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ESTIMATION OF TAIL-PROBABILITY AND RELIABILITY IN EXPONENTIATED PARETO CASE

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ABSTRACT

An exponentiated Pareto distribution is defined. We then consider maximum likelihood estimator (MLE) of the threshold parameter β with known parameters α and c for the exponentiated Pareto distribution in (2.1) and then obtain the MLE of the tail-probability of the exponentiated Pareto distribution. Finally, we consider MLE of reliability in two independent exponentiated Pareto distributions.

KEYWORDS

Exponentiated Pareto distribution; Right-tail Probability; Reliability.

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1 INTRODUCTION

Let $F(x)$ be the cumulative distribution function (cdf) of a continuous random variable X. Then $G(x) = [F(x)]^{\alpha} \equiv F^{\alpha}(x)$ is also a cdf of a continuous random variable where α is a positive real number. Hence, the distribution $G(x)$ is called an exponentiated distribution of a given cdf $F(x)$ (see Gupta (2001)). Gupta (2001) considered an exponentiated exponential family, Ali *et al* (2007) introduced several exponentiated distributions, and Ali *et al* (2006) considered the exponentiated Weibull distribution.

In this paper, after defining an exponentiated Pareto distribution, the distribution is studied numerically in terms of mean, variance, and skewness. We then consider the estimation of the tail probability, and finally the reliability in two independent exponentiated Pareto distributions.

2 Exponentiated Pareto Distribution and Tail Probability

From the cdf of the Pareto distribution an exponentiated Pareto distribution (see Ali *et al* (2007)) is defined by

$$G(x) = \left[1 - \left(\frac{\beta}{x}\right)^c\right]^\alpha, \; x \geq \beta, \; \alpha > 0, \; c > 0, \tag{2.1}$$

where the Pareto distribution is as given in Johnson *et al* (1994).

From the infinite binomial expansion 1.110 in Gradshteyn and Ryzhik (1965) and formula 3.5 in Oberhettinger and Badii (1973), the moment generating function (mgf) of the exponentiated Pareto distribution can be represented by incomplete gamma function. Hence, existence of the moments of the exponentiated Pareto distribution can be guaranteed by existence of the moment generating function.

From formula 3.194(1) in Gradshteyn and Ryzhik (1965), we can obtain the kth moment of the exponentiated Pareto random variable X as follows.

$$E(X^k) = \frac{\Gamma(\alpha+1)\Gamma(1-\frac{k}{c})}{\Gamma(\alpha+1-\frac{k}{c})} \cdot \beta^k, \; \text{if } c > k = 1,2,3,\ldots, \tag{2.2}$$

where $\Gamma(x)$ is a Gamma function (see Ali *et al* (2007)). From (2.2) the mean and the variance of the exponentiated Pareto distribution are given by,

$$E(X) = \frac{\Gamma(\alpha+1)\Gamma(1-\frac{1}{c})}{\Gamma(\alpha+1-\frac{1}{c})}\beta, \; \text{and}$$

$$Var(X) = \Gamma(\alpha+1)\left[\frac{\Gamma(1-\frac{2}{c})}{\Gamma(\alpha+1-\frac{2}{c})} - \frac{\Gamma(\alpha+1)\Gamma^2(1-\frac{1}{c})}{\Gamma^2(\alpha+1-\frac{1}{c})}\right]\beta^2. \tag{2.3}$$

From (2.2) and (2.3), Table 1 below gives the means, variances, and also the skewness of the distribution (2.1) with $\beta = 1$. From Table 1, we observe the following.

Fact 1. (a) For fixed α, when the parameter c is decreasing, the density corresponding to (2.1) looks more skewed and the density has larger mean and variance. (b) For fixed c, when the parameter α is decreasing, the density corresponding to (2.1) looks similar to those of (a).

Table 1. Means, variances, and skewness of the distribution (2.1) ($\beta = 1$).

