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ON CHARACTERIZATIONS OF DISCRETE DISTRIBUTION THROUGH REGRESSIONS OF RECORD VALUES

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ABSTRACT

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables taking on values 0,1,... with common distribution function F such that F(n) < 1 for any n = 0,1… and $EX_1^2 < \infty$. Let $X_{U(n)}$ be the n-th upper record value and $\{A_s\}_{s=0}^{\infty}$ be any sequence of positive numbers, such that $2 + 2A_{s+1} - A_s - A_{s+2} > 0$ and $A_s \geq 1$ for all s. In this study we show that or any given $p_0 = P(X_1 = 0)$ if $E\left(\left(X_{U(2)} - X_1\right)^2 \middle| X_1 = s\right) = A_s$ for all $s \geq 0$, then F(x) is unique. By using this result we may derive characterization theorems for many known discrete distributions. As examples we give characterization theorems for Poisson distribution and some other distributions. Our result is generalization of theorems of Balakrishnan and Balasubramanian (1995) and Wu (2001).

KEYWORDS

Records, characterization of discrete distributions, Poisson distribution.

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables taking on values 0,1,... with common distribution function F such that Fn < 1 for any n = 0,1,… and $EX_1^2 < \infty$. Define the sequence upper record times $U_{(n)}$ and upper record values $X_{U(n)}$ as follows:

$$U(1) = 1,\ U(n+1) = \min\{j > U(n) : X_j > X_{U(n)}\},\ n = 1,2,\ \dots$$

and denote $p_k = P(X_1 = k)$. The following are well-known:

1) X_1 has two parameter geometric distribution $p_k = (1-p_0)pq^{k-1}\ k \geq 1$, $0 < p_0 < 1$ is arbitrary), if and only if $E\left((X_{U(2)} - X_1)^2 \middle| X_1 = s\right)$ = const for s $\geq$ 0 (Balakrishnan and Balasubramanian (1995)).

2) Characterization of mixtures of geometric distributions when conditional expectations $E\left((X_{U(2)} - X_1)^2 \middle| X_1 = s\right)$ are some functions of parameters of distribution and failure rate function is given in the work of Wu (2001).

In this paper we give a characterization of discrete distribution in terms of $E\left((X_{U(2)} - X_1)^2 \middle| X_1 = s\right)$.

Various developments in characterization theory dealing with regressions of record values of discrete random variables and related topics have been reviewed by a number of authors including Ahsanullah (1995), Ahsanullah and Holland (1984,1987), Aliev (1998, 1999), Franco and Ruiz (2001), Galambos and Kotz (1978), Korwar (1984), Nagaraja (1988), Nevzorov (1987), Srivastava (1979), Stepanov (1994), Wesolowski and Ahsanullah (2001) and others.

2. CHARACTERIZATION THEOREMS

Theorem 1.

Let $\{A_k\}_{k=0}^{\infty}$ be any sequence of numbers, such that

$$2 + 2A_{s+1} - As - A_{s+2} \geq 0 \ \text{ and } \ A_s \geq 1 \text{ for all s.} \tag{1}$$

For any given $p_0\ (0 < p_0 < 1)$ if there exists F(x) such that

$$E\left(\left(X_{U(2)} - X_1\right)^2 \middle| X_1 = s\right) = A_s \text{ for } s = 0,1, \ldots.. \qquad (2)$$

then F(x) is unique.

Remark 1.

By means of this theorem we may derive characterization theorems for many known discrete distributions. For this characterization, there is no need to show the existence part for such distribution. Here we will illustrate this for some discrete distributions. However, it is possible to extend the result to other discrete distributions by calculating $E\left(\left(X_{U(2)} - X_1\right)^2 \middle| X_1 = s\right)$.

Remark 2.

Special case $A_s = \text{const}$ gives the result of Balakrishnan and Balasubramanian (1995) and the case

$$A_s = \left(\sum_{i=1}^{m} \theta_i \beta_i^{-2} (1-\beta_i)^s (1-\beta_i)\right) \Big/ \left(\sum_{i=1}^{m} \theta_i (1-\beta_i)^s\right)$$

corresponds to the result of Wu (2001) $p_k = (1-p_0) \sum_{i=1}^{m} \theta_i \beta_i (1-\beta_i)^{k-1}$.

Remark 3.

If c > 1 and 0 < a < 1 then the sequence $A_s = as^2 + bs + c$ satisfies (1).

Theorem 2.

