\mathbf{Y} = (Y_i)_{i\geq 1}$ be a sequence of i.i.d. rv's with d.f. F which has atoms at the points $a_1, a_2, \ldots, a_l$ and consider a r.v. X from the same distribution independent of **Y**. Consider the statistics $S_m(X)$ which denotes the number of Y's falling below the threshold X among the first m Y's in the sequence **Y**.

Taking $n = r = 1$ in (2) we obtain the distribution of $S_m(X)$ in the following.

$$P\{S_m(X) = k\} = \binom{m}{k}\left[\sum_{i=0}^{l} \int_{F(a_i)}^{F(a_{i+1}-0)} u^k (1-u)^{m-k}\, du\right.$$

$$\left. + \sum_{i=1}^{l} F^k(a_i - 0)(1 - F(a_i - 0))^{m-k}\left(F(a_i) - F(a_i - 0)\right)\right].$$

It is clear that for only one atom

$$P\{S_m(X) = k\} = \binom{m}{k}\left[\int_0^{F(a-0)} u^k (1-u)^{m-k}\, du + \int_{F(a)}^{1} u^k (1-u)^{m-k}\, du\right.$$

$$\left. F^k(a-0)(1-F(a-0))^{m-k}\left(F(a) - F(a-0)\right)\right].$$

In the following theorem, we give the asymptotic distribution of r.v. $\dfrac{S_m(X)}{m}$.

Theorem 1. The asymptotic distribution of r.v. $\dfrac{S_m(X)}{m}$ for large m is

$$F^*(x)=\begin{cases} 0 & , \quad x<0 \\ x & , \quad F(a_{i-1})\le x<F(a_i-0)\ ,\ i=1,2,\dots,l+1 \\ \dfrac{F(a_i-0)+F(a_i)}{2} & , \quad F(a_i-0)\le x<F(a_i)\ ,\ i=1,2,\dots,l \\ 1 & , \quad x\ge 1. \end{cases}$$

where $a_0=-\infty, a_{l+1}=\infty$.

Proof. For simplicity we shall prove the theorem in the case when the set M contains only one atom a. The characteristic function (c.f.) of $\frac{S_m(X)}{m}$ is

$$j_{\frac{S_m(X)}{m}}(t)=\sum_{k=0}^{m} e^{it\frac{k}{m}} P\{S_m(X)=k\}$$

$$=\sum_{k=0}^{m} e^{it\frac{k}{m}}\binom{m}{k}\left[\int_0^{F(a-0)} u^k(1-u)^{m-k}\,du+\int_{F(a)}^{1} u^k(1-u)^{m-k}\,du\right.$$

$$\left.+F^k(a-0)(1-F(a-0))^{m-k}\left(F(a)-F(a-0)\right)\right]$$

$$=\int_0^{F(a-0)}\left[\sum_{k=0}^{m}\binom{m}{k}(e^{\frac{it}{m}}u)^k(1-u)^{m-k}\right]du+\int_{F(a)}^{1}\left[\sum_{k=0}^{m}\binom{m}{k}(e^{\frac{it}{m}}u)^k(1-u)^{m-k}\right]du$$

$$+\left(F(a)-F(a-0)\right)\sum_{k=0}^{m}\binom{m}{k}(e^{\frac{it}{m}}F(a-0))^k(1-F(a-0))^{m-k}\quad j_{\frac{S_m(X)}{m}}(t)$$

$$=\int_0^{F(a-0)}(1-u+ue^{i\frac{t}{m}})^m\,du+\int_{F(a)}^{1}(1-u+ue^{i\frac{t}{m}})^m\,du$$

$$+\left(F(a)-F(a-0)\right)(1-F(a-0)+F(a-0)e^{i\frac{t}{m}})^m \qquad (4)$$

Taking limit for $m \to \infty$ in (4) we obtain

$$\varphi_{\frac{S_m(X)}{m}}(t) \xrightarrow[m\to\infty]{} \varphi^*(t)$$

where

$$\varphi^*(t) = \frac{1}{it}\left(e^{it} - 1\right) + \frac{1}{it}\left(e^{itF(a-0)} - e^{itF(a)}\right) + (F(a) - F(a-0))e^{itF(a-0)}.$$

The c.f. $\varphi^*(t)$ can also be represented as

$$\varphi^*(t) = \varphi_1(t) - (F(a) - F(a-0))\varphi_2(t) + (F(a) - F(a-0))\varphi_3(t)$$

where $\varphi_1(t)$ is the c.f. of r.v. which has a uniform distribution on $(0,1)$, $\varphi_2(t)$ is the c.f. of r.v. which has a uniform distribution on $(F(a-0), F(a))$ and $\varphi_3(t)$ is the c.f. of degenerate r.v. at point $F(a-0)$. By using inversion formula we obtain

$$F^*(y) - F^*(x)$$

$$= (F_1(y) - F_1(x)) - (F(a) - F(a-0))(F_2(y) - F_2(x)) + (F(a) - F(a-0))(\tilde{F}_3(y) - \tilde{F}_3(x))$$

where $\tilde{F}_3(x) = \frac{1}{2}\left[F_3(x-0) + F_3(x)\right]$ and F_1, F_2, F_3 are corresponding distribution functions for $\varphi_1, \varphi_2, \varphi_3$.

