Pak. J. Statist.
2010, Vol.26(1), 281-292

THE MOST POWERFUL TEST IN SOME CYCLIC PREDATOR-PREY POPULATIONS

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ABSTRACT

In this article, we consider the Lotka-Volterra ordinary differential equations system (ODEs) where the actual population sizes are viewed as random perturbations of the solutions to the ODEs. Further, we assume that the perturbations follow correlated Ornstein-Uhlenbeck processes. For this model, we establish the most powerful test for testing change in the ODEs parameters.

KEYWORDS

Ornstein-Uhlenbeck processes ; Gaussian processes ; uniformly most powerful unbiased test ; Conversion efficiency ; Lotka-Volterra ODEs

2000 Mathematics Subject Classification : 62M10 - 62F03 - 92D25

1 INTRODUCTION

In this paper, we are interested in predator-prey system as described by the Lotka-Volterra system of ordinary differential equations (Lotka, 1926 ; Volterra, 1931),

$$\frac{dx(t)}{dt} = (\eta - \beta y(t))x(t)\,, \quad \frac{dy(t)}{dt} = (\gamma x(t) - \delta)y(t)\,, \quad (x(0), y(0)) = (x_0, y_0) \text{ fixed}, \quad (1.1)$$

where $\eta > 0, \beta > 0, \gamma > 0, \delta > 0, x_0 > 0, y_0 > 0$. We assume that the initial value (x_0, y_0) is different from an equilibrium point. Accordingly, the trajectory of the ordinary differential equations system (ODEs) (1.1) is assumed to be nontrivial. It is well known that the trajectory $(x(t), y(t))$ lies on a closed curve and that it is a periodic function, whose period is a function of $(\gamma, \beta, \delta, \eta, x_0, y_0)$.

The $x(t)$ and $y(t)$ are interpreted as the population sizes of the prey and the predator, respectively, at time t. The parameter η is the birth rate of the prey when the predator is absent, parameter δ is the death rate of the predator when the prey is absent, and β and γ are the interaction parameters.

Following Froda and Nkurunziza (2007), we assume that, we have N pairs of observations $(X_i, Y_i)_{i=1,2,\ldots,N}$ recorded at discrete times t_i, where $0 < t_i < t_{i+1}$; X_i and Y_i, represent respectively the sizes of the prey and the predator observed at time t_i, $i = 1, 2, \ldots, N$. Further, we consider a measurement error model where the logarithm of population sizes are viewed as logarithm of the solution of ODE (1.1) plus Gaussian error process. Such model has the desired feature of preserving the periodic behaviour as well as the observed irregularities of the trajectory ODEs. Indeed, the trajectory of many animal population sizes has oscillatory behaviour (see e.g. Kendall et al., 1999, Ginzburg and Taneyhill, 1994). However, the trajectory of the observed predator-prey population sizes are not as smooth and regular as the trajectory of the ODEs (1.1).

In this work, we assume that each component of the error is Ornstein-Uhlenbeck process. The choice of this error structure is motivated by the fact that Ornstein-Uhlenbeck process is the continuous version of a first-order autoregressive model AR(1) in discrete times (see for example, Nkurunziza, 2008-2009). Further, Royama (1992), Berryman (1995), and Kendall et al. (1999) argued that the statistical model which is commonly used in cyclic populations is the linear autoregressive (AR) model, with an order less than or equal to 2. Here the periodicity is captured by the solution of the ODEs (1.1) and therefore, we reduce the order by considering an AR(1) model, for the sake of simplicity.

The point estimation problem of the parameters $\eta, \beta, \gamma, \delta$ is considered by Froda and Colavita (2005) as well as in Froda and Nkurunziza (2007). Further, Nkurunziza (2008, 2009) proposed the uniformly most powerful test (UMP), and the likelihood ratio test for the interaction parameters β, γ. In this paper, we consider the following testing problem

$$H_0 : (\gamma, \beta, \delta, \eta) = (\gamma_0, \beta_0, \delta_0, \eta_0) \quad \text{versus} \quad H_1 : (\gamma, \beta, \delta, \eta) = (\gamma_1, \beta_1, \delta_1, \eta_1) \tag{1.2}$$

where $(\gamma_0, \beta_0, \delta_0, \eta_0)$ and $(\gamma_1, \beta_1, \delta_1, \eta_1)$ are known. This test is useful in testing for example if after a certain period of time, there is a change in parameters.