Let $\lambda > 0$ and $f(\lambda) = 1 \big/ \left(e^{\lambda} - 1\right)$ and denote $A_s = \left(\sum_{j=1}^{\infty} j^2 \lambda^j \big/ (s+j)!\right) \Big/ \left(\sum_{j=1}^{\infty} \lambda^j \big/ (s+j)!\right)$. If $\{A_s\}_{s=0}^{\infty}$ satisfies (1) then necessary and sufficient condition for a r.v. X_1 to have a distribution of the form $P_k = (1-p_0) \frac{\lambda^k}{k!} f(\lambda)$ for k > 1 (p_0 is arbitrary number $(0 < p_0 < 1)$) is $E\left(\left(X_{U(2)} - X_1\right)^2 \middle| X_1 = s\right) = A_s$ for all s ≥ 0.

Remark 4.

The case $p_0 = e^{-\lambda}$ in Theorem 2 corresponds to Poisson distribution.

Theorem 3.

Let p_0, α, p be any three numbers between 0 and 1 and denote $A_s = \begin{cases} \alpha - 4\alpha/p + 4(2-p)/p^2, \\ \left(3\alpha + 1 - 4(\alpha+2)/p + 8/p^2\right)/(1-\alpha p), s = 1,3,5,\ldots \end{cases}$ If $\{A_s\}_{s=0}^{\infty}$ satisfies (1) then necessary and sufficient condition for a r.v. X_1 to have a distribution of the form

$$p_{2k-1} = (1-p_0)\alpha p(1-p)^{k-1},\ p_{2k} = (1-p_0)(1-\alpha)p(1-p)^{k-1} \tag{3}$$

for k $\geq$ 1 is $E\left((X_{U(2)} - X_1)^2 \middle| X_1 - s\right) = A_s$ for all s $\geq$ 0.

Theorem 4.

Let 0 < p_0 < 1, a > 0, b > 1 and $A_s = as + b$. The necessary and sufficient condition for a r.v. X_1 to have a distribution of the form

$$p_{k+2} = [(ak + b + 1)p_{k+1} - 2(1 - p_0 - p_1 - \ldots p_k)]/[(k+2)a + b] \tag{4}$$

for k $\geq$ 0 is $E\left((X_{U(2)} - X_1)^2 \middle| X_1 = s\right) = A_s$ for all s $\geq$ 0. In this case for each p_0 by Theorem 1 there is only p_1 in (4) satisfying condition $\sum_{j=0}^{\infty} p_j = 1$ and $p_2, p_3, \ldots$ can be uniquely obtained recurrently from (4).

3. PROOFS OF THEOREMS

Before going into the proofs in detail note that [see, for example, Ahsanullah (1995)] $P\left((X_{U(2)} - X_1)^2 = m \middle| X_1 = s\right) = p_{s+m}/P(X_1 \geq s)$ and

$$E\left(\left(X_{U(2)}-X_1\right)^2 \middle| X_1=s\right)=\left(\sum_{j=1}^{\infty} j^2 p_{s+j}\right) \Big/ \left(\sum_{j=1}^{\infty} p_{s+j}\right). \tag{5}$$

Proof of Theorem 1.

Let p_0 is any number such that $0<p_0<1$. Assume that there exist F(x) with $E\left(\left(X_{U(2)}-X_1\right)^2 \middle| X_1=s\right)=A_s$ for all s $\geq$ 0.

Together with (5) it gives

$$\sum_{j=1}^{\infty} j^2 p_{s+j}=A_s P\left(X_1 \geq s+1\right) \text{ for all s.} \tag{6}$$

Rewriting (6) for s=s+1 we have $\sum_{j=1}^{\infty} (j-1)^2 p_{s+j}=A_{s+1} P\left(X_1 \geq s+2\right)$ for all s. After extracting it from (6) and simplifying it is easy to see that

$$\sum_{j=1}^{\infty} (2j-1)^2 p_{s+j}=A_s P\left(X_1 \geq s+1\right)-A_{s+1} P\left(X_1 \geq s+2\right). \tag{7}$$

As in above case, let us rewrite (7) s=s+1 and extract it from (7). After simplifying we have recurrent relation for probabilities

$$p_{s+2}=\frac{1+2A_{s+1}-A_{s+2}}{A_{s+2}} p_{s+1}-\frac{2+2A_{s+1}-A_{s+2}}{A_{s+2}}\left(1-p_0-p_1-\ldots-p_s\right) \tag{8}$$

Since both coefficients $\left(1+2A_{s+1}-A_{s+2}\right)/A_{s+2}$ and $\left(2+2A_{s+1}-A_s-A_{s+2}\right)/A_{s+2}$ in (8) are positive, it means that p_{s+2} is increasing (decreasing) if p_{s+1} increases (decreases) for all s $\geq$ 0. It means that for any given p_0 all probabilities $p_2, p_3, \ldots$ increases when p_1 increases. Together with $\sum_{j=0}^{\infty} p_j=1$ we conclude that for any given p_0 we have only one p_1 which satisfies (8). Consequently, we have only F(x) which satisfies (2).

