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BAYESIAN ANALYSIS OF GAMMA MODELS FOR PAIRED COMPARISONS

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ABSTRACT

The method of Paired Comparisons (PC) is a technique used to prioritize items through subjective assessments. Stern (1990) develops gamma models for PC to comparing treatments on the basis of time taken in a point scoring competition. In this study, an effort is made to analyze gamma models for PC through Bayesian approach incorporating prior uncertainty existing in the model parameters. Uninformative and informative priors are used to serve the purpose. Hyperparameters of the informative priors are elicited. We use a real data set on five top-ranked one-day international cricket teams for illustration.

KEY WORDS

Paired Comparisons; Gamma Models; Worth Parameters; Bayesian Analysis; Uniform, Jeffreys, Dirichlet and Conjugate Priors; Elicitation of Hyperparameters.

1. INTRODUCTION

In the method of Paired Comparisons (PC), treatments (items, options, stimuli, objects or individuals) are ranked on the basis of the number of preferences in their pair wise encounters. Treatments are presented in pairs to judges (jurists, subjects, panelists, respondents or raters) and they are asked to pick the better one. The experiment is repeated with a fixed number of judges and the data thus obtained is presented in the form of a preference matrix, which is then analyzed via the PC models to yield ranking of the treatments. A detailed discussion on the method is given in Bradley (1976) and David (1988). It is primarily used for subjective judgments where quantitative measurement is impossible or impracticable. The most frequent application has been to sensory testing; especially taste testing, to consumer tests, personal rating, and choice behavior. Thurstone (1927) assumes the judges' responses to follow the Gaussian distribution, and Bradley (1953), the Logistic distribution in developing their PC models. Stern (1990) considers an approach to build PC models by comparing two gamma random variables. The probability that one treatment is preferred to a second one is taken to be the probability that one gamma random variable with shape parameter is less than a second independent gamma random variable with the same shape parameter but a different scale parameter. Different values of the shape parameter provide different PC models.

The aim of this study is to accommodate into analysis the uncertainty appearing in the parameters of the Stern's gamma PC models through their prior distribution via the Bayesian frame work. Stern (1990) does not consider such uncertainty and studies his gamma models through classical approach. Main difference between the Bayesian and the classical approaches is that the former associates randomness with parameters and formally incorporates any prior information pertaining to the parameters into their analysis in addition to the current information (data). The prior information about parameters is updated with the current one to yield a posterior distribution, which is the work-bench for the Bayesians. However, the latter bases only on the current information and no concepts of the utility of the prior information are there. Adams (2005) throws light on the advantages of the Bayesian approach in an explained way.

The breakup of the study is as follows: In Section 2, the gamma PC models are discussed. Section 3 deals with notations used in this study and the likelihood function. Section 4 constitutes the prior distributions to be used in the analysis. Elicitation of hyperparameters of informative priors is carried on in Section 5. Bayesian analysis of the model is performed in Section 6. Section 7 addresses the appropriateness of the model. The entire procedure is illustrated using a real data set on five top-ranked one-day international cricket teams of Australia, India, New Zealand, Pakistan and South Africa and we use the subscripts 1, 2, 3, 4 and 5 for the respective teams. All the results are discussed along with concluding remarks in Section 8.

2. THE GAMMA MODELS FOR PAIRED COMPARISONS

According to Stern (1990), we let $p_{ij}^{(r)}$ denote the probability that treatment i is preferred to j in a paired comparison indexed by shape parameter r. Let X_i and X_j be independent gamma variables (treatments) with common shape parameter r and different scale parameters θ_i and θ_j. Then according to Stern (1990),

$$p_{ij}^{(r)} = P\left(X_i < X_j\right) = \int_0^{\infty}\int_0^{x_j} \frac{\theta_i^r x_i^{r-1}\exp\left(-\theta_i x_i\right)}{\Gamma\left(r\right)}\frac{\theta_j^r x_j^{r-1}\exp\left(-\theta_j x_j\right)}{\Gamma\left(r\right)} dx_i dx_j = f\left(r, \frac{\theta_i}{\theta_j}\right). \tag{2.1}$$

Obviously (2.1) is a function of the shape parameter r and ratio of the scale parameters θ_i/θ_j. The scale parameters θ_i and θ_j vary from variable to variable and define the value of preference probabilities $p_{ij}^{(r)}$ for comparing any pair of treatments (players in Stern's words) (i, j) and hence can rightly be declared as worth or strength of the treatments i and j.

