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A UNIFIED MIXTURE MODEL BASED ON THE INVERSE GAUSSIAN DISTRIBUTION

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ABSTRACT

In this paper, we introduce a new class of mixture models based on the inverse Gaussian distribution, which is highly flexible and contains several well-known probability models. The new class of models is generated from symmetric distributions around zero by using the connection between the inverse Gaussian and standard normal distributions. We illustrate the obtained results by means of two real data sets through likelihood, goodness-of-fit and diagnostic methods. This illustration indicates the adequacy of the new model.

KEYWORDS

Birnbaum-Saunders distribution; Elliptic distributions; Kurtosis; Length-biased distributions; Mixture distributions; Moments; Robustness; Skewness.

1 INTRODUCTION

The normal (Gaussian) model has dominated the landscape of distribution theory and statistical applications for over 100 years. The remarkable properties of this distribution are well-known and widely used in theoretical and practical statistics. On the other hand, it is interesting to note the similarity between the inverse Gaussian (IG) distribution and normal distribution. In fact, Folks (2007) provided a table containing 42 analogies of these distributions.

Mixture models provide powerful and popular tools for generating flexible distributions with attractive properties; see McLachlan and Peel (2008). Specifically, if $0 < p < 1$ is a mixing parameter and $f_{X_1}(x)$ and $f_{X_2}(x)$ are the densities of the random variables X_1 and X_2, respectively, then using the obvious notation, the probability density function (pdf) of the random variable (r.v.) X determined by the mixture between X_1 and X_2 is given by

$$f_X(x) = [1-p]\, f_{X_1}(x) + p\, f_{X_2}(x), \quad x > 0. \tag{1.1}$$

In this paper, a new mixture model is presented which is named as the generalized mixture inverse Gaussian (GMIG) distribution. A motivation for this new model is based on the observation that various well-known life distributions are particular cases of this class of models. In addition, the properties of these well-known distributions could be extended to the case of GMIG models. The new model contains as particular cases the distributions derived by Díaz and Leiva (2005) and Sanhueza et al. (2008) related to Birnbaum-Saunders (BS) and IG models, respectively. For more details of the BS distribution, see Johnson et al. (1995, pp. 651-662), Marshall and Olkin (2007, pp. 451-472), Saunders (2007) and Leiva et al. (2009).

Next, we describe the connecting parts of the new model developed in this paper. These pieces represent constructive outlines of the developments of the model and indicate directions taken by several researchers dealing with this topic. The first ingredient for the generation of GMIG distributions is the IG model. Several books have appeared in the last 20 years devoted to this subject, e.g., Chhikara and Folks (1989), Seshadri (1993, 1999) and Johnson et al. (1994, pp. 259-297). Specifically, the two-parameters IG distribution is denoted by $T \sim \mathrm{IG}(\mu,\lambda)$, where both parameters, μ (the mean) and λ (scale), are positive. The pdf and the cumulative distribution function (cdf) of the IG model are given by

$$f_T(t) = \phi\big(a(t)\big)\frac{\sqrt{\lambda}}{\sqrt{t^3}} \quad \text{and} \quad F_T(t) = \Phi\big(a(t)\big) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\big(-b(t)\big), \quad t > 0, \tag{1.2}$$

respectively, where $a(t) = a(t;\mu,\lambda) = \sqrt{\lambda}[t-\mu]/[\mu\sqrt{t}]$, $b(t) = b(t;\mu,\lambda) = \sqrt{\lambda}[t+\mu]/[\mu\sqrt{t}]$ and $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal pdf and cdf, respectively. The IG pdf has a non-negative support and it is unimodal and positively skewed. This distribution is a member of the exponential family, it is related to the chi-square distribution and closed under convolution. The name of the IG distribution is due to Tweedie (1957), who found an inverse relationship between the cumulant generating functions of the IG and normal models. This model is also known as the Wald distribution. However, the IG model was originally derived by Schrödinger (1915) as the first passage time distribution of the Brownian motion with a positive drift. This model has been applied in diverse fields including agriculture, demography, ecology, engineering, genetics, meteorology and the internet; see, for example, Seshadri (1993). At the present, the IG model continues providing a rich avenue for further research.

If $T \sim \mathrm{IG}(\mu,\lambda)$, then the length-biased (LB) version of the distribution of T is given by the r.v. $L = \mu^2/T$ (also called complementary reciprocal of T) and denoted by $L \sim \mathrm{LBIG}(\mu,\lambda)$. The corresponding pdf of L is given by

$$f_L(l) = \phi\big(a(l)\big)\frac{\sqrt{\lambda}}{\mu\sqrt{l}}, \quad l > 0, \mu > 0, \lambda > 0, \tag{1.3}$$

where $a(l)$ is given as in Eq. (1.2) and $\mu = \mathbb{E}[T]$. These results are due to Jörgensen et al. (1991) and Gupta and Akman (1995). The LBIG distribution serves as the second ingredient for generating the new model.

The third ingredient is based on a mixture of IG and LBIG models called the mixture inverse Gaussian (MIG) distribution. Thus, if $T \sim \mathrm{IG}(\mu,\lambda)$ and $L \sim \mathrm{LBIG}(\mu,\lambda)$, the corresponding pdf of the r.v. M with MIG distribution is given according to Eq. (1.1) by

$$f_M(m) = \phi\big(a(m)\big)\frac{\sqrt{\lambda}}{\sqrt{m^3}}\left[[1-p] + p\,\frac{m}{\mu}\right], \quad m > 0, \mu > 0, \lambda > 0, 0 < p < 1, \tag{1.4}$$

where $a(m)$ is given as in Eq. (1.2). In this case, the notation $M \sim \mathrm{MIG}(\mu,\lambda,p)$ is used. It follows from Eq. (1.4) that the MIG distribution is related to the normal model.

