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A GENERALIZED EXPONENTIAL-TYPE DISTRIBUTION

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ABSTRACT

This article proposes a five-parameter exponential-type distribution whose exact moments and normalizing constant are obtained as inverse Mellin transforms and expressed as generalized hypergeometric functions. Several commonly used statistical distributions such as the gamma, Weibull and half-normal turn out to be particular cases of this generalized distribution. It is shown in a numerical example that the proposed distribution produces a better fit than the Weibull or inverse Gaussian distributions when used for modeling a data set consisting of maximum flood levels over a certain time period.

KEYWORDS

Exponential-type distributions, moments, generalized hypergeometric functions, inverse Mellin transform, goodness-of-fit.

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1 INTRODUCTION

A generalized exponential-type distribution is being introduced. Its density function is given by

$$f_A(x) = c\,x^{\zeta+\delta} e^{-\nu x^{\delta}} e^{-\tau x^{-\rho}}\, I_{\mathbb{R}^+}(x), \tag{1.1}$$

where $I_{\mathcal{B}}(x)$ denotes the indicator function of the set $\mathcal{B}$, $\mathbb{R}^+$ is the set of real positive numbers and c is a normalizing constant. While ζ can take on any real value, the parameters ν, δ, τ and ρ are always nonnegative. The extension proposed in this paper is more general than the generalized inverse Gaussian model introduced by Jørgensen (1982). As pointed out in Section 2, several distributions such as the gamma, Weibull, Maxwell, Rayleigh and half-normal can be obtained as particular cases.

The inverse Mellin transform technique will be used in Section 3 to determine the moments and the normalizing constant of the proposed distribution. A brief introduction to this transform and its inverse is hereby provided.

If $f(x)$ is a real piecewise smooth function that is defined and single valued almost everywhere for $x>0$ and such that $\int_0^\infty x^{k-1}|f(x)|\,\mathrm{d}x$ converges for some real value k, then $\mathcal{M}_f(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$ is the Mellin transform of $f(x)$. Whenever $f(x)$ is continuous, the corresponding inverse Mellin transform is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathcal{M}_f(s)\,\mathrm{d}s \tag{1.2}$$

which, together with $\mathcal{M}_f(s)$, constitute a transform pair. The path of integration in the complex plane is called the Bromwich path. Equation (1.2) determines $f(x)$ uniquely if the Mellin transform is an analytic function of the complex variable s for $c_1 \le \Re(s) = c \le c_2$ where c_1 and c_2 are real numbers and $\Re(s)$ denotes the real part of s. In the case of a continuous nonnegative random variable whose density function is $f(x)$, the Mellin transform is its moment of order $(s-1)$ and the inverse Mellin transform yields $f(x)$.

Letting

$$\mathcal{M}_f(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+B_js)\right\}\left\{\prod_{i=1}^{n}\Gamma(1-a_i-A_is)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\right\}\left\{\prod_{i=n+1}^{p}\Gamma(a_i+A_is)\right\}} \equiv h(s) \tag{1.3}$$

where m,n,p,q are nonnegative integers such that $0\le n\le p$, $1\le m\le q$, A_i, $i=1,\ldots,p$, B_j, $j=1,\ldots,q$, are positive numbers and a_i, $i=1,\ldots,p$, b_j, $j=1,\ldots,q$, are complex numbers such that $-A_i(b_j+\nu)\ne B_j(1-a_i+\lambda)$ for ν, $\lambda=0,1,2,\ldots$, $j=1,\ldots,m$, and $i=$

$1,\ldots,n$, the $\mathcal{H}$–function can be defined as follows in terms of the inverse Mellin transform of $\mathcal{M}_f(s)$:

$$f(x) = \mathcal{H}^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(s)\,x^{-s}\mathrm{d}s \tag{1.4}$$

where $h(s)$ is as defined in (1.3) and the Bromwich path $(c-i\infty,\, c+i\infty)$ separates the points $s=-(b_j+\nu)/B_j,\quad j=1,\ldots,m,\quad \nu=0,1,2,\ldots,$ which are the poles of $\Gamma(b_j+B_js),\quad j=1,\ldots,m,$ from the points $s=(1-a_i+\lambda)/A_i,\quad i=1,\ldots,n,\ \lambda=0,1,2,\ldots,$ which are the poles of $\Gamma(1-a_i-A_is),\quad i=1,\ldots,n$. Thus, one must have

$$\mathcal{M}ax_{\,1\le j\le m}\,\Re\{-b_j/B_j\} < c < \mathcal{M}in_{\,1\le i\le n}\,\Re\{(1-a_i)/A_i\}. \tag{1.5}$$

