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ON THE LAW OF LARGE NUMBERS FOR NEGATIVELY DEPENDENT RANDOM VARIABLES

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ABSTRACT

Strong law of large numbers concerning negatively dependent random variables are obtained and then they are utilized to establish stability results.

KEY WORDS

Strong law of large numbers, Negatively dependent

INTRODUCTION

The history and literature on laws of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concepts of negative dependence is much more limited but still very interesting. Lehmann (1966) provides an extensive introductory overview of various concepts of positive and negative dependence in the bivariate case. Negative dependence has been particularly useful in obtaining laws of large numbers (cf : Matula (1992), Qi (1995), Chandra and Ghosal (1996), Bozorgnia, et al (1996a,1996b), and Amini (2000)).

Throughout this work $\{X_n; n > 1\}$ is a sequence of pairwise negatively dependent random variables defined on a probability space (Ω, F, P) and let $S_n = \sum_{i=1}^{n} X_i$. In the resent work [2], we have shown that the classical law of large number for pairwise identically distributed random variables can be extended in an elementary fashion to the case where the random variables ND identically distributed. Following the ideas in that paper we study the stability of S_n for the case where the summand are non-negative.

Namely, we give sufficient condition so that $\frac{S_n - ES_n}{n} \to 0 \quad a.e.$ The conditions are weak enough to prove the classical law of large numbers.

In the following we present some background materials on negative dependence which will be used in obtaining the SLLN in the next section, (see Ebrahimi and Ghosh (1981)).

Definition. Random variables X and Y are negatively dependent (ND) if

$$P\{X \le x; Y \le y\} \le P\{X \le x\}P\{Y \le y\} \tag{1}$$

for all x , y º R. A collection of random variables is said to be pairwise ND (PND) if every pair of random variables in the collection satisfies (1).

Lemma 1. If $\{X_n; n > 1\}$ is a sequence of negatively dependent random variables, then

$$(a) \qquad E(X_i X_j) \le E(X_i)E(X_j), \qquad i \ne j$$

$$(b) \qquad Cov(X_i, X_j) \le 0, \qquad i \ne j.$$

Lemma 2. If $\{X_n; n > 1\}$ is a sequence of negatively dependent random variables, and s a $\{f_n\}$ sequence of monotone increasing, (or monotone decreasing) Borel functions, then $\{f_n(X_n)\}$ is a sequence of negatively dependent random variables.

Corollary 1. If $\{X_n; n > 1\}$ is a sequence of negatively dependent random variables, then $\{X_n^+; n > 1\}$ and $\{X_n^-; n > 1\}$ are.

Corollary 2. If $\{X_n; n > 1\}$ is a sequence of negatively dependent random variables and

$Y_n = X_n I_{\{X_n \le n\}} + n I_{\{X_n > n\}}$, where I is an indicator function. Then $\{Y_n; n \ge 1\}$ is a sequence of negatively dependent random variables.

MAIN RESULTS

The extension of the Etemadi's (1983) SLLN from non-negative random variables to pairwise negatively dependent is as follows.

Theorem 1. Let $\{X_i; i > 1\}$ be a sequence of negatively dependent random variables with finite second moments such that:

$$(a) \quad \sup_{i>0} E|X_i| < \infty, and$$

$$(b) \quad \sum_{i=1}^{\infty} \frac{VarX_i}{i^2} < \infty.$$

Then as $n \to \infty$,

$$\frac{S_n - ES_n}{n} \to 0 \quad a.e.$$

Proof. Since $\{X_n^+; n > 1\}$ and $\{X_n^-; n > 1\}$ are sequences of negatively dependent random variables so they satisfy the assumption of theorem and $X_i = X_i^+ - X_i^-$. Thus without loss of generality we can assume that $X_i \ge 0$. Then for every subsequence $\{k_n\}$ of positive integers such that $\liminf_{n\to\infty} \frac{k_n}{k_{n-1}} > 1$ and for any $e > 0$ by using Chebyshev's inequality we obtain:

