**Pak. J. Statist.**
**2011 Vol. 27(3), 213-219**

# GENERALIZED CONVERGENCE CHARACTERIZATIONS OF FELLER OPERATOR

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## ABSTRACT

This paper is mainly connected with the generalization of approximation properties of Feller operator. We first introduce the probabilistic properties of the Feller operator and its convergence characterization. We also compute its rates of convergence by means of modulus continuity and moments of random variable $X_n$ . The main theorem is the extension to $f^{(m)}$ derivatives of function $f \in C_B^{(m)}[0,\infty)$ . Finally, we are interested in convergence characterization of Laplace transform via Feller operator.

## 1. INTRODUCTION

Recent studies demonstrate that the theory of Korovkin type approximation plays an important role in analytic number theory and theoretical physicists. Also, many theoretical physicians and El Naschie obtained some important results in this theory [4-5]. For example, various applications of this theory have appeared in the study of hypergeometric series, in the approximation theory while other important applications are related with the quantum theory [6]; we first recall some basic definitions used in the paper. The Feller operator is defined for $f \in C[0,\infty)$ , $x \geq 0$ and $n \geq 1$ as follows:

$$F_n(f,x) = \int_0^\infty f(t) v_n(t)\,dt \ , \ n \geq 1, \tag{1.1}$$

where the kernel of the Feller operator, $v_n(t)$ is defined as:

$$v_n(t) = \frac{n^n}{x^n \Gamma(n)} t^{n-1} e^{-\frac{nt}{x}} , \ t \geq 0 \ .$$

This satisfies the following two rules:

$$v_n(t) \geq 0, \ t \geq 0 \ , \ \int_0^\infty v_n(t)\,dt = 1 \ .$$

In this case, note that $v_n$ is the probability density function of the random variable $X_n$ . The Feller operator may be expressed in terms of the expectation operator $E$ in a compact form as:

$$F_n(f,x) = E\left[f\left(X_n\right)\right].$$

An expectation operator *E* is a well-known positive linear operator in probability theory. Most of the linear positive operators used in the approximation theory are expectation operator. In particular, some linear positive operators were introduced and studied by the following authors Altomare F. and Campiti M., [1], Hirschman and Widder [11], Gelfond [7], Baskakov [2], Doğru O. and Duman O., [3], King [12], Stancu [8], Shahbazov [9-10], and others.

Feller has evaluated the $\sup\left|F_n(f)(x)-f(x)\right|<C_n$ for each $f\in C[0,\infty)$ and showed $C_n\to 0$ as $n\to\infty$ [1].

In this study it is proposed to extend the Feller theorem to following upper bound by using the $f,f',\dots,f^{(m)}$ for each $f\in {C_B}^{(m)}[0,\infty)$,

$$\sup_{x\geq 0}\left|F_n(f,x)-\left[a_0f(x)+a_1f^{/}(x)+\cdots+a_mf^{(m)}(x)\right]\right|<C_n\,.$$

## 2. CONVERGENCE CHARACTERIZATION OF A FELLER OPERATOR

We will obtain the difference between $F_n(f,x)$ and *f* by using expectation operator *E*. For this, we utilize the Taylor expansion of function $f$ at the $x\in[a,b]\subset R$. The Taylor expansion may be evaluated as:

$$f(t)-\sum_{k=0}^{m}\frac{f^{(k)}(x)}{k!}(t-x)^k=R_m(t,x)\,, \tag{2.1}$$

where $R_m(t,x)$ is Lagrange term and can be expanded in the following form:

$$R_m(t,x)=\frac{(t-x)^m}{m!}\left[f^{(m)}\left(x+\theta(t-x)\right)-f^{(m)}\left(x\right)\right],\ 0<\theta<1\,. \tag{2.2}$$

The modulus of continuity of the function $f\in C[0,\infty)$ is defined by

$$w(\delta)=\sup_{|x-y|\leq\delta}\left|f(x)-f(y)\right|.$$

At the moment we emphasize that, well known necessary and sufficient condition for a function $f\in C[a,b]$ is

$$\lim\nolimits_{\delta\to 0}w\left(f,\delta\right)=0$$

The same result is currently for $f\in {C_B}^{(m)}[0,\infty)$, which is class of continuous and bounded function on $[0,\infty)$.