If, for certain parameter values, an $\mathcal{H}$–function remains positive on the entire domain, then whenever the existence conditions are satisfied, a probability density function can be generated by normalizing it. For example, the Weibull, chi-square, half–normal and F distributions can all be expressed in terms of $\mathcal{H}$–functions. For the main properties of the $\mathcal{H}$–function as well as its applicability to various disciplines, the reader is referred to Mathai and Saxena (1978) and Mathai (1993).

When $A_i = B_j = 1$ for $i=1,\ldots,p$ and $j=1,\ldots,q$, the $\mathcal{H}$–function reduces to Meijer's $\mathcal{G}$–function, that is,

$$\mathcal{G}^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right) \equiv \mathcal{H}^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(a_1,1),\ldots,(a_p,1)\\(b_1,1),\ldots,(b_q,1)\end{matrix}\right). \tag{1.6}$$

Moreover, the $\mathcal{G}$–function satisfies the following identity:

$$\mathcal{G}^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right) = \mathcal{G}^{n,m}_{q,p}\left(\frac{1}{x}\,\middle|\,\begin{matrix}1-b_1,\ldots,1-b_q\\1-a_1,\ldots,1-a_p\end{matrix}\right). \tag{1.7}$$

It is shown in Section 4 that the density functions of products and ratios of certain exponential-type distributions can be expressed in terms of the moments of the proposed five-parameter exponential-type distribution, and thus, too, have a closed form representation.

A certain data set on maximum flood levels is successfully fitted by the proposed model in Section 5. Two goodness-of-fit measures are being utilized, namely the Anderson-Darling and the Cramér-von Mises statistics. The generalized exponential-type model turns out to provide the best fit among several other models.

2 PARTICULAR CASES OF INTEREST

Many distributions are special cases of the generalized exponential-type distribution introduced in Section 1. For instance, the following distributions arise as particular cases of (1.1) wherein $\tau = 0$:

(i) The gamma distribution, denoted $\Gamma(\theta, \phi)$, with density function

$$f(x) = \frac{x^{\theta-1}\exp(-x/\phi)}{\phi^{\theta}\Gamma(\theta)} \; I_{\mathbb{R}^+}(x), \; \theta, \phi > 0,$$

is obtained by letting $\zeta = \theta - 2$, $\nu = 1/\phi$ and $\delta = 1$.

(ii) The Weibull distribution with density function

$$f(x) = \theta\phi x^{\phi-1}\exp(-\theta x^{\phi}) \; I_{\mathbb{R}^+}(x),$$

is obtained by letting $\zeta = -1$, $\nu = \theta$ and $\delta = \phi$.

(iii) The Maxwell distribution with density function

$$f(x) = \frac{4x^2\exp(-x^2/\theta^2)}{\theta^3\sqrt{\pi}} \; I_{\mathbb{R}^+}(x),$$

is obtained by letting $\zeta = 0$, $\nu = 1/\theta^2$ and $\delta = 2$.

(iv) The half-normal distribution with density function

$$f(x) = \frac{2\exp(-x^2/(2\theta^2))}{\theta\sqrt{2\pi}} \; I_{\mathbb{R}^+}(x), \; \theta > 0,$$

is obtained by letting $\zeta = -2$, $\nu = 1/2\theta^2$ and $\delta = 2$.

(v) The exponential distribution with density function

$$f(x) = \frac{\exp(-x/\kappa)}{\kappa} \; I_{\mathbb{R}^+}(x), \; \phi > 0,$$

is obtained by letting $\zeta = -1$, $\nu = 1/\kappa$ and $\delta = 1$.