The rest of this paper is organized as follows. Section 2 presents the statistical model and notations. In Section 3, we present the solution to the testing problem (1.2) that is the most powerful test. Finally, Section 4 gives some concluding remarks.

2 Preliminaries and Statistical Model

In this section, we present the statistical model, notations, and the assumptions made in order to simplify some computations. Let $(x(t), y(t))$ be the solution of the ODEs (1.1), and as mentioned above, let $(X_i, Y_i)_{i=1,2,\dots,N}$ be N pairs of observations that are collected at discrete times $0 < t_1 < t_2 < \cdots < t_N$; with $X_i \equiv X(t_i)$, $\quad Y_i \equiv Y(t_i)$.

We assume that $\{(X(t), Y(t)), 0 \le t \le T\}$ satisfies

$$\log X_t = \log x(t) + e_t^X, \quad \log Y_t = \log y(t) + e_t^Y, \tag{2.1}$$

where each noise component of $\left\{\left(e_t^X, e_t^Y\right), 0 \le t \le T\right\}$ is Ornstein-Uhlenbeck process (Kutoyants, 2004, p. 51), with a particular dependance structure as described in Froda and Nkurunziza (2007), and in Nkurunziza (2008, 2009). Let

$$de_t^X = -c\, e_t^X dt + \tau dW_t^X, \quad de_t^Y = -c\, e_t^Y dt + \tau dW_t^Y, \quad c, \tau > 0, \tag{2.2}$$

where $\left\{W_t^X, t \geqslant 0\right\}$ and $\left\{W_t^Y, t \geqslant 0\right\}$ are Wiener processes (Kutoyants, 2004, p. 18) satisfying Assumption $(\mathcal{C}_1)$ as given in Froda and Nkurunziza (2007), and in Nkurunziza (2008, 2009).

Assumption $(\mathcal{C}_1)$ *The Wiener processes* $\left\{W_t^X, t \ge 0\right\}$ *and* $\left\{W_t^Y, t \ge 0\right\}$ *are jointly Gaussian and, for all* $i, j = 1, 2, 3, \dots,$

$$Cov(W_{t_i}^X, W_{t_j}^Y) = \rho \min(t_i, t_j), \text{ where } |\rho| < 1.$$

The parameter ρ captures the correlation between the initial random variables e_0^X and e_0^Y as well as the correlation between the two Wiener processes $\{W_t^X, t \geqslant 0\}$ and $\{W_t^Y, t \geqslant 0\}$. The initial random variables e_0^X and e_0^Y are assumed to satisfy the following assumption $(\mathcal{C}_2)$.

Assumption $(\mathcal{C}_2)$ *The random vector* $\left(e_0^X, e_0^Y\right)'$ *is independent of* $\left\{(W_t^X, W_t^Y), t \geqslant 0\right\}$. *Also,*

$$\left(e_0^X, e_0^Y\right) \sim \mathcal{N}_2(\mathbf{0}, \mathbf{\Sigma}), \quad \text{where} \quad \mathbf{\Sigma} = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{with} \quad \sigma^2 = \tau^2/2c.$$

Assumption $(\mathcal{C}_2)$ guarantees the stationarity of the noise process. For simplicity, the noise parameters, ρ, σ, c are assumed to be known or to be estimated from the previous investigations. For more details about the estimation method of these parameters, the reader is referred to Froda and Nkurunziza (2007).

3 The most powerful test

We consider the testing problem in (1.2). In solving this testing problem, the following result plays a central role.

Proposition 3.1. *Let* $(x(t;\gamma,\beta,\delta,\eta), y(t;\gamma,\beta,\delta,\eta))$ *be the nontrivial trajectory of the ODEs given in* (1.1)*. Then,* $(x(t;\gamma,\beta,\delta,\eta), y(t;\gamma,\beta,\delta,\eta))$ *is an injective function with respect to the parameters* $(\gamma,\beta,\delta,\eta)$.