Different values of the shape parameter r lead to different PC models. At $r = 1$ and $r = \infty$, we get the Bradley-Terry and the Thurstone-Mosteller models respectively. Moreover its integral values lead to Poisson process. For $r = 1/2$, gamma distribution transforms to the chi-square distribution. Assuming $\theta_i \le \theta_j$, the probability that the treatment X_i is preferred to X_j is

$$p_{ij} = P\left(X_i < X_j\right)$$
$$= \int_0^{\infty}\int_0^{x_j} \frac{\theta_i^{\frac{1}{2}} x_i^{\frac{1}{2}-1} \exp\left(-\theta_i x_i\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\theta_j^{\frac{1}{2}} x_j^{\frac{1}{2}-1} \exp\left(-\theta_j x_j\right)}{\Gamma\left(\frac{1}{2}\right)} dx_i dx_j = \frac{2\tan^{-1}\left(\sqrt{\theta_i / \theta_j}\right)}{\pi}, \qquad (2.2)$$

where $\pi \approx 3.141593$. Since the Bradley-Terry, the Thurstone-Mosteller and the Poisson PC models are extensively studied in literature, we proceed with analyzing model (2.2) under the name of *the chi-square model*.

3. NOTATIONS AND LIKELIHOOD FUNCTION

As mentioned earlier, p_{ij} denotes the probability of preferring treatment i over j with its complement being $p_{ji} = 1 - p_{ij}$, $n_{ij} = n_{ji}$ be the total number of comparisons made between the treatments i and j, a_{ij} stands for the number of preferences of treatment i over j with $a_{ji} = n_{ij} - a_{ij}$. Moreover $a_{ii} = a_{jj} = 0$, for the reason that a comparison of a treatment with itself makes no sense.

For the present situation, the trials of comparing treatments are independent with only two categories of the outcomes for all the trial, i.e., preferring the treatment i over j and the vice versa with a constant probability p_{ij} of preferring the treatment i over j for $i(\neq j) = 1, 2, ..., t$. The paired comparison experiment is performed a fixed number of times, i.e., n_{ij}. So the random variable a_{ij} follows a binomial distribution $B(a_{ij}; n_{ij}, p_{ij})$. Hence, the likelihood function $L(a;\theta) = \prod_{i(<j)=1}^{t} C_{a_{ij}}^{n_{ij}} p_{ij}^{a_{ij}} p_{ji}^{a_{ji}}$ with all predefined notations and θ denotes the vector of unknown worth or strength parameters with θ_i referring to the worth of the ith treatment. Here as a particular case, i will run from 1 through 5 denoting the five one day international cricket teams mentioned in Section 1, i.e., $t \equiv 5$.

4. PRIOR DISTRIBUTIONS OF THE MODEL'S WORTH PARAMETERS

For the case of uninformative priors of the model parameters vector θ, we use the uniform prior as $p_U(\theta) = 1$, for $\theta \in \Omega$, the parametric space, and the Jeffreys prior as $p_J\left(\theta\right) \propto \sqrt{\det\left\{I\left(\theta\right)\right\}}$, where $I\left(\theta\right) = -E\left\{\frac{\partial^2 \ln L(a;\theta)}{\partial\theta^2}\right\}$ is the Fisher Information matrix. The conjugate (informative) prior to be used here is