(vi) The chi-square distribution with density function

$$f(x) = \frac{x^{\nu/2-1}\exp(-x/2)}{2^{\nu/2}\,\Gamma(\nu/2)}\ I_{\mathbb{R}^+}(x),\ \ \nu > 0\,,$$

is obtained by letting $\zeta = \nu/2-2$, $\nu = 1/2$ and $\delta = 1$.

(vii) The Rayleigh distribution with density function

$$f(x) = \frac{x\,\exp(-x^2/(2a^2))}{a^2}\ I_{\mathbb{R}^+}(x)\,,$$

is obtained by letting $\zeta = -1$, $\nu = 1/(2a^2)$ and $\delta = 2$.

The density function of the inverse Gaussian distribution with real parameters $\mu \in \mathbb{R}$ and $\lambda > 0$ has the following form:

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\ \exp(-\lambda(x-\mu)^2/(2x\mu^2))\ I_{\mathbb{R}^+}(x)\,. \tag{2.1}$$

We note that several other parameterizations are possible and that, in this case, $\rho \neq 0$. This density function is a particular case of the density function given in (1.1) with $\zeta = -5/2$, $\nu = \lambda/(2\mu^2)$, $\delta = 1$, $\tau = \lambda/2$ and $\rho = 1$.

Jørgensen (1982) proposed the so-called generalized inverse Gaussian distribution whose density function is given by

$$f(x) = \frac{(\phi/\theta)^{\lambda/2}}{2\,K_\lambda(\sqrt{\theta\phi})}\ x^{\lambda-1}\exp(-(\theta x^{-1}+\phi x)/2)\ I_{\mathbb{R}^+}(x)\,, \tag{2.2}$$

where $K_\lambda(\cdot)$ is a Bessel function of the second kind. The density function given in (2.2) is a special case of the five-parameter exponential distribution with $\zeta = \lambda-2$, $\nu = \phi/2$, $\delta = 1$, $\tau = \theta/2$ and $\rho = 1$.

One can also obtain special cases from the symmetrized form of (1.1), that is,

$$f_S(x) = \frac{f_A(|x|)}{2}\ I_{\mathbb{R}}(x)\,.$$

For instance, the normal distribution with density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ \exp(-x^2/(2\sigma^2))\ I_{\mathbb{R}}(x)\,,\ \sigma > 0\,,$$

is obtained by letting $\tau = 0$, $\zeta = -2$, $\nu = 1/(2\sigma^2)$ and $\delta = 2$. The lognormal(μ,σ) distribution is then obtained via the transformation $y = e^x$. Another example is the double-exponential distribution with density function

$$f(x) = \frac{\theta}{2}\,\exp(-\theta\,|x|)\,I_{\mathbb{R}}(x),\,\theta > 0\,,$$

which can be obtained as a particular case of $f_S(x)$ by letting $\tau = 0$, $\zeta = -1$, $\nu = \theta$ and $\delta = 1$. Moreover, a location parameter m can be readily incorporated in the density functions by replacing x with $x-m$.

3 THE MOMENTS OF THE GENERALIZED EXPONENTIAL-TYPE DISTRIBUTION

Consider a random variable X whose density function is given by (1.1). In order to determine its h^{th} moment, one has to evaluate the integral,

$$c\int_0^{\infty} x^{\zeta+\delta+h}\,e^{-\nu x^{\delta}}\,e^{-\tau x^{-\rho}}\,\mathrm{d}x\,. \tag{3.1}$$

To this end, we define two random variables such that the density function of their product can be expressed as an integral of the type given in (3.1). By also determining the density function of the product as an inverse Mellin transform, a closed form representation of the integral is then obtained. So, let X_1 and X_2 be independently distributed random variables whose density functions are

$$g_1(x_1) = c_1\,x_1^{\varepsilon}\,e^{-\nu {x_1}^{\delta}}\,I_{\mathbb{R}^+}(x_1)$$

and

$$h_1(x_2) = c_2\,e^{-x_2^{\rho}}\,I_{\mathbb{R}^+}(x_2)\,,$$

whose $(t-1)^{\text{th}}$ moments are respectively

$$c_1\,\frac{\nu^{-\frac{t+\varepsilon}{\delta}}\Gamma\left(\frac{t+\varepsilon}{\delta}\right)}{\delta}$$