α	c	Mean	Variance	Skewness
1/4	3.5	1.13176	1.12464	17.13230
	5.0	1.08398	0.03831	7.14614
	7.5	1.05242	0.01271	5.21859
	10.0	1.03812	0.00623	4.65141
1/2	3.5	1.23654	0.22161	13.77010
	5.0	1.14964	0.06557	5.59483
	7.5	1.09290	0.02119	3.99897
	10.0	1.06738	0.01034	3.52524
2	3.5	1.63889	0.59889	10.69780
	5.0	1.38889	0.15432	4.11132
	7.5	1.23626	0.04508	2.79023
	10.0	1.16958	0.02095	2.38914
4	3.5	1.94413	0.92820	10.14740
	5.0	1.56642	0.21728	3.82686
	7.5	1.33838	0.05903	2.54848
	10.0	1.24094	0.02647	2.15694

2.1 Right-tail probability

For given α and c, we want to estimate the unknown parameter $\beta > 0$ for the cdf (2.1). Let $X_1, X_2, \ldots, X_m$ be a random sample from the cdf (2.1). Then the maximum likelihood estimator (MLE) $\hat{\beta}$ of β is given by

$$\hat{\beta} = X_{(1)},$$

where$X_{(1)}$ is the first order statistic of the random sample, and the cdf of $X_{(1)}$ is given by

$$F_{X_{(1)}}(x) = 1 - (1 - (1 - (\beta/x)^c)^\alpha)^m, \ \ x \geq \beta. \tag{2.4}$$

The kth moment of $\hat{\beta} = X_{(1)}$ can be obtained as

$$E(X_{(1)}^k) = m\alpha\Gamma(1-k/c)\sum_{j=0}^{m-1}(-1)^j \begin{pmatrix} m-1 \\ j \end{pmatrix} \frac{\Gamma(\alpha(j+1))}{\Gamma(\alpha(j+1)+1-k/c)}\beta^k. \tag{2.5}$$

Define $C(m,\alpha;-k/c) \equiv m\alpha\Gamma(1-k/c)\sum_{j=0}^{m-1}(-1)^j \begin{pmatrix} m-1 \\ j \end{pmatrix} \frac{\Gamma(\alpha(j+1))}{\Gamma(\alpha(j+1)+1-k/c)}$. Then

$$E(X_{(1)}^k) = C(m,\alpha;-k/c)\beta^k, \ \text{ if } c > k = 1,2,\ldots. \tag{2.6}$$

For $c > k$ where $k = 1,2$, we have,

$$E(X_{(1)}) = C(m,\alpha;-1/c)\beta \ \text{ and } \mathrm{Var}(X_{(1)}) = [C(m,\alpha;-2/c) - C^2(m,\alpha;-1/c)]\beta^2. \tag{2.7}$$

From (2.7), we define an unbiased estimator $\tilde{\beta}$ of β as follows.

$$\tilde{\beta} = \hat{\beta}/C(m,\alpha;-1/c).$$

The variance of the unbiased estimator $\tilde{\beta}$ is then

$$\mathrm{Var}(\tilde{\beta}) = [C(m,\alpha;-2/c)/C^2(m,\alpha;-1/c) - 1]\beta^2. \tag{2.8}$$

From (2.7) and (2.8), Table 2 provides the numerical MSE's of the MLE $\hat{\beta}$ and the unbiased estimator $\tilde{\beta}$ and we observe the following.

Fact 2. An unbiased estimator $\tilde{\beta}$ is more efficient in the sense of MSE than the MLE $\hat{\beta}$ when the cdf (2.1) has $c = 3$ and α = 1/4, 1/2, 2, 4.

From the cdf (2.1), the right-tail probability of exponentiated Pareto random variable is given by

$$R(t;\beta) = 1 - [1 - (\frac{\beta}{t})^c]^\alpha, \ \ t \geq \beta, \ \alpha > 0, \ c > 0.$$

When parameters c and α are given, the $R(t;\beta)$ is a monotone function of β. Hence, by the results of McCool (1991), inference on $R(t;\beta)$ is equivalent to inference on β. Hence from Fact 2, we have the following.