$p_C(\theta) = \prod_{i(<j)=1}^{t} \binom{n_{ij}}{a_{ij}^o} p_{ij}^{\;a_{ij}^o} p_{ji}^{\;n_{ij}-a_{ij}^o}$, where p_{ij}, the preference probability defined in (2.2), is a function of the parameters vector θ and a_{ij}^o, for $i(<j)=1,2,...,t$, denotes the hyperparameters of the conjugate prior. Moreover, the parameters θ_i, for $i=1,2,...,t$, showing the strength or worth of treatments are actually the scale parameters of the gamma distribution and attain values ranging from zero to infinity. But in practical situations, worth can never be infinity. So to develop determinacy, we impose the restriction of sum of the worth parameters to be unity and may let the parameters θ_i range from zero to one. Hence, we consider a Dirichlet (multivariate beta) informative prior for the model's parameters θ_i for $i=1,2,...,t$. The Dirichlet (informative prior) distribution, $p_I(\theta)$, is

$$p_I(\theta) = \frac{\Gamma(b_1+b_2+...+b_t)}{\Gamma(b_1)\Gamma(b_2)...\Gamma(b_t)} \prod_{i=1}^{t} \theta_i^{b_i-1}, \; b_i>0, \; 0 \le \theta_i \le 1, \; \sum_{i=1}^{t}\theta_i = 1, \tag{4.1}$$

where $b_i, \forall i=1,2,...,t$, be the hyperparameters and $\Gamma(.)$ stands for the well known gamma function.

5. ELICITATION OF HYPERPARAMETERS OF INFORMATIVE PRIORS

When substantial amount of information is available about parameter(s), we quantify the information in the form of a (prior) probability distribution. The process of quantifying the prior information accurately is known as elicitation. Garthwaite et al. (2005) give a detailed discussion on the elicitation process. We may use Minimum Chi-square Method due to Abbas and Aslam (2009) or the Prior Predictive Probabilities methods due to Aslam (2003) to elicit the hyperparameters for the conjugate and the Dirichlet priors. The underlying logic of both the elicitations methods is to minimize the difference between the elicited and the fitted/predicted probabilities.

In Prior Predictive Probabilities method, we assume that the probabilities satisfy the laws of probability for consistent eliciting. We may ask experts to elicit/suggest probabilies for the result of a particular paired comparison (wining a match by any team against its competeror in the current scenario) or may use data for convenience and avoiding subjectivism without consulting experts. Then the elicited probabilities are evaluated at a particular number of successes (may be one or more). These elicited probabilies are then compared with the prior predictive probabilities found using the same range of data values and the unknown values of the hyperparameters of the prior used in deriving the prior predictive distribution. The values of the hyperparemters which minimize the distance between the elicited and the corresponding predictive probabilies are chosen as the desires estimates of the hyperparameters. For a detailed discussion on the Prior Predictive Probabilities method, see Aslam (2003).

Suppose b_i and b_j, for $i(\neq j) = 1, 2, ..., t$, are the hyperparameters to be elicited. Then these are to be chosen by minimizing the function

$$\xi\left(b_i, b_j\right) = \min_{b_i, b_j} \sum_{i(<j)=1}^{t} \left\{ \frac{p\left(a_{ij}\right) - p_o\left(a_{ij}\right)}{p\left(a_{ij}\right)} \right\}^2, \tag{5.1}$$

where $p(a_{ij})$ denotes the prior predictive probabilities characterized by the hyperparameters b_i and b_j, and $p_o(a_{ij})$ be the corresponding elicited probabilities for the h th pair of the $t(t-1)/2$ paired comparisons at a specific value or in any specified range of the data values a_{ij}. The random variable a_{ij} may be discrete or continuous and the elicited probabilities are found accordingly.