and

$$c_2 \, \frac{\Gamma\left(\frac{t}{\rho}\right)}{\rho} \, ,$$

c_1 and c_2 being normalizing constants. Thus the $(t-1)^{\text{th}}$ moment of $U = X_1 X_2$ can be expressed as

$$k(t) = c_1 \, c_2 \, \frac{\nu^{-\frac{t+\varepsilon}{\delta}} \, \Gamma\left(\frac{t+\varepsilon}{\delta}\right) \Gamma\left(\frac{t}{\rho}\right)}{\delta \rho} \, .$$

The density function of $U = X_1 X_2$ obtained by taking the inverse Mellin transform of $k(t)$, that is,

$$\frac{1}{2\pi i} \int_C u^{-t} k(t) \, \mathrm{d}t \, ,$$

where $i = \sqrt{-1}$ and $\mathcal{C}$ is a contour of integration which encompasses the poles of $\Gamma\left(\frac{t}{\rho}\right)$ and $\Gamma\left(\frac{\varepsilon}{\delta} + \frac{t}{\delta}\right)$, can be expressed as

$$\frac{c_1 \, c_2 \, \nu^{-\frac{\varepsilon}{\delta}}}{\delta \rho} \frac{1}{2\pi i} \int_C \left(u \nu^{\frac{1}{\delta}}\right)^{-t} \Gamma\left(\frac{t+\varepsilon}{\delta}\right) \Gamma\left(\frac{t}{\rho}\right) \, \mathrm{d}t \tag{3.2}$$

$$= \frac{c_1 \, c_2 \, \nu^{-\frac{\varepsilon}{\delta}}}{\delta \rho} \mathcal{H}_{0,2}^{2,0}\left(u \nu^{\frac{1}{\delta}} \, \middle| \, \begin{matrix} \\ (0, 1/\rho), (\frac{\varepsilon}{\delta}, \frac{1}{\delta}) \end{matrix} \right), \tag{3.3}$$

where the $\mathcal{H}$-function is as defined in the Introduction.

When δ and ρ are rational numbers such that $\delta = v/d$ and $\rho = w/r$, where v, d, w and r are positive integers, one can express the integral in (3.2) as a Meijer's $\mathcal{G}$-function by letting $z = t/(v\,w)$ and making use of the Gauss-Legendre multiplication formula,

$$\Gamma(r + qs) = (2\pi)^{\frac{1-q}{2}} q^{r+qs-\frac{1}{2}} \prod_{k=0}^{q-1} \Gamma\Big(\frac{k+r}{q} + s\Big). \tag{3.4}$$

The density function of U then becomes

$$c_1\, c_2\, d\, r\, \nu^{-\frac{d\varepsilon}{v}}(2\pi)^{-\frac{rv}{2}-\frac{dw}{2}+1}(d\,w)^{\frac{d\varepsilon}{v}-\frac{1}{2}}(r\,v)^{-1/2}$$
$$\times \frac{1}{2\pi i}\int_{C'}\left(\frac{u^{vw}\nu^{wd}}{(rv)^{rv}(dw)^{dw}}\right)^{-z}\left\{\prod_{k=0}^{dw-1}\left(z+\frac{k+\frac{d\varepsilon}{v}}{dw}\right)\right\}\left\{\prod_{k=0}^{rv-1}\Gamma\left(\frac{k}{rv}+z\right)\right\}\mathrm{d}z\,, \quad (3.5)$$

that is,

$$c_1\, c_2\, d\, r\nu^{-\frac{d\varepsilon}{v}}(2\pi)^{-\frac{rv}{2}-\frac{dw}{2}+1}(d\,w)^{\frac{d\varepsilon}{v}-\frac{1}{2}}(r\,v)^{-1/2}$$
$$\times\, \mathcal{G}^{dw+rv,0}_{0,dw+rv}\left(\frac{u^{vw}\nu^{wd}}{(rv)^{rv}(dw)^{dw}}\,\middle|\, \frac{k+\frac{d\varepsilon}{v}}{dw},\ k=0,\ldots,dw-1;\, \frac{k}{rv},\ k=0,\ldots,rv-1\right). \quad (3.6)$$