It is well-known that maximum likelihood (ML) estimation under the normal distribution is quite sensitive to extreme observations. This can also occur for the MIG model and for some other models that are frequently used for describing skewed data. In the normal case, Lange et al. (1989), Lange and Sinheimer (1993) and Lucas (1997) proposed to use models with greater kurtosis than the normal model in order to solve this problem and obtain qualitatively robust estimators. We use a similar idea for the case of the MIG distribution. Thus, the final ingredient for understanding the structure of GMIG distributions leads us to the general class of symmetric distributions in $\mathbb{R}$ (or elliptically contoured univariate models or elliptic ones). This class includes distributions with different levels of kurtosis, where the normal distribution is a particular case; see Fang et al. (1990). This family of symmetric distributions around zero contains a special subclass called scale mixtures of normal distributions or normal scale mixture (NSM) containing the Laplace, logistic, normal and Student-t with ν degree of freedom (d.f.), or t_ν, models as particular cases. NSM models are important for applications with heavy tailed distributions and, in addition, this subclass produces robust parameter estimates; see Balakrishnan et al. (2009)

and references therein. Other subclasses of symmetric distributions are the Kotz type (KT) and Pearson VII (PVII) sub-families, which are also considered in this paper.

Using the connection between the MIG and normal models given in Eq. (1.4), we can generalize the MIG distribution from the elliptic class. This is achieved by replacing the standard normal pdf, $\phi(\cdot)$, in Eq. (1.4) by a general pdf of standard symmetric distributions around zero, denoted by $f(\cdot)$. This generalization of the MIG distribution preserves many of its initial properties. In addition, this new model contains a large class of life distributions, including those that produce robust parameter estimates in the presence of atypical data. For more details about life distribution, see Marshall and Olkin (2007).

The remaining part of the paper is organized as follows. Section 2 introduces the new model and presents its pdf, the mode, a graphical analysis and some of its properties and transformations as well as its cdf and moments. Section 3 applies the derived results for GMIG distributions to two real data sets using likelihood, goodness-of-fit and diagnostic methods. One of these sets involves data from behavior sciences that have not been analyzed before. Some conclusions are presented in the final section.

2 A NEW MIXTURE MODEL

An r.v. T follows a GMIG distribution, which is denoted by $T \sim \mathrm{GMIG}(\mu, \lambda, p; f)$, iff its pdf is given by

$$f_T(t) = f\big(a(t)\big)\,\frac{\sqrt{\lambda}}{\sqrt{t^3}}\left[[1-p] + p\,\frac{t}{\mu}\right], \quad t > 0, \mu > 0, \lambda > 0, 0 < p < 1, \tag{2.1}$$

where $a(t)$ is given in Eq. (1.2) and $f(\cdot)$ is the pdf of the associated symmetric distribution around zero. The mode(s) of T, denoted by t_m, is (are) given by the solution(s) of

$$\frac{d}{dt_m}\log\big(f\big(a(t_m)\big)\big) = \frac{3[1-p]\,\mu + p\,t_m}{2t_m[[1-p]\,\mu + p\,t_m]}. \tag{2.2}$$

Example 1. Let $T \sim \mathrm{GMIG}(\mu, \lambda, p; f)$. The densities of T following GMIG distributions generated from (i) Cauchy, (ii) Kotz type, (iii) Laplace, (iv) logistic, (v) normal, (vi) Pearson VII and (vii) t_ν models are, respectively, given by

(i)

$$f_T(t) = \frac{\sqrt{\lambda}}{\pi\sqrt{t^3}}\left[1 + \frac{\lambda}{\mu}\left[\frac{t}{\mu} + \frac{\mu}{t} - 2\right]\right]^{-1}\left[[1-p] + p\,\frac{t}{\mu}\right],\ t > 0;$$

(ii)

$$f_T(t) = \frac{s r^{\frac{2q-1}{2s}}}{\Gamma([2q-1]/2s)} \frac{\lambda^{q-1}}{\mu^{2[q-1]}} \frac{[t-\mu]^{2[q-1]}}{t^{q-1}} \exp\left(-\frac{r\lambda^s}{\mu^s}\left[\frac{t}{\mu}+\frac{\mu}{t}-2\right]^s\right) \frac{\sqrt{\lambda}}{\sqrt{t^3}}\left[[1-p]+p\frac{t}{\mu}\right],$$

$$t > 0, q > 1/2, r > 0, s > 0;$$

(iii)

$$f_T(t) = \frac{\sqrt{\lambda}}{2\sqrt{t^3}} \exp\left(-\frac{\sqrt{\lambda}}{\sqrt{\mu}}\left|\sqrt{\frac{t}{\mu}}-\sqrt{\frac{\mu}{t}}\right|\right)\left[[1-p]+p\frac{t}{\mu}\right], \quad t > 0;$$