With us the random variable a_{ij} follows binomial distribution $B\left(a_{ij}; n_{ij}, p_{ij}\right)$ and we find the elicited probabilities $p_o(a_{ij}) = P\left(l < a_{ij} < u\right) = \sum_{a_{ij}=l}^{u} B\left(a_{ij}; n_{ij}, p_{ij}\right)$ at an interval $\left[l, u\right]$ using p_{ij} as the probability of success (wining a match by team *i* against *j*). The probabilities p_{ij}, for $i(< j) = 1, 2, ..., t$, may be asked from an expert. The corresponding predictive probabilities at the same interval $\left[l, u\right]$ are

$$p(a_{ij}) = \sum_{a_{ij}=l}^{u} \left\{ \int_\theta p\left(\theta\right) \binom{n_{ij}}{a_{ij}} p_{ij}^{a_{ij}} \left(1 - p_{ij}\right)^{n_{ij} - a_{ij}} d\theta \right\},$$

where $p\left(\theta\right)$ is an informative prior characterized by the hyperparameters b_i and b_j to be elicited, and p_{ij} denotes the preferrence probability produced by a PC model. Then an algorithm is devised to find the estimates of the hyperparameters b_i and b_j which minimize (5.1).

Being specific, we use interval [*l*, *u*] = [0, 4] for the value of the random variable a_{ij}, the prior is assumed to be Dirichlet distribution given in (4.1), p_{ij} is taken to be the odd ratio a_{ij}/n_{ij} (through the data support using Table 2 for convenience, however we may use expert opinion here). Using the Prior Predictive Probabilities method and a computer program developed in *SAS*[1] package using *PROC SYSNLIN* command, the hyperparameters thus estimated are displayed in Table 1.

[1] We have used *SAS* package and *C* language to develop extensive computer programs for estimation. The programs are not given here to save space, but may be had from the authors.

Table 1
Estimate of Hyperparameters of Informative Priors

Hyperparameters*	a_{12}^{o}	a_{13}^{o}	a_{14}^{o}	a_{15}^{o}	a_{23}^{o}
Estimates*	10.0171	10.0152	10.0055	10.0200	2.6758
Hyperparameters*	a_{24}^{o}	a_{25}^{o}	a_{34}^{o}	a_{35}^{o}	a_{45}^{o}
Estimates*	2.9218	8.87164	8.5587	8.6621	3.9628
Hyperparameters**	b_1	b_2	b_3	b_4	b_5
Estimates**	5.0039	2.6233	5.1427	0.5841	4.8589

*Conjugate Prior **Dirichlet Prior

6. BAYESIAN ANALYSIS OF THE MODEL

The joint posterior distribution $P(\theta \mid a)$ for the vector of the model's worth parameters $\theta = (\theta_1, \theta_2, ..., \theta_t)$, conditional upon data a of Table 2, is:

$$P(\theta \mid a) = K^{-1} p(\theta) L(\mathrm{a};\theta) = K^{-1} p(\theta) \prod_{i(<j)=1}^{t} \left[p_{ij}^{\ a_{ij}} p_{ji}^{\ a_{ji}} \right], \quad 0 \le \theta_i \le 1, \qquad (6.1)$$

where the normalizing constant $K = \int_{\theta} p(\theta) \prod_{i(<j)=1}^{t} \left[p_{ij}^{\ a_{ij}} p_{ji}^{\ a_{ji}} \right] d\theta$ and $p(\theta)$ denotes one of the prior distributions elaborated in Section 4. Being specific, θ_1, θ_2, θ_3, θ_4 and θ_5 denote the worth of the Australian, Indian, New Zealand, Pakistan and South African teams respectively. The real data on five top-ranked one day international (ODI) cricket teams mentioned earlier is taken from Abbas and Aslam (2009) and is given in Table 2.

Table 2
Data of ODI Cricket Matches

Teams	Australia	India	New Zealand	Pakistan	South Africa
Australia	-	15	12	10	15
India	4	-	3	9	6
New Zealand	6	7	-	6	6
Pakistan	4	8	11	-	3
South Africa	9	7	10	6	-

6.1 Marginal Posterior Distributions of the Model's Worth Parameters

Using the joint posterior distribution (6.1), the marginal posterior distribution $p(\theta_i \mid a)$ of the model parameter θ_i, conditional on data a, is

$$p(\theta_i \mid a) = \int_{\theta \neq \theta_i} p(\theta \mid a) d\theta \ , \quad 0 \le \theta_i \le 1 . \qquad (6.2)$$