Now, considering the transformation $U=X_1X_2$ and $V=X_1$, it is seen that the density function of U is also given by

$$\begin{aligned} r(u) &= \int_0^\infty \frac{1}{v}\, g_1(v)h_1(u/v)\,\mathrm{d}v \\ &= c_1\, c_2\int_0^\infty \frac{1}{v}\, v^{\varepsilon}\, e^{-\nu v^{\delta}}\, e^{-(u/v)^{\rho}}\,\mathrm{d}v\,, \end{aligned} \quad (3.7)$$

and letting $\varepsilon=\zeta+\delta+h+1$ and $u=\tau^{1/\rho}$, the integral in (3.7) is seen to coincide with that appearing in Equation (3.1). Thus the h^{th}moment of X is

$$m_X(h) = c\,\frac{\nu^{-\frac{\zeta+\delta+h+1}{\delta}}}{\delta\rho}\,\mathcal{H}^{2,0}_{0,2}\left(\tau^{1/\rho}\nu^{1/\delta}\,\middle|\, (0,1/\rho),(\tfrac{\zeta+\delta+h+1}{\delta},\tfrac{1}{\delta})\right) \quad (3.8)$$

or, in light of Equation (3.6),

$$m_X^{(R)}(h) = c^{(R)}\, d\, r\nu^{-\frac{d(\zeta+h+1)}{v}-1}(2\pi)^{-\frac{rv}{2}-\frac{dw}{2}+1}(d\,w)^{\frac{d(\zeta+h+1)}{v}+\frac{1}{2}}(r\,v)^{-1/2}$$
$$\times\, \mathcal{G}^{dw+rv,0}_{0,dw+rv}\left(\frac{\tau^{rv}\nu^{wd}}{(rv)^{rv}(dw)^{dw}}\,\middle|\, \frac{k+\frac{d(\zeta+h+1)}{v}+1}{dw},\ k=0,\ldots,dw-1;\, \frac{k}{rv},\ k=0,\ldots,rv-1\right)$$

(3.9)

when ρ and δ are rational numbers with $\delta = v/d$ and $\rho = w/r$.

Now assuming that $\delta = \rho$ and letting $w = t/\delta$, $\varepsilon = \zeta + \delta + h + 1$ and $u = \tau^{1/\rho}$ in the integral in (3.2), one obtains the h^{th} moment of X as

$$m_X^{(E)}(h) = c^{(E)}\, \frac{\nu^{-\frac{h+\zeta+1}{\delta}-1}}{\delta}\, \mathcal{G}_{0,2}^{2,0}\left(\nu\tau \,\middle|\, \begin{matrix} \\ 0,\, \frac{\zeta+h+1}{\delta}+1 \end{matrix}\right), \tag{3.10}$$

which can also be expressed in terms of $K_\lambda(\cdot)$, a Bessel function of the second kind, as

$$c^{(E)}\, \frac{2\nu^{-\frac{h+\zeta+\delta+1}{\delta}} \left(\nu^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}\right)^{\frac{h+\zeta+\delta+1}{2\delta}} K_{\frac{h+\zeta+\delta+1}{\delta}}\left(2\sqrt{\nu^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}}\right)}{\delta}. \tag{3.11}$$

Since the null moments are equal to one, the normalizing constant c is seen to be the inverse of the moment expressions $m_X(h)$, $m_X^{(R)}(h)$ and $m_X^{(E)}(h)$, wherein h is set equal to zero and c is omitted. When there are no restrictions on δ and ρ, the normalizing constant in (1.1) is

$$c = \frac{\delta\,\rho\,\nu^{\frac{\zeta+1}{\delta}+1}}{\mathcal{H}_{0,2}^{2,0}\left(\tau^{\frac{1}{\rho}}\nu^{\frac{1}{\delta}} \,\middle|\, \begin{matrix} \\ (0,1/\rho),(\frac{\zeta+1}{\delta}+1,\frac{1}{\delta}) \end{matrix}\right)}. \tag{3.12}$$