(iv)

$$f_T(t) = \frac{\sqrt{\lambda}}{\sqrt{t^3}} \frac{\exp\left(\frac{\sqrt{\lambda}}{\sqrt{\mu}}\left[\sqrt{\frac{t}{\mu}}-\sqrt{\frac{\mu}{t}}\right]\right)}{\left[1+\exp\left(\frac{\sqrt{\lambda}}{\sqrt{\mu}}\left[\sqrt{\frac{t}{\mu}}-\sqrt{\frac{\mu}{t}}\right]\right)\right]^2}\left[[1-p]+p\frac{t}{\mu}\right], \quad t > 0;$$

(v)

$$f_T(t) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}\sqrt{t^3}} \exp\left(-\frac{\lambda}{2\mu}\left[\frac{t}{\mu}+\frac{\mu}{t}-2\right]\right)\left[[1-p]+p\frac{t}{\mu}\right], \quad t > 0;$$

(vi)

$$f_T(t) = \frac{\Gamma(q)}{\sqrt{r\pi}\,\Gamma\left(q-\frac{1}{2}\right)}\left[1+\frac{\lambda}{r\mu}\left[\frac{t}{\mu}+\frac{\mu}{t}-2\right]\right]^{-q} \frac{\sqrt{\lambda}}{\sqrt{t^3}}\left[[1-p]+p\frac{t}{\mu}\right],$$

$$t > 0, q > 1/2, r > 0;$$

(vii)

$$f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\lambda}}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)\sqrt{t^3}}\left[1+\frac{\lambda}{\nu\mu}\left[\frac{t}{\mu}+\frac{\mu}{t}-2\right]\right]^{-\left[\frac{\nu+1}{2}\right]}\left[[1-p]+p\frac{t}{\mu}\right], \quad t > 0, \nu > 0.$$

A graphical representation of GMIG distributions for selected values of their parameters is shown in figures 1-2. We emphasize once again that GMIG models are flexible and represent several shapes and different degrees of kurtosis and asymmetry as well as unimodality and bimodality. Specifically, figures 1-2 show several possible shapes obtained from Eq. (2.1) involving the generating distributions given in Example 1. The parameters μ and λ are of course position and scale parameters, respectively. However, μ also behaves like a scale parameter. Thus, μ is considered equal to 1, because it is easy to verify that changes in μ do not modify the shape of the pdf of GMIG distributions. In Figure 1 (left), we take $\lambda = 1$, which leads to a strong asymmetry, while Figure 1 (right) shows a weak asymmetry for a large λ (in our case $\lambda = 32$). Thus, depending on the generator pdf (f) and the parameter λ, GMIG distributions exhibit densities which are platykurtic or leptokurtic, asymmetric or symmetric and unimodal and bimodal, modifying their shape, position, scale and modality. Densities of the GMIG distribution generated from the t_ν model (GMIG-t_ν) are displayed in Figure 2 (left). The tails of this distribution are presented to the right of the same figure being possible to exhibit heavy-tailed GMIG distributions.

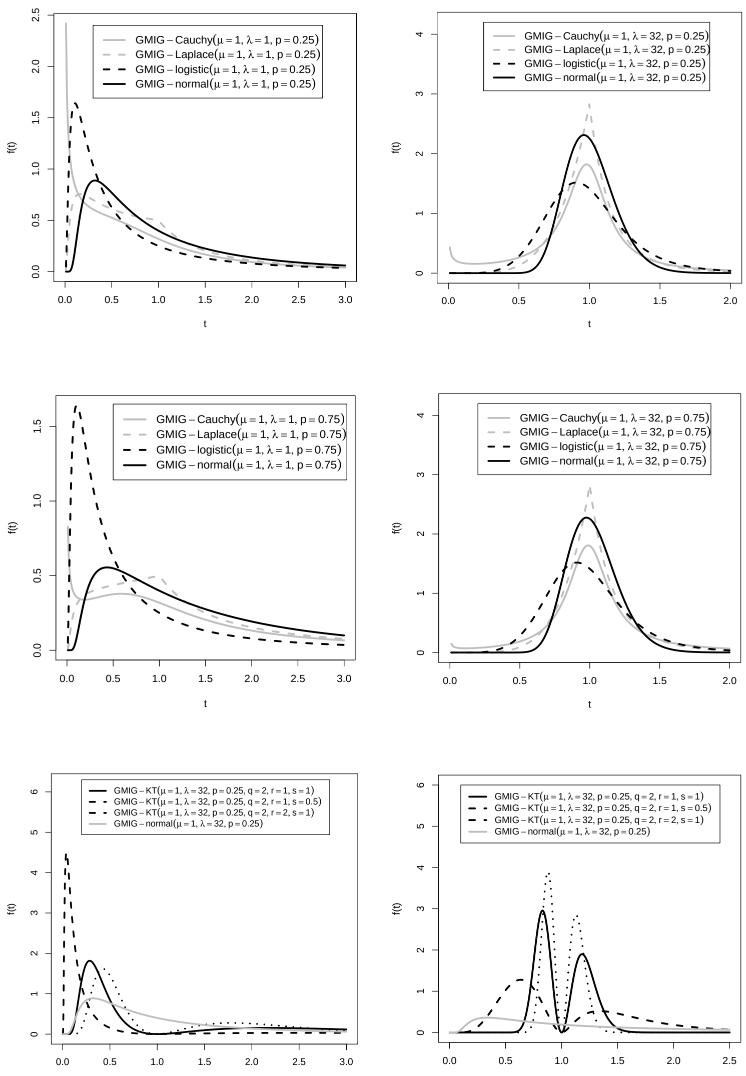

Figure 1: pdf graphs of $T \sim \text{GMIG}(\mu, \lambda, p; f)$ for the indicated cases.