Similar result is proved in Nkurunziza (2005, Proposition 1.1). However, for a smooth reading of the paper, the proof is outlined in the Appendix. It should be noticed that, under the conditions of the previous proposition, one can conclude that the trajectory is a bijective function with respect to the parameters in every compact of $(0, +\infty)^{\oplus 4}$. This follows from Proposition 1, and by using the well known Theorem of continuous dependence in parameters of the trajectory of the ODEs (1.1).

Let

$$\Gamma = (\gamma,\beta,\delta,\eta)' \text{ et } \Gamma_j = (\gamma_j,\beta_j,\delta_j,\eta_j)' \text{ pour } j = 0,1,$$

where the components of Γ and Γ_j, $j = 0,1$ are strictly positive.

Using Proposition 1, the testing problem in (1.2) is equivalent to the testing problem

$$H_0 : (\log(x(t;\Gamma)), \log(y(t;\Gamma))) = (\log(x(t;\Gamma_0)), \log(y(t;\Gamma_0)))$$

versus

$$H_1 : (\log(x(t;\Gamma)), \log(y(t;\Gamma))) = (\log(x(t;\Gamma_1)), \log(y(t;\Gamma_1)))$$

for all $t > 0$; under fixed initial value.

To simplify the notation, let

$$(x(t;\gamma_1,\beta_1,\delta_1,\eta_1), y(t;\gamma_1,\beta_1,\delta_1,\eta_1)) = (x_t, y_t),$$

and

$$(x(t;\eta_0,\gamma_0,\beta_0,\delta_0,\eta_0),y(t;\gamma_0,\beta_0,\delta_0,\eta_0)) \quad = \quad (u_t,v_t)$$

for all $t \geqslant 0$. Recall that, here the nuisance parameters σ, ρ and ϕ are assumed to be known, and let

$$D^2 = \sigma^2\left\{\left(\log\left(\frac{u_1}{x_1}\right)\right)^2 + \left(\log\left(\frac{v_1}{y_1}\right)\right)^2\right\}$$
$$+\frac{\sigma^2}{(1-\phi^2)}\sum_{i=2}^{N}\left\{\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)^2+\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right)^2\right\},$$

and

$$T = \frac{1}{D}\left\{\log\left(\frac{X_1}{u_1}\right)\log\left(\frac{u_1}{x_1}\right)+\log\left(\frac{Y_1}{v_1}\right)\log\left(\frac{v_1}{y_1}\right)\right\} \tag{3.1}$$
$$+\frac{1}{(1-\phi^2)D}\sum_{i=2}^{N}\left(\log\left(\frac{X_i}{u_i}\right)-\phi\log\left(\frac{X_{i-1}}{u_{i-1}}\right)\right)\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)$$
$$+\frac{1}{(1-\phi^2)D}\sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{v_i}\right)-\phi\log\left(\frac{Y_{i-1}}{v_{i-1}}\right)\right)\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right).$$

Further, let z_α be a real number that satisfies the equation

$$\frac{1}{\sqrt{2\pi}}\int_{\alpha}^{+\infty}e^{-\frac{1}{2}t^2}dt = \alpha. \tag{3.2}$$

The following proposition gives the solution to our testing problem.

Proposition 3.2. *Consider testing* (1.2) *at level* $0<\alpha<1$. *Under Assumptions* $(\mathcal{C}_1)$ *and* $(\mathcal{C}_2)$, *the most powerful* α*-level test exists and is given by*

$$\Psi = \begin{cases} 1 \text{ if } T < z_{1-\alpha} \\ 0 \text{ if } T > z_{1-\alpha}, \end{cases} \tag{3.3}$$

where z_α *is the real number defined in relation* (3.2).

Proof Let P_0 and P_1 be the probability measure generated $\{(\log(X_t),\log(Y_t)),$ $t=1,2,\ldots,N\}$ ($\{$ or $\left(e_t^X,e_t^Y\right),t=1,2,\ldots,N\}$), under hypotheses H_0 and H_1 respectively.