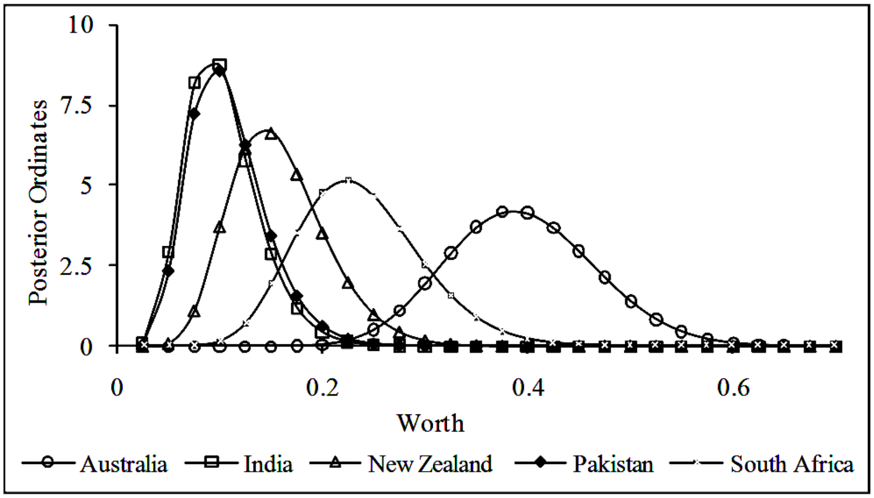

Figure 1: Marginal Posterior Distributions of the Model's Worth Parameters

Due to the complicated natures of (6.1) and (6.2), the marginal posterior distributions can not be derived analytically in closed mathematical forms. However, we may have an idea about the nature of variation of the worth parameters by plotting their ordinates found using the observed data of Table 2 against their parametric values. The resulting graphs are portrayed in Figure 1.

Having a look at the figure, it becomes evident that the marginal posterior distributions of the worth parameters are almost symmetric around their posterior modal values with varying dispersions.

6.2 Posterior Means

Using the real data set given in Table 2 and assuming a squared error loss function, we come up with the posterior means as the Bayes estimates of the worth parameter of ith team, θ_i, i.e.

$$E\left(\theta_i \mid a\right) = \int_0^1 p(\theta_i \mid a) d\theta_i \,, \quad \text{for } i = 1, 2, \ldots, 5 \,, \tag{6.3}$$

where the marginal posterior distribution $p\left(\theta_i \mid \mathrm{a}\right)$ of the model parameter θ_i is given in (6.2). Due to the complicated nature of (6.3), the posterior means can not be found analytically. We thus use the quadrature method of numerical integration to solve it by developing a computer program in C language imposing the constraint $\sum_{i=1}^{t} \theta_i = 1$ for identification. The results thus achieved are displayed in Table 3.

Table 3
Posterior Means of the Model's Worth Parameters

Teams	Parameters	UP[1]	JP[2]	DP[3]	CP[4]
Australia	θ_1	0.43116	0.48587	0.39413	0.33463
India	θ_2	0.10022	0.09460	0.10298	0.14869
New Zealand	θ_3	0.12019	0.11397	0.15758	0.17012
Pakistan	θ_4	0.13586	0.13072	0.10828	0.13482
South Africa	θ_5	0.21257	0.17485	0.23703	0.21174

[1]Uniform prior [2]Jeffreys prior [3]Dirichlet prior [4]Conjugate prior

From the estimates obtained using the noninformative uniform and the Jeffreys priors, we observe that the Kangaroos enjoy the first position, South Africans the second, Pakistanis being the third, Kiwis remain the fourth and finally come Indians with the lowest and fifth position. The results agree up to one decimal place. But for the informative Dirichlet and the conjugate priors, Australia and South Africa maintain their positions but the third position is captured by New Zealand, and Indians along with Pakistanis occupy the last fourth-fifth positions with meager differences. This change in the posterior means for the conjugate and informative priors may be due to the utility of additional information regarding the model parameters. So, we may say that a change in the nature of prior information may change the results.