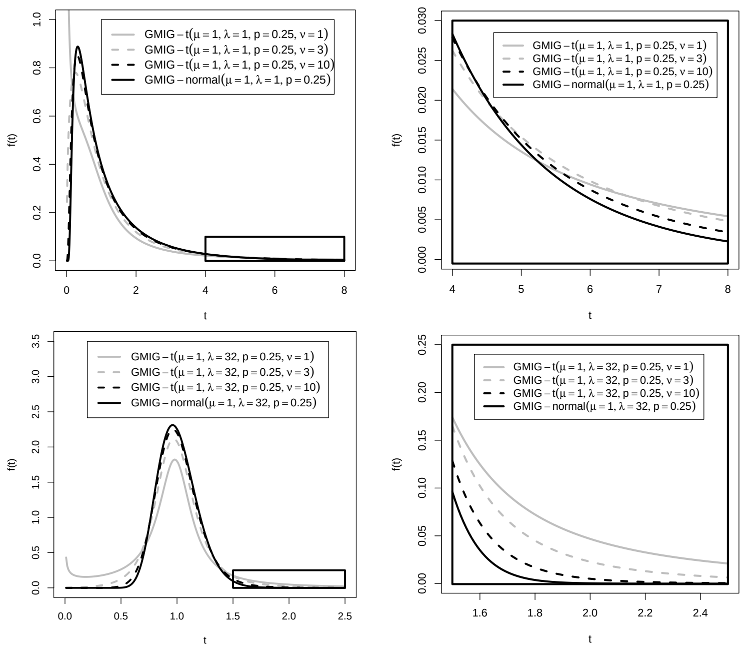

Figure 2: pdf graphs of $T \sim \text{GMIG-}t(\mu, \lambda, p; \nu)$ for the indicated values and tail zooms.

3 TRANSFORMATIONS ASSOCIATED WITH THE MODEL

In this section, we provide densities of some transformations of $T \sim \text{GMIG}(\mu, \lambda, p; f)$, including its proportional, reciprocal, logarithm and

$$U = \sqrt{\frac{\lambda}{\mu}} \left[\sqrt{\frac{T}{\mu}} - \sqrt{\frac{\mu}{T}} \right] = \frac{\sqrt{\lambda}}{\mu} \, \frac{[T - \mu]}{\sqrt{T}}. \tag{3.1}$$

Theorem 1. *Let* $T \sim \text{GMIG}(\mu, \lambda, p; f)$, *with* $\mu > 0$, $\lambda > 0$ *and* $0 < p < 1$. *Then, the pdf of*

(i) $R = 1/T$ *is given by*

$$f_R(r) = f\left(\sqrt{\tfrac{\lambda}{r\,\mu^2}}\,\left[r - \tfrac{1}{\mu}\right]\right) \sqrt{\tfrac{\lambda}{r^3 \mu^2}} \left[p + [1 - p]\,\mu\, r\right], \quad r > 0.$$

(ii) $Y = \log(T)$ *is given by*

$$\begin{aligned} f_Y(y) \quad &= \quad f\left(\tfrac{2\sqrt{\lambda}}{\sqrt{\mu}} \sinh\left(\tfrac{y - \log(\mu)}{2}\right)\right) \left[[1 - p] \exp\left(-\tfrac{1}{2}[y - \log(\lambda)]\right) + \right. \\ &\qquad \left. p \exp\left(\tfrac{1}{2}\left[y - \log\left(\tfrac{\mu^2}{\lambda}\right)\right]\right)\right], \quad -\infty < y < +\infty. \end{aligned}$$

(iii) $U = [\sqrt{\lambda}/\mu]\,[T-\mu]/\sqrt{T}$ *is given by*

$$f_U(u) = f(u)\left[1 - [1-2p]\,\frac{u}{\sqrt{u^2+4\lambda/\mu}}\right], \quad y > 0.$$

Proof. These results directly follow from Eq. (2.1) and the change-of-variable method. ■

Corollary 3.1. Let $T \sim \mathrm{GMIG}(\mu,\lambda,p;f)$ and $U = [\sqrt{\lambda}/\mu]\,[T-\mu]/\sqrt{T}$. Then, the pdf of

(i) $H = |U|$ is given by $f_H(h) = 2f(h)$, for $h > 0$.

(ii) $W = U^2$ is given by $f_W(w) = f\left(\sqrt{w}\right)\frac{1}{\sqrt{w}}$, for $w > 0$.

Remark 1. The results given in Corollary 3.1 show that the absolute value of U has a half-symmetric distribution and the square of U follows the generalized chi-square distribution with one d.f., denoted by $\mathrm{G}\chi^2(1;f)$; see Fang et al. (1990).

Let $T \sim \mathrm{GMIG}(\mu,\lambda,p;f)$. Then, the following properties are valid:

(P1) $cT \sim \mathrm{GMIG}(c\mu, c\lambda, p; f)$, with $c > 0$,

(P2) $\frac{\lambda}{\mu^2}T \sim \mathrm{GMIG}(\eta,\eta^2,p;f)$, where $\eta = \lambda/\mu$,

(P3) $\frac{1}{T} \sim \mathrm{GMIG}\left(\frac{1}{\mu},\frac{\lambda}{\mu^2},[1-p];f\right)$,

(P4) $\mu^2\,\frac{1}{T} \sim \mathrm{GMIG}(\mu,\lambda,[1-p];f)$,

(P5) If $p = \frac{1}{2}$, then $\frac{\mu^2}{T}$ has the same distribution as T, and

(P6) $W = \frac{\lambda}{\mu^2}\,\frac{[T-\mu]^2}{T} \sim \mathrm{G}\chi^2(1;f)$.