Under Assumptions $(\mathcal{C}_1)-(\mathcal{C}_2)$, P_0 and P_1 are dominated by Lebesgue measures. Let f_0 and f_1 be the density functions with respect to P_0 and P_1 respectively.

From Neyman-Pearson Lemma (see Lehmann, 1986, p. 74), the most powerful α-level test, for testing H_0 versus H_1, exists and is given by

$$\Psi(\boldsymbol{z}) = \begin{cases} 1 \text{ if } f_1(\boldsymbol{z}) > k f_0(\boldsymbol{z}) \\ 0 \text{ if } f_1(\boldsymbol{z}) < k f_0(\boldsymbol{z}), \end{cases}$$

where k is a positive real number that satisfies

$$\mathrm{E}_0\left(\Psi(\boldsymbol{Z})\right) = \alpha,$$

where E_0 refers to the expected value taken with respect to the probability measure P_0. In the previous notation, $\boldsymbol{Z}$ corresponds to $(\log(X_t), \log(Y_t)), t = 1, 2, \ldots, N)$. The test is established by finding k.

Let

$$\begin{aligned}
\zeta_1\left(\Gamma^{(0)}\right) &= \left(\log(X_1)-\log(u_1)\right)^2 + \left(\log(Y_1)-\log(v_1)\right)^2, \\
\zeta_2\left(\Gamma^{(0)}\right) &= \sum_{i=2}^{N}\left(\log\left(\frac{X_i}{u_i}\right) - \phi\log\left(\frac{X_{i-1}}{u_{i-1}}\right)\right)^2 + \sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{v_i}\right) - \phi\log\left(\frac{Y_{i-1}}{v_{i-1}}\right)\right)^2, \\
\zeta_1\left(\Gamma^{(1)}\right) &= \left(\log(X_1)-\log(x_1)\right)^2 + \left(\log(Y_1)-\log(y_1)\right)^2, \\
\zeta_2\left(\Gamma^{(1)}\right) &= \sum_{i=2}^{N}\left(\log\left(\frac{X_i}{x_i}\right) - \phi\log\left(\frac{X_{i-1}}{x_{i-1}}\right)\right)^2 + \sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{y_i}\right) - \phi\log\left(\frac{Y_{i-1}}{y_{i-1}}\right)\right)^2.
\end{aligned}$$

We have

$$f_0(\boldsymbol{z}) = \frac{1}{2\pi\sigma^2\left(2\pi\sigma^2(1-\phi^2)\right)^{N-1}} \times \exp\left\{-\frac{1}{2\sigma^2}\zeta_1\left(\Gamma^{(0)}\right) - \frac{1}{2\sigma^2(1-\phi^2)}\zeta_2\left(\Gamma^{(0)}\right)\right\},$$

and

$$f_1(\boldsymbol{z}) = \frac{1}{2\pi\sigma^2\left(2\pi\sigma^2(1-\phi^2)\right)^{N-1}} \exp\left\{-\frac{1}{2\sigma^2}\zeta_1\left(\Gamma^{(1)}\right) - \frac{1}{2\sigma^2(1-\phi^2)}\zeta_2\left(\Gamma^{(1)}\right)\right\}.$$

Using the fact that

$$\begin{aligned}
\left(\log(X_1)-\log(x_1)\right)^2 &= \left(\log(X_1)-\log(u_1)+\log(u_1)-\log(x_1)\right)^2 \\
\left(\log(Y_1)-\log(y_1)\right)^2 &= \left(\log(Y_1)-\log(v_1)+\log(v_1)-\log(y_1)\right)^2,
\end{aligned}$$

we get

$$\begin{aligned}\zeta_1\left(\Gamma^{(1)}\right) &= \left\{\left(\log\left(\frac{X_1}{u_1}\right)\right)^2+\left(\log\left(\frac{u_1}{x_1}\right)\right)^2+2\log\left(\frac{X_1}{u_1}\right)\log\left(\frac{u_1}{x_1}\right)\right\}\\ &\quad+\left\{\left(\log\left(\frac{Y_1}{v_1}\right)\right)^2+\left(\log\left(\frac{v_1}{y_1}\right)\right)^2+2\log\left(\frac{Y_1}{v_1}\right)\log\left(\frac{v_1}{y_1}\right)\right\}.\end{aligned}$$