Fact 3. An estimator $\tilde{R}(t;\tilde{\beta}) = 1-[1-(\frac{\tilde{\beta}}{t})^c]^\alpha$ is more efficient than the unbiased estimator $\hat{R}(t;\tilde{\beta}) = 1-[1-(\frac{\hat{\beta}}{t})^c]^\alpha$.

Table 2. Mean squared errors of the two estimators $\hat{\beta}$ and $\tilde{\beta}$ for $c = 3$.

m	α	$\hat{\beta}$	$\tilde{\beta}$
10	1/4	0.2807E-5	0.2691E-5
	1/2	0.1346E-3	0.1054E-3
	2	0.2043E-1	0.4759E-2
	4	0.9681E-1	0.9721E-2
20	1/4	0.3645E-7	0.3547E-9
	1/2	‘0.1116E-4	0.8988E-5
	2	0.8846E-2	0.1937E-2
	4	0.5343E-1	0.5054E-2
30	1/4	0.0000E-10	0.0000E-10
	1/2	0.2429E-5	0.1967E-5
	2	0.5213E-2	0.1198E-2
	4	0.3890E-1	0.3530E-2

3 Reliability

In this Section, we consider estimation of the reliability for two independent exponentiated Pareto distributions. Let us first consider the following Lemma 1.

Lemma 1: *Let X and Y be continuous independent random variables each with cdf F(x/θ_1), $x \geq \theta_1 > 0$, and F(x/θ_2), $x \geq \theta_2 > 0$, respectively, where θ_1 and θ_2 are scale parameters. Then, the reliability function $R(\rho) = P(Y < X)$ is a monotone function of* ρ, *where* $\rho \equiv \theta_1/\theta_2$.

Proof: (i) For $\theta_1 > \theta_2$,

$$\begin{aligned} R(\rho) &= P(Y < X) = \int_{\theta_1}^{\infty} F(x/\theta_2) dF(x/\theta_1) = \int_{\theta_1}^{\infty} F((\theta_1/\theta_2)\cdot x/\theta_1) dF(x/\theta_1) \\ &= \int_{1}^{\infty} F(\rho\cdot y) dF(y), \text{ where } y = x/\theta_1. \end{aligned}$$

(ii) For $\theta_1 < \theta_2$, in a similar manner it can be shown that

$$R(\rho) = P(Y < X) = 1 - \int_{1}^{\infty} F(y/\rho) dF(y).$$

From the results in (i) and (ii), since $\frac{d}{dR}R(\rho) > 0$, the reliability function $R(\rho) = P(Y < X)$ is a monotone function of ρ.

Now, let X and Y be independent exponentiated Pareto random variables each having unknown parameters β_1 and β_2, and corresponding cdf's $F_X(x)$ and $G_Y(y)$, respectively. Then from formula 3.197(3) in Gradshteyn and Ryzhik (1965), the reliability $P(Y < X)$ can be obtained as

$$\begin{aligned} P(Y < X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} dG_Y(y) dF_X(x) \\ &= \int_{-\infty}^{\infty} G_Y(x) dF_X(x) \\ &= \begin{cases} 1 - F(-\alpha, 1; \alpha+1, \rho^c), & \text{if } 0 < \rho < 1 \\ F(-\alpha, 1; \alpha+1; 1/\rho^c), & \text{if } \rho > 1, \end{cases} \end{aligned}$$

where $\rho \equiv \beta_1/\beta_2$ and $F(a,b;c;x)$ is the generalized hypergeometric function. The reliability $P(Y < X)$ is expressed in terms of the generalized hypergeometric function in order to put the integral expression of the reliability in simple mathematical form. If $\rho = 1$, it is obvious that the reliability is 1/2.

Hence, from Lemma 1 the reliability $P(Y < X)$ above for the exponentiated Pareto distributions is a monotone function of ρ, and, therefore, an inference on $P(Y < X)$ is equivalent to inference on ρ (see McCool (1991)). It is, therefore, sufficient for us to estimate $\rho \equiv \beta_1/\beta_2$ instead of $P(Y < X)$.