Remark 2.

(i) Property (P1) proves that GMIG models belong to the family of scale distributions.

(ii) Property (P3) implies that GMIG models belong to the family of distributions closed under reciprocation; see Saunders (1974).

(iii) Property (P4) indicates a reciprocal connection between T and $1/T$. As mentioned, the r.v. μ^2/T is known as the complementary reciprocal of T.

(iv) Property (P6) verifies that GMIG models are also related to generalized χ^2 distributions; see Fang et al. (1990).

3.1 Cumulative distribution function

We now present the cdf of the r.v. $T \sim \text{GMIG}(\mu, \lambda, p; f)$ and of some transformations of T.

Theorem 2. *Let $T \sim \text{GMIG}(\mu, \lambda, p; f)$. Then, the cdf of T is given by*

$$F_T(t) = F\left(\frac{\sqrt{\lambda}[t-\mu]}{\mu\sqrt{t}}\right) + [1-2p] \int_{b(t)}^{\infty} f\left(\sqrt{u^2 - \frac{4\lambda}{\mu}}\right) du,$$

where $b(t)$ is given in Eq. (1.2) and $F(\cdot)$ denotes the associated symmetric cdf.

Proof. Using the expression of the pdf given in Eq. (2.1), the definition of cdf and the transformation $u = [\sqrt{\lambda}/\mu][t-\mu]/\sqrt{t}$, we obtain

$$F_T(t) = \int_{-\infty}^{a(t)} f(u) \left[1 - [1-2p] \frac{u}{\sqrt{u^2+4\lambda/\mu}}\right] du = F\big(a(t)\big) - [1-2p]\, I\big(a(t)\big),$$

where $I\big(a(t)\big) = \int_{-\infty}^{a(t)} \left[f(x)\, x/\sqrt{x^2+4\lambda/\mu}\right] dx$. Considering $y = \sqrt{x^2+4\lambda/\mu}$, we have $I\big(a(t)\big) = -\int_{b(t)}^{\infty} f\left(\sqrt{y^2-4\lambda/\mu}\right) dy$, which proves the theorem. ■

Corollary 3.2. Let $T \sim \text{GMIG}(\mu, \lambda, p; f)$. Then, the cdf of

(i) $R = 1/T$ is given by

$$F_R(r) = 1 - F\left(a\left(\tfrac{1}{r}\right)\right) - [1-2p] \int_{b(1/r)}^{\infty} f\left(\sqrt{x^2 - \frac{4\lambda}{\mu}}\right) dx,$$

(ii) $Y = \log(T)$ is given by

$$F_Y(y) = F\left(a\big(\exp(y)\big)\right) + [1-2p] \int_{b(\exp(y))}^{\infty} f\left(\sqrt{x^2 - \frac{4\lambda}{\mu}}\right) dx,$$

(iii) $U = [\sqrt{\lambda}/\mu][T-\mu]/\sqrt{T}$ is given by

$$F_U(u) = F(u) + [1-2p] \int_{\sqrt{\frac{4\lambda}{\mu}+u^2}}^{\infty} f\left(\sqrt{x^2 - \frac{4\lambda}{\mu}}\right) dx,$$

where $a(1/r)$, $a\big(\exp(y)\big)$, $b(1/r)$ and $b\big(\exp(y)\big)$ are analogously defined as in Eq. (1.2) and $f(\cdot)$ and $F(\cdot)$ denote the pdf and cdf of the associated symmetric distribution.

3.2 Moments

In the case of GMIG distributions, calculation of moments is somewhat tedious, involving standard manipulations with finite sums considering combinatorial quantities and recursive relationships. Based on the moments, the coefficients of variation (CV), skewness (CS) and kurtosis (CK) are obtained.

Let $T \sim \mathrm{GMIG}(\mu, \lambda, p; f)$. Then, the following are the first four non-central moments of GMIG distributions:

(M1) $\mathbb{E}[T] = \mu + p\,\frac{\mu}{\eta}\,\omega_1$,

(M2) $\mathbb{E}[T^2] = \mu^2 + [1+2p]\,\frac{\mu^2}{\eta}\,\omega_1 + p\,\frac{\mu^2}{\eta^2}\,\omega_2$,

(M3) $\mathbb{E}[T^3] = \mu^3 + 3[1+p]\,\frac{\mu^3}{\eta}\,\omega_1 + [1+4p]\,\frac{\mu^3}{\eta^2}\,\omega_2 + p\,\frac{\mu^3}{\eta^3}\,\omega_3$ and