Let $0 < x < F(a-0)$, then

$$F^*(y) - F^*(x) = \begin{cases} -x & , \quad y < 0 \\ y - x & , \quad 0 \le y < F(a-0) \text{ and } F(a) \le y < 1 \\ \dfrac{F(a-0) + F(a)}{2} - x & , \quad F(a-0) \le y < F(a) \\ 1 - x & , \quad y \ge 1 \end{cases} \qquad (5)$$

for $x \to 0$ in (5) we obtain

$$F^*(y)=\begin{cases} 0 & , \quad y<0 \\ y & , \quad 0\le y<F(a-0) \ and \ F(a)\le y<1 \\ \dfrac{F(a-0)+F(a)}{2} & , \quad F(a-0)\le y<F(a) \\ 1 & , \quad y\ge 1. \end{cases}$$

Remark. Let in Theorem 1 $F(a_i-0)=F(a_i), i=1,2,...,l,$ i.e. F is a continuous d.f. Then from Theorem 1 one obtains that the asymptotic distribution of r.v. $\frac{S_m(X)}{m}$ is uniform on (0,1) which coincides with the result of Wesolowski and Ahsanullah (1998).

Now, let $X_1, X_2,..., X_n$ be a random sample with d.f. F which has atoms at the points $a_1, a_2,..., a_l$. Denote the order statistics of this sample by $X_{1:n}\le X_{2:n}\le ...\le X_{n:n}$. $Y_1, Y_2,..., Y_m$ be another sample from the same distribution independent of $X_1, X_2,..., X_n$. Let us consider the statistic

$$S_m^{rs}=\#\{k\le m: Y_k\in[X_{r:n}, X_{s:n}]\}$$

which shows the number of Y's falling into interval $[X_{r:n}, X_{s:n}]$. There is no difference between statistics S_m^{rs} and T since we can choose $s-r=q$. Our goal is to find the distribution of S_m^{rs}. We require the following explanations.

$P\{X_{r:n}\le x | X_{s:n}=a\}$ shows the d.f. of rth order statistic in a sample of size $s-1$ from a population with continuous distribution F truncated on the right at a, i.e. the corresponding probability density function (p.d.f.) is

$$\frac{1}{B(r,s-r)}\left(\frac{F(x)}{F(a)}\right)^{r-1}\left(1-\frac{F(x)}{F(a)}\right)^{s-r-1}\frac{f(x)}{F(a)} \quad , \quad x<a\,.$$

$P\{X_{s:n} \le x | X_{r:n} = a\}$ shows the d.f. of $(s-r)th$ order statistic in a sample of size $n-r$ from a population with continuous distribution F truncated on the left at a, i.e. the corresponding p.d.f. is

$$\frac{1}{B(s-r,n-s+1)}\left(\frac{F(x)-F(a)}{1-F(a)}\right)^{s-r-1}\left(\frac{1-F(x)}{1-F(a)}\right)^{n-s}\frac{f(x)}{1-F(a)} \quad , \quad x>a.$$

Theorem 2. The probability mass function of r.v. S_m^{rs} is

$$P\{S_m^{rs}=k\}=\binom{m}{k}\Bigg\{\frac{1}{B(r,s-r)}\frac{F_s(a)-F_s(a-0)}{(F(a))^{s-1}(1-F(a))^{n-s}}\int_0^{F(a-0)}A(u,F(a))du$$

$$+K(n,r,s)\int_0^{F(a-0)}\int_u^{F(a-0)}A(u,v)dvdu+K(n,r,s)\int_0^{F(a-0)}\int_{F(a)}^{1}A(u,v)dvdu$$

$$+\frac{1}{B(s-r,n-s+1)}\frac{F_r(a)-F_r(a-0)}{(1-F(a))^{n-r}(F(a))^{r-1}}\int_{F(a)}^{1}A(F(a),v)dv$$

$$+K(n,r,s)\int_{F(a)}^{1}\int_u^{1}A(u,v)dvdu\Bigg\} \qquad k=0,1,...,m.$$

where $K(n,r,s)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}$ and $A(u,v)=u^{r-1}(v-u)^{k+s-r-1}(1-(v-u))^{m-k}(1-v)^{n-s}$.