6.3 Preference Probabilities

The probabilities indicating the expected chances of preferences of any treatment over the other in any one comparison are termed as the preference probabilities and are calculated using the posterior means and the model (2.2). Such probabilities are calculated and are summarized in Table 4.

Table 4
The Pair-wise Preference Probabilities p_{ij}

Team-pairs	UP[1]	JP[2]	DP[3]	CP[4]
(Australia , India)	0.71400	0.73546	0.69917	0.62570
(Australia , New Zealand)	0.69074	0.71287	0.64104	0.60568
(Australia , Pakistan)	0.67436	0.69538	0.69266	0.63995
(Australia , South Africa)	0.61028	0.65601	0.58007	0.57221
(India, New Zealand)	0.47112	0.47039	0.43280	0.47859
(India, Pakistan)	0.45176	0.44874	0.49202	0.51558
(India, South Africa)	0.38305	0.40373	0.37101	0.44403
(New Zealand, Pakistan)	0.48051	0.47818	0.55938	0.53693
(New Zealand, South Africa)	0.41045	0.43239	0.43548	0.46524
(Pakistan, South Africa)	0.42934	0.48596	0.37837	0.42876

[1]Uniform prior [2]Jeffreys prior [3]Dirichlet prior [4]Conjugate prior

Obviously these probabilities are compatible with the posterior means in designating worth and ranking orders of the competing teams.

6.4 Bayesian Testing of Hypotheses

The hypotheses for comparing the worth parameters θ_i and θ_j of any two teams i and j are:

$$H_{ij}:\theta_i>\theta_j \text{ and } H_{ji}:\theta_i\le\theta_j\,,\ \forall i(<j)=1,2,3,...,t\,.$$

To test these hypotheses, we need the posterior distribution of φ_{ij} and ξ_i by reparameterization in the posterior distribution (6.1) as $\varphi_{ij}=\theta_i-\theta_j$ and $\xi_i=\theta_i$, $\forall i(<j)=1,2,...,t$ and get the transformed joint posterior distribution $p_U\left\{\left(\varphi_{ij},\xi_i,\theta'\right)|\,a\right\}$, where $\theta'\cap\xi_i\cap\varphi_{ij}=\varnothing$, the empty set and $\theta'\cup\xi_i\cup\varphi_{ij}=\Omega$, the entire vector space. The marginal posterior distribution of the parameter of interest φ_{ij}, for $i(<j)=1,2,3,...,t$, is:

$$p(\varphi_{ij}\,|\,a)=\int_{\xi_i}\int_{\theta'}p(\varphi_{ij},\xi_i,\theta'\,|\,a)d\theta' d\xi_i\,,\qquad -1\le\varphi_{ij}\le 1\,.$$

Being specific for the Australia-India teams pair, the marginal posterior distribution of φ_{12} is:

$$p(\varphi_{12}\,|\,a)=\int_{\xi_1=0}^{(1+\varphi_{12})/2}\int_{\theta_3=0}^{1-2\xi_1+\varphi_{12}}\int_{\theta_4=0}^{1-2\xi_1+\varphi_{12}-\theta_3}p(\varphi_{12},\xi_1,\theta_3,\theta_4\,|\,a)d\theta_4 d\theta_3 d\xi_1\,,-1\le\varphi_{12}\le 0$$

and

$$p(\varphi_{12}\,|\,a)=\int_{\xi_1=\varphi_{12}}^{(1+\varphi_{12})/2}\int_{\theta_3=0}^{1-2\xi_1+\varphi_{12}}\int_{\theta_4=0}^{1-2\xi_1+\varphi_{12}-\theta_3}p(\varphi_{12},\xi_1,\theta_3,\theta_4\,|\,a)d\theta_4 d\theta_3 d\xi_1\,,0\le\varphi_{12}\le +1\,.$$

Therefore the posterior probability, ϕ_{12}, of the hypothesis H_{12} is:

$$\phi_{12}=P(H_{12}\,|\,a)=P(\theta_1>\theta_2\,|\,a)=P(\varphi_{12}>0\,|\,a)=\int_{\varphi_{12}=0}^{1}p(\varphi_{12}\,|\,a)d\varphi_{12}\,. \qquad (6.4)$$

Here again (6.4) has a complicated structure and analytical solution is intractable. So we solve it through numerical integration using the Quadrature method. Computer programs are run is *SAS* package and the resulting posterior probabilities of the hypotheses are presented in Table 5.