(M4) $\mathbb{E}[T^4] = \mu^4 + 2[3+2p]\,\frac{\mu^4}{\eta}\,\omega_1 + 5[1+2p]\,\frac{\mu^4}{\eta^2}\,\omega_2 + [1+6p]\,\frac{\mu^4}{\eta^3}\,\omega_3 + p\,\frac{\mu^4}{\eta^4}\,\omega_4$,

where $\eta = \lambda/\mu$ and $\omega_r = \mathbb{E}[W^r] < +\infty$, for $r = 1,2,3,4$, with $W \sim \mathrm{G}\chi^2(1; f)$. Values of $\mathbb{E}[W^r]$, with $r = 1,2,3,4$, for the symmetric distributions specified in Example 1, can be revised in Sanhueza et al. (2008). Thus, supposing that $\mathbb{E}[T^r]$ exists, for $r = 1,2,3,4$, then the variance, CV, CS and CK of T are, respectively, given by

$$
\begin{aligned}
\mathrm{Var[T]} &= \frac{\mu^2\omega_1}{\eta} - \frac{p^2\mu^2\omega_1^2}{\eta^2} + \frac{p\mu^2\omega_2}{\eta^2}, \quad \gamma[T] = \frac{\sqrt{\lambda\omega_1 + p\,\mu[\omega_2 - p\omega_1^2]}}{[\lambda + p\omega_1]}, \\
\alpha_3[T] &= \frac{2p^3\omega_1^3 + \eta\omega_2 - 3p^2\omega_1\omega_2 + p[\eta[\omega_2 - 3\omega_1^2] + \omega_3]}{\sqrt{\left[\eta\omega_1 + p[\omega_2 - p\omega_1^2]\right]^3}}, \quad \text{and} \\
\alpha_4[T] &= \frac{6p^3\omega_1^2\omega_2 + \eta[\eta\omega_2 + \omega_3] + p^2[6\eta\omega_1^3 - 4\eta\omega_1\omega_2 - 4\omega_1\omega_3]}{\left[\eta\omega_1 + p\left[\omega_2 - p\omega_1^2\right]\right]^2} + \\
&\quad \frac{p[2\eta\omega_3 + \omega_4 - 4\eta\omega_1\omega_2] - 3p^4\omega_1^4}{\left[\eta\omega_1 + p\left[\omega_2 - p\omega_1^2\right]\right]^2}.
\end{aligned}
$$

Remark 3. If $T \sim \mathrm{GMIG}(\mu, \lambda, p; f)$ and $f(\cdot)$ is the pdf of the standard normal distribution, the mean and variance of GMIG distributions coincide with the results given in Seshadri (1993, p. 120).

4 ESTIMATION, CHECKING AND APPLICATION TO REAL DATA

In this section, for the purposes of illustration, we analyze two data sets, which come from: (i) behavior sciences and (ii) engineering. The first case involves a new data set that have not been analyzed before. The associated problems and data (with frequency in parentheses and none in parentheses when the frequency is one) are presented next. Using these data sets, we first carry out an exploratory data analysis (EDA) and then, by means of goodness-of-fit and diagnostic tools, we check the fitting of the model to the data sets. In the analyzes presented here, all the parameters of GMIG models were estimated by using ML methods.

4.1 Data sets

Depressive condition data (S1). The scale "general rating of affective symptoms for preschoolers" (GRASP) measures behavioral and emotional problems of children, who can be classified with depressive condition or not according to this scale. A study conducted by the authors in a city located at the south part of Chile has allowed to collect real data (unpublished) corresponding to the scores of the GRASP scale of children, which are: 19(16), 20(15), 21(14), 22(9), 23(12), 24(10), 25(6), 26(9), 27(8), 28(5), 29(6), 30(4), 31(3), 32(4), 33, 34, 35(4), 36(2), 37(2), 39, 42, 44.

Lifetime data (S2). Chhikara and Folks (1989, p. 139) reported data corresponding to repair times (in hours) for an airborne communication transceiver, which are: 0.2, 0.3, 0.5(4), 0.6(2), 0.7(3), 0.8(2), 1.0(4), 1.1, 1.3, 1.5(4), 2.0(2), 2.2, 2.5, 2.7, 3.0(2), 3.3(2), 4.0(2), 4.5, 4.7, 5.0, 5.4(2), 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

4.2 Exploratory data analysis

The data sets (S1 and S2) consist of 134 and 46 units of observations, respectively, with values of GRASP scale and repair times varying between 19 and 44 points, and 0.2 and 24.5 hours, respectively. The sample mode, median and mean are: 19, 24 and 25 points for S1, and 1.0, 1.8 and 3.6 hours for S2. The sample standard deviation, CV, CS and CK are: 5.36 point, 21.47%, 1.11, and 0.89 for S1, and 4.94 hours, 137.09%, 2.79 and 8.29 for S2. Figure 3 displays histograms of S1 and S2. This EDA suggests positively skewed distributions with high degrees of variability, skewness and kurtosis. GMIG models seem to take account quite well of the degrees of skewness and kurtosis presented in S1 and S2.