Proof. By definition of S_m^{rs}, one can write

$$P\{S_m^{rs}=k\}=\sum_{i_1,i_2,...,i_m}P\{A_{i_1}\cap A_{i_2}\cap...\cap A_{i_k}\cap\overline{A}_{i_{k+1}}\cap\overline{A}_{i_{k+2}}...\cap\overline{A}_{i_m}\}, \qquad (6)$$

where $A_{i_j} = \{Y_{i_j} \in [X_{r:n}, X_{s:n}]\}, i_j \in \{1,2,\ldots,m\}$. The number of summands in (6) is equal to $\binom{m}{k}$ each having the same probability. Hence

$$P\{S_m^{rs} = k\} = \binom{m}{k} P\{B^{rs}\},$$

where

$$B^{rs} \equiv \{Y_1 \in [X_{r:n}, X_{s:n}], \ldots, Y_k \in [X_{r:n}, X_{s:n}], Y_{k+1} \notin [X_{r:n}, X_{s:n}], \ldots, Y_m \notin [X_{r:n}, X_{s:n}]\}.$$

One can write for the probability of event B^{rs},

$$P\{B^{rs}\} = \int_{-\infty}^{a-0} P\{B^{rs} | X_{r:n} = x, X_{s:n} = a\} dP\{X_{r:n} \le x, X_{s:n} = a\}$$

$$+ \int_{-\infty}^{a-0} \int_{x}^{a-0} P\{B^{rs} | X_{r:n} = x, X_{s:n} = y\} dP\{X_{r:n} \le x, X_{s:n} \le y\}$$

$$+ \int_{-\infty}^{a-0} \int_{a}^{\infty} P\{B^{rs} | X_{r:n} = x, X_{s:n} = y\} dP\{X_{r:n} \le x, X_{s:n} \le y\}$$

$$+ \int_{a}^{\infty} P\{B^{rs} | X_{r:n} = a, X_{s:n} = y\} dP\{X_{s:n} \le y, X_{r:n} = a\}$$

$$+ \int_{a}^{\infty} \int_{x}^{\infty} P\{B^{rs} | X_{r:n} = x, X_{s:n} = y\} dP\{X_{r:n} \le x, X_{s:n} \le y\} \qquad (7)$$

Let us consider the each summand in (7).

$$\int_{-\infty}^{a-0} P\{B^{rs} | X_{r:n} = x, X_{s:n} = a\} dP\{X_{r:n} \le x, X_{s:n} = a\}$$

$$= \int_{-\infty}^{a-0} P\{B^{rs} | X_{r:n} = x, X_{s:n} = a\} dP\{X_{r:n} \le x | X_{s:n} = a\} P\{X_{s:n} = a\}$$

$$= \int_{-\infty}^{a-0} (F(a) - F(x))^k (1 - (F(a) - F(x)))^{m-k}$$

$$\times \frac{1}{B(r, s-r)} \left(\frac{F(x)}{F(a)} \right)^{r-1} \left(1 - \frac{F(x)}{F(a)} \right)^{s-r-1} \frac{f(x)}{F(a)} (F_s(a) - F_s(a-0)) dx \quad (8)$$

$$\int_{-\infty}^{a-0} \int_{x}^{a-0} P\{B^{rs} | X_{r:n} = x, X_{s:n} = y\} dP\{X_{r:n} \le x, X_{s:n} \le y\}$$

$$= \int_{-\infty}^{a-0} \int_{x}^{a-0} (F(y) - F(x))^k (1 - (F(y) - F(x)))^{m-k}$$

$$\times K(n, r, s) (F(x))^{r-1} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(x) f(y) dx dy \quad (9)$$

$$\int_{-\infty}^{a-0} \int_{a}^{\infty} P\{B^{rs} | X_{r:n} = x, X_{s:n} = y\} dP\{X_{r:n} \le x, X_{s:n} \le y\}$$

$$= \int_{-\infty}^{a-0} \int_{a}^{\infty} (F(y) - F(x))^k (1 - (F(y) - F(x)))^{m-k}$$

$$\times K(n, r, s) (F(x))^{r-1} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(x) f(y) dx dy \quad (10)$$

$$\int_a^\infty P\{B^{rs} | X_{r:n} = a, X_{s:n} = y\} dP\{X_{s:n} \le y, X_{r:n} = a\}$$

$$= \int_a^\infty (F(y) - F(a))^k (1 - (F(y) - F(a)))^{m-k}$$

$$\times \frac{1}{B(s-r, n-s+1)} \left(\frac{F(y) - F(a)}{1 - F(a)} \right)^{s-r-1} \left(\frac{1 - F(y)}{1 - F(a)} \right)^{n-s} \frac{f(y)}{1 - F(a)}$$

$$\times (F_r(a) - F_r(a-0)) dy \qquad (11)$$

$$\int_a^\infty \int_x^\infty P\{B^{rs} | X_{r:n} = x, X_{s:n} = y\} dP\{X_{r:n} \le x, X_{s:n} \le y\}$$

$$= \int_a^\infty \int_x^\infty (F(y) - F(x))^k (1 - (F(y) - F(x)))^{m-k}$$

$$\times K(n,r,s)(F(x))^{r-1} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(x) f(y) dx dy \quad (12)$$

By using (8)-(12) in (7) simple calculations complete the proof.

4. A NOTE FOR APPLICATION

The statistics T defined in section 1 was used to suggest a test criterion for testing the hypothesis $H_0 : F_X(x) = F_Y(x)$ (Katzenbeisser (1985), (1986), Matveychuck and Petunin (1990), (1991)). This homogeneity test is applicable when F_X and F_Y are absolutely continuous. The findings of this paper extend the applicability of these tests for data which have not continuous distributions. Statistics $S_m(X)$ and $S_m(X_{r:n})$ can also be used to inference for nature events. Because this type of data is widespread in nature.

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