In this case, if we take $m=0$ in the equality (2.2), the absolute value of the Lagrange term can be limited by the following equality,

$$\left|R_0(t,x)\right|=\left|f(x+\theta(t-x)-f(x)\right|$$

$$\leq w\left(|t-x|\right)$$

$$\leq \left(1+\delta|t-x|\right) w(1/\delta) .$$

If this inequality is used in the Lagrange term (2.1), we obtain the following inequality

$$|f(t)-f(x)| \leq \left((1+\delta|t-x|\right) w(1/\delta) . \tag{2.3}$$

If we choose $t = X_n$ in (2.3) and take the expectation operator on bots side, the following inequality is obtained

$$\left|E\left[f\left(X_n\right)\right]-f(x)\right| \leq \left(1+\delta\left|E\left(X_n\right)-x\right|\right) w(1/\delta) , \tag{2.4}$$

where $E\left(X_n\right)$ is, the expectation operator of the random variable $X_n$ , defined by

$$E\left(X_n\right) = \int_0^\infty t v_n(t) dt .$$

In particular, if the $E\left[f(X_n)\right] = F_n(f,x)$ is taken in the inequality (2.4), then we obtain

$$\left|F_n(f,x)-f(x)\right| \leq w(1/\delta) .$$

Since $w(1/\delta) \to 0$ for $\delta = n \to \infty$ the desired result is then reached.

Finally, we obtain following uniform approximation:

$$E\left[f\left(X_n\right)\right] \to f(x) , \; n \to \infty .$$

## 3. GENERALIZATION OF FELLER THEOREM

In order to prove the main theorem we will give following lemmas and proves are given.

**Lemma 1:**

Let $R_m$ be the Lagrange term in equality (2.4) and $w_m$ the modulus of continuity of function $f \in C^{(m)}[0,\infty)$ . Then following inequality holds for $m = 1,3,5,\ldots$ ,

$$E\left|R_m\left(X_n,x\right)\right| \leq \frac{1}{m!}\left\{\sqrt{E\left[\left(X_n-x\right)^{2m}\right]} + \delta E\left[\left(X_n-x\right)^{m+1}\right]\right\} w_m(1/\delta) . \tag{3.1}$$

**Proof:**

By using the property of modulus of continuity for $\lambda > 0$ and $\delta > 0$,

$$w_m(\lambda\delta) \le (1+\lambda) w_m(\delta)$$

and Cauchy inequality

$$E\left(\left|X_n - x\right|^m\right) \le \sqrt{E\left[(X_n - x)^{2m}\right]} .$$

Then, inequality (3.1) is obtained.

**Lemma 2:**

The following inequality holds for $m = 0, 2, 4, \ldots$

$$E\left[\left|R_m\left(X_n - x\right)\right|\right] \le \frac{1}{m!}\left\{E\left[\left(X_n - x\right)^m\right] + \delta\sqrt{E\left[\left(X_n - x\right)^2\right]E\left[\left(X_n - x\right)^{2m}\right]}\right\} w_m(1/\delta) . \qquad (3.2)$$

**Proof:**

By using the property of modulus of continuity and Cauchy inequality

$$E\left(\left|X_n - x\right|^{m+1}\right) \le \sqrt{E\left[\left(X_n - x\right)^2\right]E\left[\left(X_n - x\right)^{2m}\right]}$$

then, inequality (3.2) is obtained.

**Lemma 3:**

Let $v_n$ be density function of the random variable $X_n$, then the *m*-th moment of the random variable $X_n$ have

$$E\left[\left(X_n - x\right)^m\right] = x^m \sum_{k=0}^{m} \frac{(-1)^k}{n^k} C(m,k)(n+k-1)(n+k-2)\cdots(n+1)n . \qquad (3.3)$$

**Proof:**

By using the Binomial formula we get the following equality

$$E\left[\left(X_n - x\right)^m\right] = \sum_{k=0}^{m} (-1)^k C(m,k) X^{m-k} E\left(X_n^k\right) . \qquad (3.4)$$

Thus, we obtain following moment,

$$E\left(X_n^k\right) = \int_0^\infty t^k v_n(t)dt = \frac{x^k}{n^k}(n+k-1)(n+k-2)\cdots(n+1)n .$$

If this expression is substituted in the equality (3.4), the equality (3.3) is obtained.