So, we first consider the following moments to estimate $\rho \equiv \beta_1/\beta_2$. Assume $Y_1, Y_2, \ldots, Y_n$ be independent random variables each having the cdf (2.1) where α and c are known positive numbers. Then the maximum likelihood estimate of β is $\hat{\beta} = Y_{(1)}$, the corresponding first order statistic of the random sample. Replacing m by n in (2.4) and using definition of $C(m,\alpha;-k/c)$ in Section 2, the kth moment of $1/Y_{(1)}^k$ is obtained as follows.

$$E(1/Y_{(1)}^k) = C(n,\alpha;k/c)/\beta^k, \tag{3.1}$$

and hence

$$E(1/[C(n,\alpha;k/c)Y_{(1)}^k]) = 1/\beta^k,\ k = 1,2,\ldots. \tag{3.2}$$

Assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent samples each with the cdf (2.1) having parameters β_1 and β_2, respectively. Then the MLE $\hat{\rho}$ of ρ is

$$\hat{\rho} = \hat{\beta_1}/\hat{\beta_2}.$$

From the moments (3.1) and (3.2), we obtain the kth moment of $\hat{\rho}$ as follows.

$$E(\hat{\rho}^k) = C(m,\alpha;-k/c)\cdot C(n,\alpha;k/c)\cdot\rho^k,\ \ k = 1,2,\ldots,. \tag{3.3}$$

From (3.3), mean and variance of $\hat{\rho}$ are given by

$$\begin{aligned} E(\hat{\rho}) &= C(m,\alpha;-1/c)\cdot C(n,\alpha;1/c)\cdot\rho \text{ and,} \qquad (3.4)\\ \mathrm{Var}(\hat{\rho}) &= [C(m,\alpha;-2/c)\cdot C(n,\alpha;2/c) - C^2(m,\alpha;-1/c)\cdot C^2(n,\alpha;1/c)]\cdot\rho^2. \end{aligned}$$

Hence, we define an unbiased estimator $\tilde{\rho}$ of ρ as follows.

$$\tilde{\rho} = \frac{1}{C(m,\alpha;-1/c)\cdot C(n,\alpha;1/c)}\cdot\frac{X_{(1)}}{Y_{(1)}}. \tag{3.5}$$

From (3.5), variance of $\tilde{\rho}$ is given by

$$\mathrm{Var}(\tilde{\rho}) = [C(m,\alpha;-2/c)\cdot C(n,\alpha;2/c)/\{C^2(m,\alpha;-1/c)\cdot C^2(n,\alpha;1/c)\} - 1]\cdot\rho^2. \tag{3.6}$$

From (3.4) and (3.6), Table 3 below provides the mean squared errors of two estimators $\tilde{\rho}$ and $\hat{\rho}$ and hence we observe the following.

Fact 4. The estimator $\tilde{\rho}$ is more efficient than the estimator $\hat{\rho}$ in the sense of MSE when the cdf (2.1) has $c = 3$, $\alpha = 1/4, 1/2, 2, 4$ and $m(n) = 10, 20$.

Table 3. Mean squared errors of the two reliability estimators $\hat{\rho}$ and $\tilde{\rho}$ for $c = 3$.

m	n	α	$\hat{\rho}$	$\tilde{\rho}$
10	10	1/4	0.5245E-5	0.5244E-5
		1/2	0.2015E-3	0.2014E-3
		2	0.9041E-3	0.8941E-2
		4	0.1899E-1	0.1857E-2
10	20	1/4	0.2780E-5	0.2684E-5
		1/2	0.1297E-3	0.1142E-3
		2	0.8737E-e	0.6560E-2
		4	0.2035E-1	0.1454E-1
20	10	1/4	0.2657E-5	0.2566E-5
		1/2	0.1181E-3	0.1051E-3
		2	0.6788E-2	0.6107E-2
		4	0.1471E-1	0.1385E-1
20	20	1/4	0.1655E-5	0.1548E-5
		1/2	0.1190E-4	0.1074E-4
		2	0.3706E-2	0.3689E-2
		4	0.1003E-1	0.9909E-2

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