When δ and ρ are rational numbers, the normalizing constant becomes

$$c^{(R)} = \frac{\dfrac{\nu^{\frac{d(\zeta+1)}{v}+1}(2\pi)^{\frac{rv}{2}+\frac{dw}{2}-1}(dw)^{-\frac{d(\zeta+1)}{v}-\frac{1}{2}}(rv)^{1/2}}{dr}}{\mathcal{G}_{0,dw+rv}^{dw+rv,0}\left(\frac{\tau^{rv}\nu^{wd}}{(rv)^{rv}(dw)^{dw}} \,\middle|\, \begin{matrix} \\ \frac{k+\frac{d(\zeta+1)}{v}+1}{dw},\ k=0,\ldots,dw-1;\,\frac{k}{rv},\ k=0,\ldots,rv-1 \end{matrix}\right)} \tag{3.13}$$

and when $\delta = \rho$, one has

$$\begin{aligned} c^{(E)} &= \frac{\delta\nu^{\frac{\zeta++1}{\delta}+1}}{\mathcal{G}_{0,2}^{2,0}\left(\nu\tau \,\middle|\, \begin{matrix} \\ 0,\,\frac{\zeta+1}{\delta}+1 \end{matrix}\right)} && (3.14)\\ &= \frac{\delta\nu^{\frac{\zeta+\delta+1}{\delta}}}{\left(\nu^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}\right)^{\frac{\zeta+\delta+1}{2\delta}} K_{\frac{\zeta+\delta+1}{\delta}}\left(2\sqrt{\nu^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}}\right)}. && (3.15) \end{aligned}$$

4 RELATED DISTRIBUTIONAL RESULTS

It is shown in this section that the density functions of products and ratios of certain exponential-type random variables can be expressed in the integral form obtained for the moments of the proposed distribution and thus, in terms of generalized hypergeometric functions.

(i) Let $X_i \sim \Gamma(\theta_i, \phi_i)$ with p.d.f. $f_i(x_i) = x_i^{\theta_i - 1}\exp(-x_i/\phi_i)/({\phi_i}^{\theta_i}\Gamma(\theta_i))\, I_{\mathbb{R}^+}(x_i),\ \theta_i, \phi_i > 0$, $i = 1,2$, and consider the transformation of variables $z_1 = x_1 x_2$ and $z_2 = x_2$. The absolute value of the Jacobian of the inverse transformation being $\frac{1}{z_2}$, the joint p.d.f of Z_1 and Z_2 is obtained as $f_1(\frac{z_1}{z_2})f_2(z_2)\frac{1}{z_2}$. Thus, the marginal p.d.f of $Z_1 = X_1 X_2$ is

$$
\begin{aligned}
g_1(z_1) &= \int_0^\infty f_1\left(\frac{z_1}{z_2}\right) f_2(z_2)\frac{1}{z_2}\mathrm{d}z_2 \\
&= \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)\phi_1^{\theta_1}\phi_2^{\theta_2}} z_1^{\theta_1 - 1}\int_0^\infty z_2^{\theta_2 - \theta_1 - 1} e^{-\frac{1}{\phi_2}z_2} e^{-\frac{z_1}{\phi_1}z_2^{-1}}\mathrm{d}z_2 \,.
\end{aligned}
$$

This corresponds to the expression given in (3.1) with $h = 0,\ c^{-1} = \Gamma(\theta_1)\Gamma(\theta_2)\phi_1^{\theta_1}\phi_2^{\theta_2}z_1^{1-\theta_1}$, $\zeta = \theta_2 - \theta_1 - 2,\ \nu = \frac{1}{\phi_2},\ \delta = 1,\ \tau = \frac{z_1}{\phi_1}$ and $\rho = 1$. Clearly, this result also holds for chi-square and exponential distributions, which are particular cases of the gamma distribution.