Then,

$$\begin{aligned}\zeta_1\left(\Gamma^{(1)}\right) &= \zeta_1\left(\Gamma^{(0)}\right)+\left(\log\left(\frac{u_1}{x_1}\right)\right)^2+2\log\left(\frac{X_1}{u_1}\right)\log\left(\frac{u_1}{x_1}\right)\\ &\quad+\left(\log\left(\frac{v_1}{y_1}\right)\right)^2+2\log\left(\frac{Y_1}{v_1}\right)\log\left(\frac{v_1}{y_1}\right).\end{aligned}$$

Similarly, we have,

$$\begin{aligned}\zeta_2\left(\Gamma^{(1)}\right) &= \sum_{i=2}^{N}\left(\log\left(\frac{X_i}{u_i}\right)-\phi\log\left(\frac{X_{i-1}}{u_{i-1}}\right)\right)^2\\ &\quad+\sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{v_i}\right)-\phi\log\left(\frac{Y_{i-1}}{v_{i-1}}\right)\right)^2\\ &\quad+2\sum_{i=2}^{N}\left(\log\left(\frac{X_i}{u_i}\right)-\phi\log\left(\frac{X_{i-1}}{u_{i-1}}\right)\right)\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)\\ &\quad+2\sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{v_i}\right)-\phi\log\left(\frac{Y_{i-1}}{v_{i-1}}\right)\right)\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right).\\ &\quad+\sum_{i=2}^{N}\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)^2\\ &\quad+\sum_{i=2}^{N}\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right)^2.\end{aligned}$$

Hence,

$$\begin{aligned}\zeta_2\left(\Gamma^{(1)}\right) &= \zeta_2\left(\Gamma^{(0)}\right)+\sum_{i=2}^{N}\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)^2\\ &\quad+\sum_{i=2}^{N}\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right)^2\\ &\quad+2\sum_{i=2}^{N}\left(\log\left(\frac{X_i}{u_i}\right)-\phi\log\left(\frac{X_{i-1}}{u_{i-1}}\right)\right)\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)\\ &\quad+2\sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{v_i}\right)-\phi\log\left(\frac{Y_{i-1}}{v_{i-1}}\right)\right)\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right).\end{aligned}$$

Therefore,

$$\frac{f_1(\boldsymbol{Z})}{f_0(\boldsymbol{Z})} = e^{-\frac{S}{\sigma^2}} \times e^{-\frac{1}{2\sigma^2}\left\{\left(\log\left(\frac{u_1}{x_1}\right)\right)^2+\left(\log\left(\frac{v_1}{y_1}\right)\right)^2\right\}} \times e^{-\frac{1}{2\sigma^2(1-\phi^2)}\sum_{i=2}^{N}\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)^2} \times e^{-\frac{1}{2\sigma^2(1-\phi^2)}\sum_{i=2}^{N}\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right)^2},$$

where

$$\begin{aligned} S &= \log\left(\frac{X_1}{u_1}\right)\log\left(\frac{u_1}{x_1}\right)+\log\left(\frac{Y_1}{v_1}\right)\log\left(\frac{v_1}{y_1}\right) \\ &+\frac{1}{(1-\phi^2)}\sum_{i=2}^{N}\left(\log\left(\frac{X_i}{u_i}\right)-\phi\log\left(\frac{X_{i-1}}{u_{i-1}}\right)\right)\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right) \\ &+\frac{1}{(1-\phi^2)}\sum_{i=2}^{N}\left(\log\left(\frac{Y_i}{v_i}\right)-\phi\log\left(\frac{Y_{i-1}}{v_{i-1}}\right)\right)\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right). \end{aligned}$$