Another exploratory tool that we consider is the Mahalanobis distance (see Cook and Weisberg (1982)), which is useful for detecting the influence of atypical data on the ML

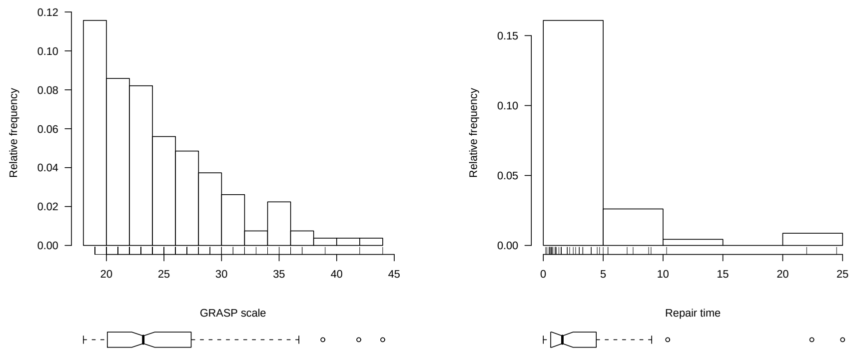

Figure 3: histograms of S1 (left) and S2 (right).

estimates. In this case, the Mahalanobis distances based on the expression given in (P6) are shown in Figure 4 for the GMIG-normal (left) and GMIG-t_ν (right) cases associated with the $\chi^2(1)$ and $F(1,5)$ distributions, respectively, using the corresponding 95th percentiles as benchmark and S1 and S2. From these figures, we note that exist a less pronounced influence in the case of the GMIG-t_ν model than in the GMIG-normal model.

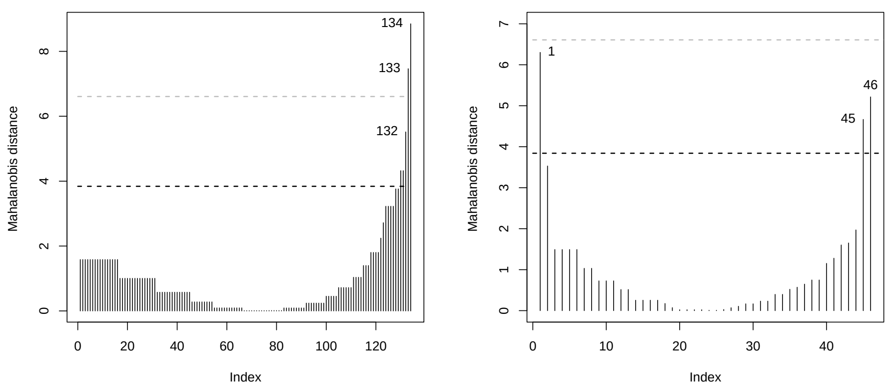

Figure 4: Mahalanobis distance for S1 (left) and S2 (right) in GMIG-normal (dotted line in bold) and GMIG-t_5 (dotted line in gray) models.

Remark 4. In Figure 4, we consider the GMIG-t_5 distribution as comparative model related to the GMIG-normal distribution because several works recommend using $\nu = 5$; see Lange et al. (1989). In addition, in the confirmatory analysis for S1 and S2 that we present in the next section, effectively, ν results to be equal to 5.

4.3 Estimation

The log-likelihood function based on a random sample $T = [T_1, \ldots, T_n]^\top$, where $T_i \sim \text{GMIG}(\mu, \lambda, p; f)$, for $\theta = [\mu, \lambda, p]^\top$ and $i = 1, 2, \ldots, n$, is expressed as

$$\ell(\theta) \propto \frac{n}{2}\log(\lambda) + \sum_{i=1}^{n} \log(f(a(t_i))) + \sum_{i=1}^{n} \log\left([1-p] + \tfrac{p}{\mu} t_i\right). \tag{4.1}$$

The score vector for θ is $\dot{\ell}_\theta = \left[\dot{\ell}_\mu, \dot{\ell}_\lambda, \dot{\ell}_p\right]^\top = [\partial\ell(\theta)/\partial\mu, \partial\ell(\theta)/\partial\lambda, \partial\ell(\theta)/\partial p]^\top$, where

$$\dot{\ell}_\mu = -\frac{\lambda}{\mu^2} \sum_{i=1}^{n} \frac{f'(a(t_i))}{f(a(t_i))}\sqrt{t_i} - \frac{p}{\mu^2} \sum_{i=1}^{n} \frac{t_i}{[1-p] + \frac{p}{\mu} t_i}, \tag{4.2}$$

$$\dot{\ell}_\lambda = \frac{n}{2\lambda} + \frac{1}{2\lambda} \sum_{i=1}^{n} \frac{f'(a(t_i))}{f(a(t_i))} a(t_i), \quad \text{and} \quad \dot{\ell}_p = \sum_{i=1}^{n} \frac{t_i - \mu}{[1-p]\mu + p t_i}. \tag{4.3}$$

The ML estimates of the parameters μ, λ and p are solutions of the equations $\dot{\ell}_\mu = 0$, $\dot{\ell}_\lambda = 0$ and $\dot{\ell}_p = 0$. However, these equations do not provide analytical solutions. Thus, it becomes necessary to use iterative methods to find the roots.

We propose the GMIG-t_ν model to describe S1 and S2 due to their kurtosis levels. Now, we must discuss how to handle the ν parameter of the t_ν model. The question is whether ν should be estimated or not. Several authors have treated this topic for the t_ν distribution and noticed that there are problems of unbounded and local maximum in the likelihood function when ν is estimated. For this reason, it is better to fix ν and assume that it is a known value. For more details, see Lange et al. (1989), Lucas (1997) and Barros et al. (2008). In order to simultaneously estimate μ, λ and p of the GMIG-t_ν distribution, we fix different values for ν and minimize the negative value of the likelihood function given in Eq. (4.1). This is carry out by the command `constrOptim()` available in the `R` software (`www.R-project.org`). This command utilizes an adaptive barrier algorithm based on the Nelder-Mead method that is a type of simplex procedure used in nonlinear optimization subject to linear inequality constraints. As starting values for the iterative method, we consider the ML estimates of μ, λ and p of the MIG distribution; see Seshadri (1993, p. 145). We chose the value of ν that maximizes the likelihood function for $\nu \in [2, 100]$, which turns to be $\nu = 5$ and produces a high level of kurtosis and parameter estimates that are qualitatively robust. The ML estimates of μ, λ and p are: $\hat{\mu} = 23.04$, $\hat{\lambda} = 617.33$ and $\hat{p} = 0.02$ for S1, and $\hat{\mu} = 3.46$, $\hat{\lambda} = 2.22$, and $\hat{p} = 0.98$ for S2.

