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STATISTICAL INFERENCE IN CONNECTION WITH THE WEIBULL MODEL USING TYPE-II PROGRESSIVELY CENSORED DATA WITH RANDOM SCHEME

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ABSTRACT

This paper discusses statistical inference in connection with a Weibull distribution model, using Type-II progressively censored data with random scheme. We used the maximum likelihood procedure to derive both point and interval estimates of the unknown parameters included in the model. The expected termination point to complete the censoring test is computed and analyzed under different censoring schemes. A numerical study is presented to illustrate the application of the theoretical results presented.

KEYWORDS

Maximum likelihood procedure, Exponential distribution, expected experiment time.

2000 Mathematics Subject Classification: Place Classification here. Leave as is, if there is no classification.

1 INTRODUCTION

There are several situations in life-testing, in reliability experiments, and survival analysis in which units are lost or removed from the experiments while they are still alive. The loss may occur out of control or be preassigned. The out of control case can happen when an individual under study (testing) drops out or so. The other case may occur because

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of limitation of funds or to save the time. For more details we refer to Balakrishnan and Aggarwala [**?**] (2000) and the references therein. In such situations, progressive censoring schemes take place.

Recently, the estimation of parameters from different lifetime distributions based on progressive Type-II censored samples have been studied by several authors including Childs and Balakrishnanm (2000), Balakrishnanm and Kannan (2001), Mousa and Jaheen (2002), Ng, et al. (2002), Balakrishnanm, et al. (2003) and Soliman (2005).

But in some reliability experiments, the number of patients who dropped out of the experiment cannot be pre-fixed and they are random. In such situations, progressive censoring schemes with random removals are needed. Sarhan and Abuammoh (2008) used progressive Type-II censoring of the data with random removals to estimate the parameters included in an exponentially distributed reliability model. In this paper, we estimate the unknown parameters included in a Weibull reliability model using Type-II progressively censored data with random removals. It is assumed in the Weibull reliability model that the life time of the object under investigation follows a Weibull distribution. The Weibull distribution is widely used in reliability for many reasons. Among these reasons are: (1) it has one shape and one scale parameters; and (2) based on the shape parameter, its failure rate function may be increasing or decreasing or constant. We will derive both point and interval estimates of the unknown parameters using the maximum likelihood method. Further, the expected termination point of the test under the type-II progressive censoring scheme is discussed in this paper. The results presented in this paper generalize those given in Sarhan and Abuammoh (2008).

The rest of this paper is organized as follows. Section 2 presents the Type-II progressive censoring scheme with random removals and the likelihood function. Maximum likelihood procedure is used in Section 3 to derive both point and interval estimates of the unknown parameters included in the model. Section 4 discusses the expected termination point of the tests assuming different scenarios of sampling. Finally, a numerical studies and a conclusion are included in Section 5.

2 The Model

Let random variable X have a Weibull distribution with parameter θ. The probability density function of X takes the following form:

$$f(x) = \alpha\beta x^{\beta-1}\exp\{-\alpha x^{\beta}\},\ x \geq 0, \alpha, \beta > 0. \tag{2.1}$$

The survival function of X is then

$$S(x) = \exp\{-\alpha x^{\beta}\}. \tag{2.2}$$

Let (X_1, R_1), $(X_2, R_2), \cdots$, (X_m, R_m), denote a progressively type II censored sample, where $X_1 < X_2 < \cdots < X_m$. With pre-determined number of removals, say $R_1 = r_1$, $R_2 = r_2, \ldots R_m = r_m$, the conditional likelihood function can be written as, Cohen (1963),

$$L(\theta; \mathbf{x}|\mathbf{R} = \mathbf{r}) = c^* \prod_{i=1}^{m} f(x_i) \, [S(x_i)]^{r_i}, \tag{2.3}$$

where $c^* = n(n - r_1 - 1)(n - r_1 - r_1 - 2) \ldots (n - r_1 - \ldots - r_m - m + 1)$, and $0 \le r_i \le (n - m - r_1 - \ldots - r_{i-1})$, for $i = 1, 2, \ldots, m - 1$.