Thus, $f_1(\boldsymbol{Z})/f_0(\boldsymbol{Z})$ is large if and only if S is small. Accordingly, we reject the null hypothesis, at α-level, if and only if

$$S < c_\alpha$$

where c_α is such that

$$P_0\{S < c_\alpha\} = \alpha.$$

Further, under the null hypothesis,

$$S \sim \mathcal{N}(0, \mathrm{Var}(S))$$

where

$$\begin{aligned} \mathrm{Var}(S) &= \sigma^2\left\{\left(\log\left(\frac{u_1}{x_1}\right)\right)^2+\left(\log\left(\frac{v_1}{y_1}\right)\right)^2\right\} \\ &+\frac{\sigma^2}{(1-\phi^2)}\sum_{i=2}^{N}\left\{\left(\log\left(\frac{u_i}{x_i}\right)-\phi\log\left(\frac{u_{i-1}}{x_{i-1}}\right)\right)^2+\left(\log\left(\frac{v_i}{y_i}\right)-\phi\log\left(\frac{v_{i-1}}{y_{i-1}}\right)\right)^2\right\}. \end{aligned} \tag{3.4}$$

Let

$$D^2 = \mathrm{Var}(S) \text{ and } T = \frac{S}{D}.$$

One can verify that, under the null hypothesis,

$$T \sim \mathcal{N}(0,1).$$

Then,

$$P_0\{S < c_\alpha\} = \alpha$$

if and only if

$$P_0\left\{T < \frac{c_\alpha}{D}\right\} = 1 - P_0\left\{T \geqslant \frac{c_\alpha}{D}\right\} = 1 - \frac{1}{\sqrt{2\pi}}\int_{\frac{c_\alpha}{D}}^{+\infty} e^{-\frac{1}{2}t^2}dt = \alpha,$$

with

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{c_\alpha}{D}}^{+\infty} e^{-\frac{1}{2}t^2}dt = 1-\alpha.$$

That gives

$$k = \frac{c_\alpha}{D} = z_{1-\alpha},$$

which completes the proof. ■

Remark : In establishing the test Ψ, given by Proposition 2, we assumed without loss of generality that $\rho = 0$ in order to simplify computations. In fact, suppose that $\rho \neq 0$ but known. One can consider the following transformation of the original process $\left\{\left(e_t^X, e_t^Y\right), t \geqslant 0\right\}$,

$$\widetilde{e}_t^X = \frac{e_t^X + e_t^Y}{\sqrt{2(1+\rho)}}, \quad \widetilde{e}_t^Y = \frac{e_t^Y - e_t^X}{\sqrt{2(1-\rho)}}. \tag{3.5}$$

Further, as in Nkurunziza (2005, Proposition 1.2), from the process $\left\{\left(W_t^X, W_t^Y\right), t \geqslant 0\right\}$ that satisfies Assumption $(\mathcal{C}_1)$, one get a bivariate Wiener process $\left\{\left(Z_t^X, Z_t^Y\right), t \geqslant 0\right\}$, using the following transformation

$$Z_t^X = \frac{W_t^X + W_t^Y}{\sqrt{2(1+\rho)}}, \quad Z_t^Y = \frac{W_t^Y - W_t^X}{\sqrt{2(1-\rho)}}.$$

It follows that $\left\{\widetilde{e}_t^Y, t \geqslant 0\right\}$ is Ornstein-Uhlenbeck process that is independent of the Ornstein-Uhlenbeck process $\{(\widetilde{e}_t^X, t \geqslant 0\}$.

Thus, from (3.5), the transformation from $\left(e_t^X, e_t^Y\right)$ to $\left(\widetilde{e}_t^X, \widetilde{e}_t^Y\right)$ is bijective, subject to $|\rho| < 1$. Then, it suffices to apply the reciprocal transformation of (3.5) to the logarithms of the observations $(\log(X_i), \log(Y_i))$ as well as to logarithms of the deterministic trajectories $(\log(x_i), \log(y_i))$ and $(\log(u_i), \log(v_i))$.