(ii) Let $X_i \sim$ Weibull(θ_i, ϕ_i) with p.d.f. $f_i(x_i) = \theta_i\,\phi_i\, x_i^{\phi_i - 1}\exp(-\theta_i\, x_i^{\phi_i})\, I_{\mathbb{R}^+}(x_i),\ i = 1,2$. Again, letting $z_1 = x_1 x_2$ and $z_2 = x_2$, the absolute value of the Jacobian of the inverse transformation is $\frac{1}{z_2}$ and the marginal p.d.f of $Z_1 = X_1 X_2$ is then given by

$$
\begin{aligned}
g_2(z_1) &= \int_0^\infty f_1\left(\frac{z_1}{z_2}\right) f_2(z_2)\frac{1}{z_2}\mathrm{d}z_2 \\
&= \phi_1\,\phi_2\,\theta_1\,\theta_2\, z_1^{\phi_1 - 1}\int_0^\infty z_2^{\phi_2 - \phi_1 - 1} e^{-\theta_2 z_2^{\phi_2}} e^{-\theta_1 z_1^{\phi_1} z_2^{-\phi_1}}\mathrm{d}z_2 \ ,
\end{aligned}
$$

which is also in the form of the expression specified by (3.1) with $h = 0,\ c = \phi_1\phi_2\theta_1\theta_2{}_1^{\phi_1 - 1}$, $\zeta = -\phi_1 - 1,\ \delta = \phi_2,\ \nu = \theta_2\,, \tau = \theta_1\, z_1^{\phi_1}$, and $\rho = \phi_1$. This result holds as well for the Rayleigh and exponential distributions, which are particular cases of the Weibull distribution with $\phi = 2,\ \theta = 1/(2a^2)$ and $\phi = 1,\ \theta = 1/\kappa$, respectively.

(iii) Now, if one lets $X_i,\ i=1,2$, be distributed as in (1.1) with common parameters $(\zeta_1, \delta_1, \nu_1, \tau_1, \rho_1)$, then, on letting $z_1=\frac{x_1}{x_2}$ and $z_2=x_2$, one finds that the absolute value of the Jacobian of the inverse transformation is z_2 and that the p.d.f of $Z_1=\frac{X_1}{X_2}$ is

$$\begin{aligned} g_3(z_1) &= \int_0^\infty f_1(z_1z_2)f_2(z_2)z_2 \mathrm{d}z_2 \\ &= c_1\,c_2\,z_1{}^{\zeta_1+\delta_1}\int_0^\infty z_2^{2\zeta_1+2\delta_1+1}e^{-(\nu_1 z_1^{\delta_1}+\nu_1)z_2^{\delta_1}}e^{-(\tau_1 z_1{}^{-\rho_1}+\tau_1)z_2^{\rho_1}}\mathrm{d}z_2\,, \end{aligned}$$

which corresponds to the expression appearing in (3.1) with $h=0,\ \zeta=2\zeta_1+\delta_1+1,\ \nu=\nu_1 z_1^{\delta_1}+\nu_1,\ \delta=\delta_1,\ \tau=\tau_1 z_1^{-\rho_1}+\tau_1$, and $\rho=\rho_1$.

Thus, in light of the representations of the moments of the generalized exponential-type distribution provided in (3.8) and (3.9), the density functions $g_i(z_1),\ i=1,2,3$, respectively obtained in (i), (ii) and (iii) are seen to be expressible in terms of generalized hypergeometric functions.

5 A NUMERICAL EXAMPLE

In order to assess the goodness-of-fit of a model with respect to a given data set, one can make use of the following statistics among others:

(i) The Anderson-Darling statistic denoted by A_0^2 and given by

$$A_0^2=\Big(1+\frac{0.2}{\sqrt{n}}\Big)\Big(-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(z_i(1-z_{n+1-i})\right)\Big)$$

where $z_i=\mathrm{cdf}(x_i)$, the x_i's, $i=1,\ldots,n$, being the *ordered* observations;

(ii) The Cramér-von Mises statistic, which is

$$W_0^2=\Big(1+\frac{0.2}{\sqrt{n}}\Big)\Big(\sum_{i=1}^{n}\Big(z_i-\frac{2i-1}{2n}\Big)^2+\frac{1}{12n}\Big).$$

The smaller these statistics are, the better the fit.

The inverse Gaussian, Weibull, lognormal, generalized inverse Gaussian and the proposed generalized exponential-type distributions were fitted to the data set presented in

Table 1, which consist of maximum flood levels (in millions cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods, *cf.* Dumonceaux and Antle (1973).