Table 5:
Pair-wise Posterior Probabilities of Hypotheses ϕ_{ij}

Hypotheses	UP[1]	JP[2]	DP[3]	CP[4]
$\theta_1 > \theta_2$	0.99883	0.99916	0.99940	0.98948
$\theta_1 > \theta_3$	0.99654	0.99797	0.98830	0.97250
$\theta_1 > \theta_4$	0.99166	0.99709	0.99887	0.99386
$\theta_1 > \theta_5$	0.94154	0.97889	0.90637	0.90693
$\theta_2 > \theta_3$	0.31268	0.31314	0.14750	0.32438
$\theta_2 > \theta_4$	0.23253	0.27309	0.39631	0.41251
$\theta_2 > \theta_5$	0.05605	0.07909	0.02382	0.13517
$\theta_3 > \theta_4$	0.35875	0.37738	0.76109	0.42681
$\theta_3 > \theta_5$	0.10307	0.12521	0.14384	0.22905
$\theta_4 > \theta_5$	0.16635	0.16147	0.03320	0.09048

[1]Uniform prior [2]Jeffreys prior [3]Dirichlet prior [4]Conjugate prior

Following rule applies to draw conclusions about the hypotheses regarding the teams being compared. Let

$$s = \min\left(\phi_{ij}, \phi_{ji}\right).$$

If ϕ_{ij} is small, then H_{ji} is accepted with high probability. On the other hand, if $\phi_{ji} = 1 - \phi_{ij}$ is small, then H_{ij} is accepted with high probability. This implies that if $'s'$ is small, we can reject one hypothesis otherwise if $s > 0.1$ (say), then the evidence is inconclusive.

From the results of Table 5, we see that the posterior probabilities of the hypotheses are in accordance with the posterior means. For instance, while testing the hypotheses H_{12}, H_{13}, H_{14} and H_{15}, there exist high probabilities in favor of the Kangaroos against all the competing teams for all the priors used, which lead us to the acceptance of all the hypotheses in favor of the Kangaroos and declare the Kangaroos unbeaten. The probabilities of the hypotheses for rest of the pairs may similarly be interpreted.

6.5 The Predictive Probabilities

The predictive distribution represents our current predictions of the variable a_{ij} taking into account both the uncertainty about the parameters θ and the residual uncertainty about the variable a_{ij} when the parameters θ are unknown, (Lee, 1989). The predictive probability $P_{(ij)}$ is the probability (unconditional on the parameters vector θ)

that the treatment i is preferred to j in a single future comparison of these treatments, and is determined as:

$$P_{(ij)} = p(i > j) = \int_{\theta} p_{ij}\, p(\theta \mid a)\, d\theta, \quad \text{for } \forall i(< j) = 1, 2, ..., t, \qquad (6.5)$$

where p_{ij} is the probability (conditional on θ) of preferring the treatment i over j and is given by the model (2.2). Moreover, $P_{(ji)} = 1 - P_{(ij)}$. The predictive probabilities $P_{(ij)}$, $\forall i(< j) = 1, 2, ..., t$, are determined evaluating (6.5) using the numerical method of integration and the resulting estimates are shown in Table 6.