More precisely, note that (3.5) is equivalent to

$$e_t^X = \frac{1}{2}\left[\sqrt{2(1+\rho)}\,\widetilde{e}_t^X + \sqrt{2(1-\rho)}\,\widetilde{e}_t^Y\right], \quad e_t^Y = \frac{1}{2}\left[\sqrt{2(1+\rho)}\,\widetilde{e}_t^X - \sqrt{2(1-\rho)}\,\widetilde{e}_t^Y\right];$$

whenever $|\rho| < 1$. Then, we can rewrite the relation in (3.1) that defines the statistic T, by replacing

$$\begin{pmatrix} \log(X_i) \\ \log(Y_i) \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \frac{1}{2}\left[\sqrt{2(1+\rho)}\log(X_i)+\sqrt{2(1-\rho)}\log(Y_i)\right] \\ \frac{1}{2}\left[\sqrt{2(1+\rho)}\log(X_i)-\sqrt{2(1-\rho)}\log(Y_i)\right] \end{pmatrix},$$

$$\begin{pmatrix} \log(x_i) \\ \log(y_i) \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \frac{1}{2}\left[\sqrt{2(1+\rho)}\log(x_i)+\sqrt{2(1-\rho)}\log(y_i)\right] \\ \frac{1}{2}\left[\sqrt{2(1+\rho)}\log(x_i)-\sqrt{2(1-\rho)}\log(y_i)\right] \end{pmatrix}$$

and

$$\begin{pmatrix} \log(u_i) \\ \log(v_i) \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \frac{1}{2}\left[\sqrt{2(1+\rho)}\log(u_i)+\sqrt{2(1-\rho)}\log(v_i)\right] \\ \frac{1}{2}\left[\sqrt{2(1+\rho)}\log(u_i)-\sqrt{2(1-\rho)}\log(v_i)\right] \end{pmatrix}.$$

4 Conclusion

In this article, we studied testing problem concerning the Lotka-Voltarra ODEs parameters. Using the property of the trajectory of the Lotka-Voltarra ODEs, we derived the most powerful test for testing change in the ODEs parameters.

4.1 Acknowledgements

The authors would like to thank Prof. R. Ferland for helpful comments and suggestions. Also, the authors would like to acknowledge the financial support received from Natural Sciences and Engineering Research Council of Canada.

A Outline of proof of Proposition 1

Proof Suppose that, for (x_0, y_0) fixed,

$$(x(t_0;\gamma,\beta,\delta,\eta), y(t_0;\gamma,\beta,\delta,\eta)) = (x(t_0;\gamma_1,\beta_1,\delta_1,\eta_1), y(t_0;\gamma_1,\beta_1,\delta_1,\eta_1))$$

for some $t_0 > 0$ where $(x(t_0;\gamma,\beta,\delta,\eta), y(t_0;\gamma,\beta,\delta,\eta))$ is a nontrivial solution to the ODEs (1.1). Then, by continuity of the solution of the ODEs (1.1), we have

$$(x(t;\gamma,\beta,\delta,\eta), y(t;\gamma,\beta,\delta,\eta)) = (x(t;\gamma_1,\beta_1,\delta_1,\eta_1), y(t;\gamma_1,\beta_1,\delta_1,\eta_1))$$

for all $t > 0$. To simplify the notation, let us write (x_t, y_t) to denote the trajectory $(x(t;\gamma,\beta,\delta,\eta), y(t;\gamma,\beta,\delta,\eta))$.

We get

$$\eta - \beta y_t = \eta_1 - \beta_1 y_t, \quad \gamma x_t - \delta = \gamma_1 x_t - \delta_1,$$

and then,

$$\eta - \eta_1 = (\beta - \beta_1) y_t, \quad \delta - \delta_1 = (\gamma - \gamma_1) x_t. \tag{A.1}$$

Further, suppose that $(\beta - \beta_1) \neq 0$ or $(\gamma - \gamma_1) \neq 0$. Straightforward computations show that

$$x_t = \frac{\delta - \delta_1}{\gamma - \gamma_1}, \quad y_t = \frac{\eta - \eta_1}{\beta - \beta_1},$$

and this is a contradiction with the fact that (x_t, y_t) is a nontrivial solution to the ODEs (1.1). Therefore,

$$(\beta - \beta_1) = 0, \quad (\gamma - \gamma_1) = 0.$$

Further, from the relation in (A.1), we conclude that

$$\eta = \eta_1 \quad \text{and} \quad \delta = \delta_1$$

and finally

$$(\gamma, \beta, \delta, \eta) = (\gamma_1, \beta_1, \delta_1, \eta_1).$$

that completes the proof. ■

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