**Theorem:**

Let us $E\left[f\left(X_n\right)\right]$ be a Feller operator. Then the difference between $E\left[f\left(X_n\right)\right]$ and $\sum_{k=0}^{m}\frac{\alpha_k}{k!}f^{(k)}\left(x\right)$ has a upper bound for $m=1,3,5,\ldots$ as follows

$$\left|E\left[f\left(X_n\right)\right]-\sum_{k=0}^{m}\frac{\alpha_k}{k!}f^{(k)}\left(x\right)\right|\le\frac{1}{m!}\left(\sqrt{\alpha_{2m}}+\delta\alpha_{m+1}\right)w_m(1/\delta).$$

and $m=0,2,4,\ldots$ as follows

$$\left|E\left[f\left(X_n\right)\right]-\sum_{k=0}^{m}\frac{\alpha_k}{k!}f^{(k)}\left(x\right)\right|\le\frac{1}{m!}\left(\alpha_m+\delta\sqrt{\alpha_2\alpha_{2m}}\right)w_m(1/\delta),$$

where $\alpha_m$ is in the form below

$$\alpha_m=x^m\sum_{k=0}\frac{(-1)^k}{n^k}C(m,k)(n+k-1)\cdots(n+1)n. \tag{3.5}$$

**Proof:**

If the equality (1.1) is kept in mind, the desired result will be provided from Lemma 1 and Lemma 2. By using the Lemma 3, we find equality (3.5). Thus the proof is completed.

Now, by using this theorem, let us find the difference between the special sum which includes the second derivative of the Taylor series and the Feller operator $E\left[f\left(X_n\right)\right]$. In particular, if we take $m=2$ in the theorem, we will get the following inequality:

$$\left|E\left[f\left(X_n\right)\right]-\left[\alpha_0 f(x)+\alpha_1 f'(x)+\frac{\alpha_2}{2}f''(x)\right]\right|\le C_n\ , \tag{3.6}$$

where it is obtained as $\alpha_0=1,\alpha_1=0,\ \alpha_2=x^2/n$. In addition it is regarded $\alpha_4$ as following

$$\alpha_4=x^4\left[-3+\frac{6(n+1)}{n}-\frac{4(n+2)(n+1)}{n^2}+\frac{(n+3)(n+2)(n+1)}{n^3}\right].$$

It is easily seen that as $n\to\infty$,

$$C_n=\frac{1}{2}\left(\alpha_2+\delta\sqrt{\alpha_2\alpha_4}\right)w_2(1/\delta)\to 0.$$

Even though $n\ C_n\to 0$.

If the values of $\alpha_0,\alpha_1,\ \alpha_2$ are substituted in the inequality (3.6), then we obtain

$$\left| E\left[ f\left( X_n \right) \right] - \left[ f(x) + \frac{x^2}{2n} f''(x) \right] \right| \le C_n . \tag{3.7}$$

From this equation the following asymptotic result is obtained. From the inequality (3.7), we obtain

$$\lim_{n\to\infty} \left\{ nE\left[ f\left( X_n \right) \right] - nf(x) \right\} = \frac{x^2}{2} f''(x) . \tag{3.8}$$

**Example:**

Let us consider as $f(x) = \exp(\lambda x)$, then, from the expression (3.7), we can write

$$nE\left[ f\left( X_n \right) \right] - nf(x) = \frac{n^{n+1}}{(n-\lambda x)^n} - n e^{\lambda x} .$$

Further, from the inequality (3.8)

$$\frac{x^2}{2} f''(x) = \frac{(\lambda x)^2}{2} e^{\lambda x} .$$

Hence, we obtain the following inequality

$$\lim_{n\to\infty} \left[ \frac{n^{n+1}}{(n-\lambda x)^n} - n e^{\lambda x} \right] = \frac{(\lambda x)^2}{2} e^{\lambda x} .$$

## 4. APPLICATION TO LAPLACE TRANSFORM

If in equality (1.1) we consider the variable $nt/x$, it can be obtained the following result:

$$F_n(f,x) = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} f(xt/n) e^{-t} dt$$

$$= \frac{1}{\Gamma(n)} L\left\{ t^{n-1} f(xt/n) \right\}_{s=1} .$$

So the operator $(\Gamma(n))^{-1} L\left\{ t^{n-1} f(xt/n) \right\}_{s=1}$ converges uniformly to $f$ on $[0,\infty)$. As usual we use the test functions $e_i(x) = x^i$, $i = 0,1,2$ as following:

$$\frac{1}{\Gamma(n)} L\left\{ t^{n-1} \right\}_{s=1} = 1$$

$$\frac{1}{\Gamma(n)} L\left\{ t^{n-1} (xt/n) \right\}_{s=1} = \left\{ \frac{x\, n!}{n\, \Gamma(n)\, s^{n+1}} \right\}_{s=1} = x$$

$$\frac{1}{\Gamma(n)}L\left\{t^{n-1}(xt/n)^2\right\}_{s=1}=\left\{\frac{x^2(n+1)!}{n^2\Gamma(n)s^{n+1}}\right\}_{s=1}=\frac{x^2(n+1)}{n}\to x^2 .$$

Finally, we point out that this result may be useful in applied mathematics when inverse Laplace transformation is very difficult for some functions.

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