Substituting (2.1) and (2.2) into (2.3), we get

$$L(\theta; \mathbf{x}|\mathbf{R} = \mathbf{r}) = c^* \alpha^m \beta^m \prod_{i=1}^{m} x_i^{\beta-1} \exp\left\{-\alpha \sum_{i=1}^{m} (1 + r_i) x_i^{\beta}\right\}. \tag{2.4}$$

Suppose that an individual unit being removed from the test at the i^{th} failure, $i = 1, 2, \ldots, m - 1$, is independent of the others but with the same probability p. That is, the number R_i of units removed at the i^{th} failure, $i = 1, 2, \ldots, m - 1$, follows a binomial distribution with parameters $n - m - \sum_{\ell=1}^{i-1} r_\ell$ and p. Therefore,

$$P(R_1 = r_1) = \binom{n - m}{r_1} p^{r_1} (1 - p)^{n-m-r_1}, \tag{2.5}$$

and for $i = 1, 2, \ldots, m - 1$,

$$P(R_i = r_i | R_{i-1} = r_{i-1}, \ldots, R_1 = r_1) = \binom{n - m - \sum_{\ell=1}^{i-1} r_\ell}{r_i} p^{r_i} (1 - p)^{n-m-\sum_{\ell=1}^{i} r_\ell}. \tag{2.6}$$

Now, we further suppose that R_i is independent of X_i for all i. Then the full likelihood function takes the following form

$$L(\theta, p; \mathbf{x}, \mathbf{r}) = L(\theta; \mathbf{x}|\mathbf{R} = \mathbf{r}) P(\mathbf{R} = \mathbf{r}), \tag{2.7}$$

where

$$\begin{aligned} P(\mathbf{R} = \mathbf{r}) \quad = \quad & P(R_1 = r_1) P(R_2 = r_2 | R_1 = r_1) P(R_3 = r_3 | R_2 = r_2, R_1 = r_1) \ldots \\ & P(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \ldots, R_1 = r_1),. \end{aligned} \tag{2.8}$$

Substituting (2.5) and (2.6) into (2.8), we get

$$P(\mathbf{R}=\mathbf{r}) = \frac{(n-m)!\, p^{\sum_{i=1}^{m-1} r_i}\,(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)!\,\prod_{i=1}^{m-1} r_i!}\,. \tag{2.9}$$

Now, using (2.4), (2.7) and (2.9), we can write the full likelihood function in the following form

$$L(\theta, p; \mathbf{x}, \mathbf{r}) = A L_1(\theta)\, L_2(p)\,, \tag{2.10}$$

where $A = \frac{c^*(n-m)!}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)!\,\prod_{i=1}^{m-1} r_i!}$ does not depend on the parameters θ and p,

$$L_1(\alpha, \beta) = \alpha^m\, \beta^m \exp\left\{-\alpha \sum_{i=1}^{m} (1+r_i) x_i^{\beta}\right\}\,, \tag{2.11}$$

and

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i}\,(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}\,. \tag{2.12}$$

3 Maximum Likelihood Estimation

This section discuss the process of obtaining the maximum likelihood estimates of the parameters α, β and p based on progressively Type-II censored data with binomial removals. Both point and interval estimations of the parameters are derived.

3.1 Point Estimations

Clearly, L_1 does not involve p. Therefore, the maximum likelihood estimators (MLE) of the parameters α, β can be derived by maximizing (2.11) directly. The log-likelihood function of L_1 takes the following form:

$$\mathcal{L}_1(\alpha, \beta) = m \ln \alpha + m \ln \beta + (\beta - 1) \sum_{i=1}^{m} \ln x_i - \alpha \sum_{i=1}^{m} (1+r_i) x_i^{\beta}. \tag{3.1}$$