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{S_{k_n} - ES_{k_n}}{k_n}\right| > e\right\} \le \sum_{n=1}^{\infty} \frac{VarS_{k_n}}{e^2 k_n^2}$$

$$= c\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{VarX_i}{k_n^2}$$

$$\leq c\sum_{n=1}^{\infty}\sum_{n=i}^{\infty}\frac{VarX_i}{k_n^2}$$

$$\leq c(1-b^{-2})^{-1}\sum_{i=1}^{\infty}\frac{VarX_i}{i^2} < \infty, \qquad (2)$$

where $b = \liminf_{n\to\infty}\frac{k_n}{k_{n-1}} > 1$,and for more detail about $\sum_{n=i}^{\infty}\frac{1}{k_n^2}$ see Candera and Goswami (1992) page 217.Therefore by the Borel Cantelli Lemma we have:

$$P\left\{\left|\frac{S_{k_n} - ES_{k_n}}{k_n}\right| > e \quad i.o.\right\} = 0$$

and this is equivalent to

$$\frac{S_{k_n} - ES_{k_n}}{k_n} \to 0 \quad a.e.$$

(see Chung (1974) theorem 4.2.2 p,73). Now for any positive number k such that, $k_n \leq k \leq k_{n+1}$, we have

$$\frac{S_k - ES_k}{k} \leq \frac{S_{k_{n+1}} - ES_{k_{N+1}}}{k_{n+1}}.\frac{k_{n+1}}{k_n} + \frac{ES_{k_{n+1}} - ES_{k_n}}{k_n} \qquad (3)$$

$$\frac{S_{k_n} - ES_{k_n}}{k_n}.\frac{k_n}{k_{n+1}} - \frac{ES_{k_{n+1}} - ES_{k_n}}{k_{n+1}} \leq \frac{S_k - ES_k}{k} \qquad (4)$$

Then by using (3),(4), and (a) it follows that

$$(1-1/a)(\sup_{i>0} EX_i) \leq \liminf_{n\to\infty}\frac{S_k - ES_k}{k} \leq \limsup_{n\to\infty}\frac{S_k - ES_k}{k} \leq (a-1)\sup_{i>0} EX_i \quad (5)$$

for every $a > 1$ which concludes the proof.

Corollary 3. Let $\{X_i; i > 1\}$ be a sequence of negatively dependent random variables with finite second moments such that:

$$(a) \quad \sup_{i>0} E|X_i - EX_i| < \infty, and$$

$$(b) \quad \sum_{i=1}^{\infty} \frac{VarX_i}{i^2} < \infty.$$

Then as $n \to \infty$,

$$\frac{S_n - ES_n}{n} \to 0 \qquad \text{a.e.}$$

Proof. It is obvious by definition that $\{(X_i - EX_i); i > 0\}$ is a sequence of negatively dependent random variables, then $\{(X_i - EX_i)^+; i > 0\}$ and $\{(X_i - EX_i)^-; i > 0\}$ are sequences of negatively dependent random variables so they satisfy the assumption of the corollary and $X_i - EX_i = (X_i - EX_i)^+ - (X_i - EX_i)^-$. Let $Y_i = (X_i - EX_i)^+$ and $Z_i = (X_i - EX_i)^-$ and set $S_n^* = \sum_{i=1}^{n} Y_i$ and $S_n^{**} = \sum_{i=1}^{n} Z_i$. Since $Var(X_i - EX_i)^+ \le E(X_i - EX_i)^{+2} \le VarX_i$ and sequence $\{Y_i\}$ satisfy the assumption of the Theorem 1, then we clearly have

$$\frac{S_n^* - ES_n^*}{n} \to 0 \tag{6}$$

A similar consideration for negative part gives

$$\frac{S_n^{**} - ES_n^{**}}{n} \to 0 \tag{7}$$

Now (6) and (7) together with the fact that $ES_n^* - ES_n^{**} = 0$ complete the proof.