Table 6
Pair-wise Predictive Probabilities $P_{(ij)}$

Team-pairs	UP[1]	JP[2]	DP[3]	CP[4]
(Australia , India)	0.71573	0.74694	0.70101	0.62616
(Australia , New Zealand)	0.69221	0.72179	0.64242	0.60601
(Australia , Pakistan)	0.67629	0.70999	0.69456	0.64046
(Australia , South Africa)	0.61161	0.61260	0.58102	0.57253
(India, New Zealand)	0.47091	0.46626	0.43238	0.47830
(India, Pakistan)	0.45241	0.45159	0.49232	0.51555
(India, South Africa)	0.38278	0.34400	0.37013	0.44372
(New Zealand, Pakistan)	0.48142	0.48509	0.56017	0.53701
(New Zealand, South Africa)	0.41029	0.37454	0.43499	0.46500
(Pakistan, South Africa)	0.42852	0.38869	0.37734	0.42835

[1]Uniform prior [2]Jeffreys prior [3]Dirichlet prior [4]Conjugate prior

Obviously these probabilities for all the priors are also compatible with the preference and posterior probabilities and second the ranking order of the competing teams established via the posterior means.

7. PLAUSIBILITY OF THE MODEL

A model is said to give appropriate or good fit to the observed data if the expected frequencies obtained using the model are considerably close to the observed frequencies of the data. To test the hypothesis of goodness of fit of a model, we use χ^2 test. The smaller the value of χ^2, the better would be the fit. Let the ordered pairs $\left(a_{ij}, \hat{a}_{ij}\right)$ and $\left(a_{ji}, \hat{a}_{ji}\right)$, where $\hat{a}_{ij} = n_{ij}.p_{ij}$ and $\hat{a}_{ji} = n_{ij}.p_{ji}$, denote the observed and the corresponding expected frequencies of the preferences of treatments i over j and the vice versa. Then the χ^2-statistic attains the form:

$$\chi^2 = \sum_{i<j}^{t} \left\{ \frac{(a_{ij} - \hat{a}_{ij})^2}{\hat{a}_{ij}} + \frac{(a_{ji} - \hat{a}_{ji})^2}{\hat{a}_{ji}} \right\}, \tag{7.1}$$

with $(t-1)(t-2)/2$ degrees of freedom, [Stern (1990)]. The null hypothesis H_0 and the alternative hypothesis H_1 are formed as:

H_0 : The model is plausible for any value of the parameter $\theta = \theta_0$.
H_1 : The model is not plausible for any value of the parameter θ.

The expected frequencies are calculated as $\hat{a}_{ij} = n_{ij} \dfrac{2\tan^{-1}\sqrt{\theta_i/\theta_j}}{\pi}$. The p-values associated with the values of the χ^2 statistics for all the priors are found using (7.1) and are presented in Table 7.

Table 7
***p*-values**

UP[1]	JP[2]	DP[3]	CP[4]
0.65501	0.64926	0.47630	0.27333

[1]Uniform prior [2]Jeffreys prior [3]Dirichlet prior [4]Conjugate prior

From the *p*-values, it is quite evident that under all circumstances, the Chi-square PC model under study is plausible to the data.

8. CONCLUDING REMARKS WITH DISCUSSION

Bayesian analysis of gamma/chi-square model is given. Having a view of the facts and figures of the analysis given in the form of the posterior means found via the noninformative priors, we see that the five ODI cricket teams under study may be ranked as Australia being the number one, South Africa the second, Pakistan being the third, New Zealand with the fourth position and finally India being the fifth and the last one. It is also worth mentioning that the posterior means produced using the uniform and the Jeffreys priors agree up to one decimal place. By switching over to using informative priors from their noninformative counterparts, the ranking order is changed by dragging the Kiwis to the third position and placing the Indians and the Pakistanis at equivalent fourth-fifth positions. It means that inference is changed by changing the prior types and this distinction goes to the Bayesians for utilizing priors. The marginal posterior distributions of the worth parameters show a symmetric behavior of the model’s worth parameters around their modal values. The preference and the predictive probabilities also favor the same ranking order established by the posterior means. The highest position of the Kangaroos is also established through the posterior probabilities of the hypotheses. The test for goodness of fit of the model conducted through the χ^2 statistic also indicates the appropriateness of the Chi-square model for paired comparisons under study with highly insignificant *p*-values.

As a future study, the accommodation of ties and triple comparisons may be considered.

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