4.4 Model checking and goodness-of-fit

To check the adequacy of GMIG models, we use the relative changes (RC), in percentage, of each parameter estimate given by: $\mathrm{RC}_{\theta_j} = |[\hat{\theta}_j - \hat{\theta}_{j(\mathrm{I})}]/\hat{\theta}_j| \times 100\%$, where $\hat{\theta}_{j(\mathrm{I})}$ denotes the ML estimate of θ_j after the set I of observations has been removed, with $\theta_1 = \mu$, $\theta_2 = \lambda$ and $\theta_3 = p$. We evaluate the effect of the potentially influential observations (see box-plot in Figure 3). We compute the RC for $\theta = [\mu, \lambda, p]^\top$ using the parameter estimates of the GMIG-normal and GMIG-t_ν models. We note that the mixing parameter estimate, $\hat{p}$, provides the most important changes. However, these changes are significantly reduced when we replace the MIG model by the GMIG-t_ν one, which shows that the GMIG-t_ν distribution indeed produces less sensitive parameter estimates to atypical data. To check the goodness-of-fit of the new model to S1 and S2, we produce quantile-quantile (QQ) plots with generated envelopes based on the MIG and GMIG-t_5 distributions. These graphical plots are shown in Figures 5 and 6. From both plots, it is possible to highlight that a heavy-tailed distribution might be more appropriate than the GMIG-normal distribution. The QQ plot with generated envelope obtained from the GMIG-t_5 distribution behaves in a better way; see Fig. 5. This is an indication on the adequacy of the GMIG-t_5 model to S1 and S2.

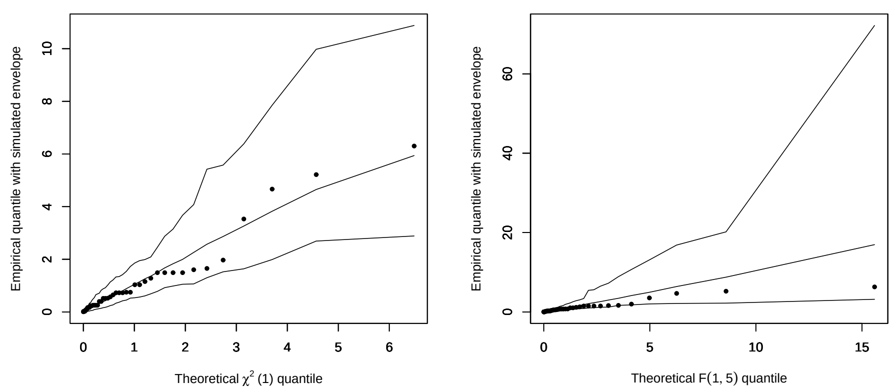

Figure 5: QQ plots with envelopes in GMIG-normal (left) and GMIG-t_5 models for S1.

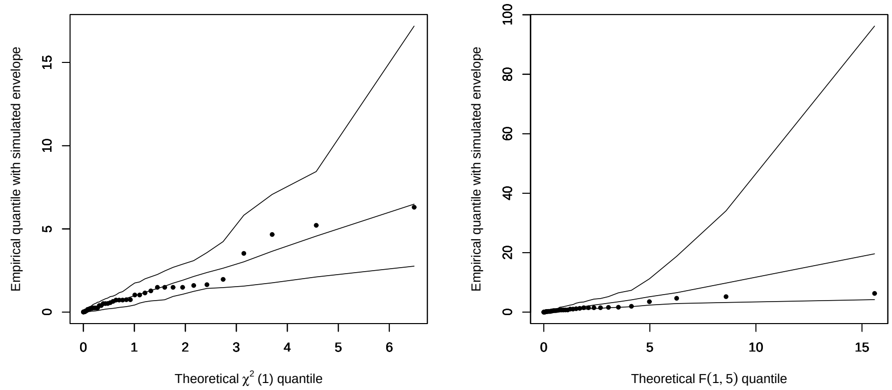

Figure 6: QQ plots with envelopes in GMIG-normal (left) and GMIG-t_5 models for S2.

5 CONCLUDING REMARKS

In this paper, we have developed a generalization of the mixture inverse Gaussian distribution utilizing symmetric distributions around zero. The use of this general class of symmetric models with different degrees of kurtosis for generating the new distribution renders a high flexibility. Moreover, depending on the specific pdf of the symmetric model and the scale parameter of GMIG distributions, the asymmetry also becomes flexible. We have obtained the pdf and mode of GMIG model and provided a graphical analysis for some particular cases showing how the kurtosis influences their behavior. We have also computed the cdf and moments and outlined some properties and transformations of this model. This probability model possesses specific characteristics no available in other classical models of this type. For instance, it contains distributions with robust parameter estimate in the presence of atypical data, bimodality and the absence of moments. Two examples with real data, one of which is novel, show the adequacy and versatility of the new model.

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