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MEAN RESIDUAL LIVES OF SOME DISCRETE DISTRIBUTIONS

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ABSTRACT

We present some distributional properties of the mean residual lives of some discrete distributions. It has been found that for most of these distributions the mean residual lives are related to their hazard rates. Using these relations several characterizations of the discrete distributions are presented.

KEY WORDS

Mean residual lives; Discrete distributions; Conditional expectation; Hazard rate and Characterization.

1. INTRODUCTION

Characterizations of distributions have always been an important aspect of statistical theory. Mean residual life or conditional expectation has been widely used in reliability. Actuaries use tail conditional expectation and tail conditional variance as a measure of risk. Ahmed (1991) characterized beta, binomial, and Poisson distributions by connecting conditional expectation with hazard rate. The characterization of negative binomial distribution by Osaki and Li (1988) is a special case of our generalized result. For various results on the characterization of discrete distributions see Arnold (1980), Galambos and Kotz (1978), Panjer (2006), Rao and Rubin (1964), Srivastava (1981), and Zilstra (1983).

In this paper we present characterizations of some discrete distributions and point out some errors in the proof of the results of Ahmed (1991).

Section 2 states a theorem which is used to present characterizations of five well-known discrete distributions. In this section we also point out the error made in the earlier paper, Ahmed (1991), with respect to characterization of binomial and Poisson distributions using conditional expectation $E\left(X \mid X > t\right)$.

2. MAIN RESULTS

We present here the following theorem which will be used to characterize several discrete distributions.

Theorem 1.

Let X be a nonnegative integer valued random variable with probability mass function (pmf) $p(x)$ for $x = m, m+1, m+2, \ldots, \beta$, where m may be zero and β may be ∞. Further we assume $E(X) < \infty$ and

$$E(X \mid X > t) = \alpha + \omega(t) h(t), \tag{2.1}$$

where α is a constant, $\omega(t)$ is a function of t and $h(t) = \dfrac{p(t)}{\sum_{x=t+1}^{\beta} p(x)}$, then

$$p(x) = \begin{cases} \dfrac{1}{1+g(m)+g(m)g(m+1)+\cdots} & \text{if} \quad x = m \\[2ex] \dfrac{g(m)g(m+1)\cdots g(x-1)}{1+g(m)+g(m)g(m+1)+\cdots} & \text{if} \quad x = m+1, m+2, \ldots \end{cases}$$

where

$$g(x) = \frac{\omega(x)}{1-\alpha+t+\omega(x+1)}. \tag{2.2}$$

Proof:

By definition

$$\mathrm{E}(X \mid X > t) = \frac{\sum_{x=t+1}^{\beta} x p(x)}{\sum_{x=t+1}^{\beta} p(x)}. \tag{2.3}$$

Equating equations (2.1) and (2.3), we get

$$(t+1)p(t+1) + (t+2)p(t+2) + \cdots = \alpha\left[p(t+1) + p(t+2) + \cdots\right] + \omega(t)p(t)$$

Taking first difference with respect to t, we obtain

$$p(t+1) = g(t)p(t), \tag{2.4}$$

where $g(t) = \dfrac{\omega(t)}{t+1-\alpha+\omega(t+1)}$.

We obtain from (2.4)

$$p(x) = g(m)g(m+1)\cdots g(x-1)p(m), \text{ where } x = m+1, m+2, \ldots$$

Since $1 = \sum_{m}^{\infty} p(x) = p(m)\left[1+g(m)+g(m)g(m+1)+\cdots\right]$, we must have

$$p(m) = \frac{1}{1 + g(m) + g(m)g(m+1) + \cdots}$$

and

$$p(x) = \frac{g(m)g(m+1)\cdots g(x-1)}{1 + g(m) + g(m)g(m+1) + \cdots} \quad \text{where } x = m+1, m+2, \ldots$$

which proves the theorem. □

Using Theorem 1, we present several characterizations of discrete distributions in the following table.

Table 1

Probability Mass Function	**Iff Characterization Condition**
$Poisson(x;\lambda) = \begin{cases} \frac{\lambda^x}{x!}e^{-\lambda} & x = 0,1,2,\ldots, \text{ and } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$	$E(X \mid X > t) = \lambda + \lambda h(t)$. The condition is obtained from (2.1) by substituting $\alpha = \omega(t) = \lambda(\lambda > 0)$ in (2.2) and assuming $m = 0$ in Theorem 1.
$\log arithmic(x;p) = \begin{cases} \frac{p^x}{-x\ln(1-p)} & \text{for } x = 1,2,\ldots; \\ 0 & \text{otherwise} \end{cases}$	$E(X \mid X > t) = \frac{tp}{1-p}h(t)$. The condition is obtained from (2.1) by substituting $\alpha = 0,\ \omega(t) = \frac{p}{1-p}t$ in (2.2) and assuming $m = 1$ in Theorem 1.
$negativebinomial(x;n,p) = \begin{cases} \binom{n+x-1}{x} p^x (1-p)^n & \text{for } x = 0,1,2,\ldots; \\ 0 & \text{otherwise} \end{cases}$	$E(X \mid X > t) = \frac{np}{1-p} + \frac{(n+t)p}{1-p}h(t)$ The condition is obtained from (2.1) by substituting $\alpha = \frac{np}{1-p},\ \omega(t) = \frac{p}{1-p}(n+t)$ in (2.2) and assuming $m = 0$ in Theorem 1.
$binomial(x;n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0,1,2,\ldots; \\ 0 & \text{otherwise} \end{cases}$	$E(X \mid X > t) = np + (n-t)ph(t)$ The condition is obtained from (2.1) by substituting $\alpha = np,\ \omega(t) = p(n-t)$ in (2.2) and assuming $m = 0$ in Theorem 1.
$uniform(x,N) = \begin{cases} \frac{1}{N} & \text{for } x = 1,2,\ldots,N \\ 0 & \text{otherwise} \end{cases}$	$E(X \mid X > t) = \frac{(N+t+1)(N-t)}{2}h(t)$. This is obtained from (2.1) by substituting $\alpha = 0,\ \omega(t) = \frac{(N+t+1)(N-t)}{2}$ in (2.2) and assuming $m = 1$ in Theorem 1.

Note that in Ahmed (1991), the theorem 3 statement is erroneous. Lemma 7 should be $\overline{F}(m+1)=\overline{F}(m)-p(m+1)$ and not $\overline{F}(m+1)=\frac{(m+1)}{\lambda}p(m+1)+\overline{F}(m)$ as claimed.

Note that in Ahmed (1991), the claim in the statement of Theorem 2 is erroneous. This is explained in the following numerical example.

Suppose $X \sim bin(5,0.8)$, then by Ahmed (1991)

$$E(X \mid X>3)=np+q(3+1)h(3+1)$$
$$=5(0.8)+(0.2)(4)\frac{p(4)}{p(5)}=5$$

The correct result is

$$E(X \mid X>3)=\frac{(4)p(4)+(5)p(5)}{p(4)+p(5)}=4.44\ldots$$

Our result, $E(X \mid X>t)=np+(n-t)ph(t)$, achieves the correct answer.

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