Proof of Theorems 2 and 4.

Necessity part may be taken from (5) by trivial verification. Sufficiency part follows from Theorem 1.

Proof of Theorem 3.

For s=0,2,4,... using (5) and (3) we have

$$A_s = A_0 = \sum_{j=s/2+1}^{\infty} \left[(2j-s-1)^2 p_{2j-1} + (2s-j)^2 p_{2j}\right] / \sum_{j=s/2+1}^{\infty} \left(p_{2j-1} + p_{2j}\right)$$

$$= \sum_{j=1}^{\infty} \left[(2j-1)^2 \alpha p(1-p)^{j-1} + j^2 (1-\alpha) p (1-p)^{j-1}\right] / \sum_{j=1}^{\infty} \left(\alpha p(1-p)^{j-1} + (1-\alpha) p (1-p)^{j-1}\right)$$

$$= \alpha - 4\alpha / p + 4(2-p)/p^2 .$$

By the similar way for s=1, 3, 5,... taking 2j-s-1=2t one can write

$$A_s = A_1 =$$

$$\left(p_{s+1} + \sum_{j=(s+3)/2}^{\infty} \left[(2j-s-1)^2 p_{2j-1} + (2s-j)^2 p_{2j}\right]\right) / \left(p_{s+1} + \sum_{j=(s+3)/2}^{\infty} \left(p_{2j-1} + p_{2j}\right)\right)$$

$$= \left((1-\alpha)p + \sum_{t=1}^{\infty} \left[(2t)^2 \alpha p (1-p)^t + (2t+1)^2 (1-\alpha)(1-p)^t\right]\right) / \left((1-\alpha)p + 1 - p\right)$$

$$= \left(3\alpha + 1 - 4(\alpha+2)/p + 8/p^2\right) / (1-\alpha p)$$

Sufficiency part can be obtained directly from Theorem 1.

REFERENCES

1. Ahsanullah, M. (1995). *Record Statistics*. Nova Science Publishers, Inc., Commack, NY.
2. Ahsanullah, M. and Holland, B. (1984). Record values and the geometric distribution. *Statistische Hefte*, V. 25, 319-327.
3. Ahsanullah, M. and Holland, B. (1987). Distributional properties of record values from the geometric distribution. *Statistica Neerlandica*, 41, 129-137.

4. Aliev, F.A. (1998). Characterization of distributions through weak records. *J. App. Statist. Sc.*, V. 8, N. 1, 13-16.
5. Aliev, F.A. (1999). New characterization of discrete distributions through weak records. *Theory Probab. Appl.,* V. 44, N. 4, 756-761.
6. Balakrishnan, N. and Balasubramanian, K. (1995). A characterization of geometric distribution based on record values. *J. App. Statist. Sc.*, V.2, N.3, 277-282.
7. Franco, M. and Ruiz, J. M. (2001). Characterization of discrete distributions based on conditional expectations of record values. *Statistical Papers,* V. 42, 101-110.
8. Galambos, J. and Kotz, S. (1978). Characterizations of probability distributions. *Lecture Notes in Mathematics* No. 675, Springer-Verlag, New-York.
9. Korwar, R.M. (1984). On characterizing distributions for which the second record value has a linear regression of the first. *Sankhya*, V. 46, B, 108-109
10. Nagaraja, H.N. (1988). Record values and related statistics, A review, *Comm. Statist. Theory Methods*, Ser. A, 17, 2223-2238.
11. Nevzorov, V.B. (1987). Records, *Theory Probab. Appl.*, V. 32, 201-228.
12. Srivastava, R.C. (1979). Two characterization of the geometric distribution by record values. *Sankhya*, V. 40, B, 276-278.
13. Stepanov, A.V. (1994). A characterization theorem for weak records. *Theory Probab. Appl.*, V. 39, 762-764.
14. Wesolowski, J. and Ahsanullah, M. (2001). Linearity of regressions for non-adjacent weak record. *Statistica Sinica*, V. 11, 39-52.
15. Wu, J. Characterization of generalised mixtures of geometric and exponential distributions based on upper record values. *Statistical Papers,* V. 42, 123-133.