Theorem 2. Let $\{X_i; i \geq 1\}$ be a sequence of negatively dependent random variables such that:

(a) $\sup_{i>0} E|X_i| < \infty, and$

(b) $\sum_{i=1}^{\infty} P\{|X_i| > i\} < \infty$ and $\sum_{i=1}^{n} E\left(|X_i| I_{\{|X_i|>i\}}\right)/n \to 0,$

(c) $\sum_{i=1}^{\infty} E\left(X_i^2 I_{\{|X_i| \leq i\}}\right)/i^2 < \infty.$

Then as $n \to \infty$,

$$\frac{S_n - ES_n}{n} \to 0 \qquad \text{a.e.}$$

Proof. Since $\{X_n^+; n > 1\}$ and $\{X_n^-; n > 1\}$ are sequences of negatively dependent random variables so they satisfy the assumption of theorem an $X_i = X_i^+ - X_i^-$. Thus without loss of generality we can assume that $X_i \geq 0$. Let $Y_i = X_i I_{\{X_{in} \leq i\}} + iI_{\{X_i > i\}}$ and $S_n^* = \sum_{i=1}^{n} Y_i$ Then for every subsequence $\{k_n\}$ of positive integers such that $\liminf_{n\to\infty} \frac{k_n}{k_{n-1}} > 1$ and for any $e > 0$ by using Chebyshev's inequality we obtain

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{S_{k_n}^* - ES_{k_n}^*}{k_n}\right| > e\right\} \leq \sum_{n=1}^{\infty} \frac{VarS_{k_n}^*}{e^2 k_n^2}$$

$$= c\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \frac{VarY_i}{k_n^2} \leq c\sum_{n=1}^{\infty}\sum_{n=i}^{\infty} \frac{VarY_i}{k_n^2}$$

$$= c(1-b^{-2})^{-1}\sum_{i=1}^{\infty} \frac{VarY_i}{i^2} \leq c\sum_{i=1}^{\infty} \frac{EY_i^2}{i^2}$$

$$= c_1 \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\int_0^i x_i^2 dF(x) + \int_i^{\infty} i^2 dF(x) \right)$$

$$= c_1 \sum_{i=1}^{\infty} \left(\frac{E(X_i^2 I_{\{|X_i| \le i\}})}{i^2} + \frac{1}{i^2} \int_i^{\infty} i^2 dF(x) \right)$$

$$= c_1 \sum_{i=1}^{\infty} \frac{E(X_i^2 I_{\{|X_i| \le i\}})}{i^2} + c_1 \sum_{i=1}^{\infty} P\{|X_i| > i\}$$

$< \infty$, by (b) and (c) (8)

Therefore by the Borel Cantelli Lemma we have

$$\frac{S_{k_n}^* - ES_{k_n}^*}{k_n} \to 0 \quad a.e.$$

So similary to the end of the Theorem 1 we obtain

$$\frac{S_n^* - ES_n^*}{n} \to 0 \quad a.e. \tag{9}$$

By considering first part of (b) it follows that

$$\sum_{n=1}^{\infty} P\{Y_n \ne X_n\} = \sum_{n=1}^{\infty} P\{X_n > n\} < \infty,$$

so $\{X_n\}$ and $\{Y_n\}$ are equivalent, then

$$\sum_{n=1}^{\infty} P\{Y_n - X_n\} \quad converges \quad a.e.$$

Furthermore,

$$\frac{1}{n} \sum_{n=1}^{n} P\{Y_n - X_n\} \to 0 \qquad \text{a.e.} \tag{10}$$

(see theorem 5.2.1 of Chung (1974)).

By second part of (b) we have

$$\frac{ES_n - ES_n^*}{n} = \left(\sum_{i=1}^{n} E\left(|X_i| I_{\{|X_i|>i\}}\right) - \sum_{i=1}^{n} E\left(i I_{\{|X_i|>i\}}\right) \right) / n$$
$$\leq \left(\sum_{i=1}^{n} E\left(|X_i| I_{\{|X_i|>i\}}\right) / n - \sum_{i=1}^{\infty} P\{|X_i| > i\} / n \right) \to 0 \quad (11)$$

Therefore (9), (10), and (11) give the desired result.

Corollary 4. Let $\{X_i; i \geq 1\}$ be a sequence of negatively dependent identically distributed random variables with $E|X_1| < \infty$. Then as $n \to \infty$,

$$\frac{S_n - ES_n}{n} \to 0 \quad a.e$$

Proof. A simple calculation shows that all the relevant terms in (a)-(c) of theorem 2 when applied to $\{(X_i - EX_i)^+; i > 0\}$ and are bounded by $cE|X_1|$ (see also Azarnoosh [2]), and this completes the proof.

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