Table 1: Maximum Flood Levels

.654	.613	.402	.379	.269
.740	.416	.338	.315	.449
.297	.423	.379	.3235	.418
.412	.494	.392	.484	.265

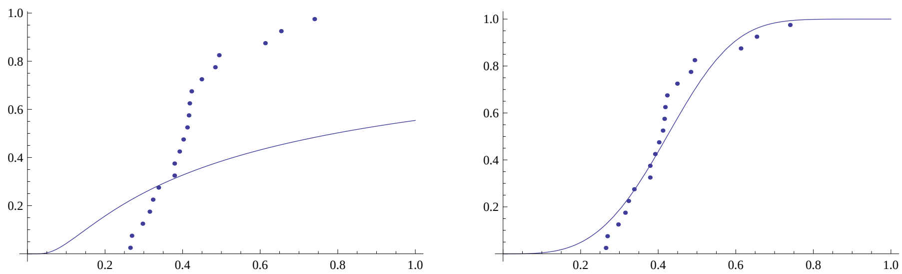

Figure 1: CDF (solid line) and empirical CDF (dots) for the fitted models. Left panel: inverse Gaussian distribution; Right panel: Weibull distribution.

We made use of the symbolic computing package *Mathematica* in which the $\mathcal{G}$–function is built in, in conjunction with the command *NMaximize* applied to the loglikelihood functions to estimate the parameters, first assuming Weibull, inverse Gaussian and lognormal models, and then making use of the generalized inverse Gaussian distribution whose density function is given in (2.2) and the general model as specified by (1.1) with $\delta = v/d = 5/2$ and $\rho = w/r = 5/2$. Plots of the cumulative distribution functions of the resulting fitted distributions are superimposed on the empirical cdf of the data in the graphs shown in Figures 1, 2 and 3. It can be seen from the results presented in Table 2 that the proposed generalized exponential-type distribution provides the best fit.

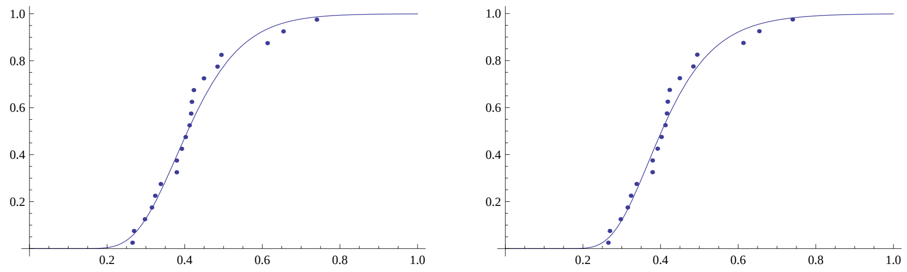

Figure 2: CDF (solid line) and empirical CDF (dots) for the fitted models. Left panel: lognormal distribution; Right panel: generalized inverse Gaussian distribution.

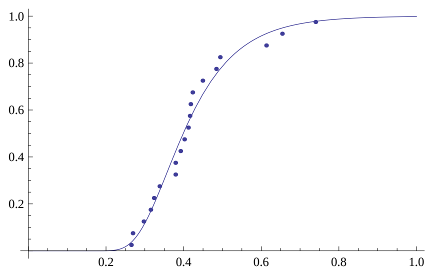

Figure 3: CDF of the generalized exponential-type distribution (solid line) and empirical CDF (dots).

Table 2: Estimates of the Parameters and Goodness-of-Fit Statistics

	$\hat{\zeta}$	$\hat{\nu}$	$\hat{\tau}$	$\hat{\phi}$	A_0^2	W_0^2
Inverse Gaussian	$-5/2$	15.745	2.819	—	7.17061	1.56141
Weibull	-1	14.450	0	3.526	0.8213	0.13998
Lognormal$(-0.898, 0.269)$	–	–	–	–	0.34701	0.05396
$\delta=\rho=1$	-16.572	0.001	5.736	—	0.28605	0.04488
$\delta=\rho=5/2$	-8.698	1.608	0.199	—	0.26880	0.04371

6 ACKNOWLEDGEMENTS

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