The normal equations become

$$\frac{\partial \mathcal{L}_1(\alpha, \beta)}{\partial \alpha} = 0 = \frac{m}{\alpha} - \sum_{i=1}^{m} (1+r_i) x_i^{\beta} \tag{3.2}$$

$$\frac{\partial \mathcal{L}_1(\alpha, \beta)}{\partial \beta} = 0 = \frac{m}{\beta} + \sum_{i=1}^{m} \ln x_i - \alpha \sum_{i=1}^{m} (1+r_i) x_i^{\beta} \ln x_i. \tag{3.3}$$

From (3.2), we obtain the MLE of α as a function of β, say $\hat{\alpha}(\beta)$, as

$$\hat{\alpha}(\beta) = \frac{m}{\sum_{i=1}^{m}(1+r_i)x_i^{\beta}}. \tag{3.4}$$

Substituting (3.4) in (3.1), we obtain the profile log-likelihood of β as

$$\begin{aligned} g(\beta) &= \mathcal{L}_1(\hat{\alpha}(\beta),\beta) \\ &= m\ln\frac{m}{\sum_{i=1}^{m}(1+r_i)x_i^{\beta}} + (\beta-1)\sum_{i=1}^{m}\ln x_i - m. \end{aligned} \tag{3.5}$$

Therefore, the MLE of β, say $\hat{\beta}$, can be obtained by maximizing (3.5) with respect to β. It can be shown that the maximum of (3.5) can be obtained as a fixed point solution of the following equation:

$$h(\beta) = \beta \tag{3.6}$$

where

$$h(\beta) = \beta + m\ln m + \sum_{i=1}^{m}\ln x_i - \frac{m\sum_{i=1}^{m}(1+r_i)x_i^{\beta}\ln x_i}{\sum_{i=1}^{m}(1+r_i)x_i^{\beta}}. \tag{3.7}$$

The very simple iterative procedure $h(\beta^{(j)}) = h(\beta^{(j+1)})$, can be used, where $\beta^{(j)}$ is the jth iterate. The iterative procedure works very well. Once $\hat{\beta}_{\text{MLE}}$ is obtained, the MLE of α, say $\hat{\alpha}_{\text{MLE}}$, can be obtained from (3.4) as $\hat{\alpha}_{\text{MLE}} = g(\hat{\beta}_{\text{MLE}})$. Note that, $\hat{\alpha}_{\text{MLE}}$ and $\hat{\beta}_{\text{MLE}}$ are not in explicit form.

Similarly, since L_2 does not involve α and β but only p, the maximum likelihood estimator of p can be derived by maximizing (2.12) directly. The log-likelihood function of L_2 takes the following form:

$$\mathcal{L}_2(p) = \ln p\sum_{i=1}^{m-1} r_i + \ln(1-p)\left[(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i\right]. \tag{3.8}$$

Maximizing this function with respect to p gives the MLE of p, say $\hat{p}_{\text{MLE}}$ as

$$\hat{p}_{MLE} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-1-i)r_i}. \tag{3.9}$$

3.2 Interval Estimation

The approximate confidence intervals of the parameters based on the asymptotic distributions of the MLE of the parameters α, β and p are derived in this subsection. For the observed information matrix for α, β and p, we find

$$
\begin{aligned}
-\frac{\partial^2 \mathcal{L}(\alpha,\beta,p)}{\partial \alpha^2} &= A_{11} = \frac{m}{\alpha^2}\,, \\
-\frac{\partial^2 \mathcal{L}(\alpha,\beta,p)}{\partial \alpha \partial \beta} &= A_{12} = A_{21} = \sum_{i=1}^{m} (1+r_i) x_i^{\beta} \ln x_i \\
-\frac{\partial^2 \mathcal{L}(\alpha,\beta,p)}{\partial \beta^2} &= A_{22} = \frac{m}{\beta^2} + \alpha \sum_{i=1}^{m} (1+r_i) x_i^{\beta} \ln^2 x_i \\
-\frac{\partial^2 \mathcal{L}(\alpha,\beta,p)}{\partial \alpha \partial p} &= A_{13} = -\frac{\partial^2 \mathcal{L}(\alpha,\beta,p)}{\partial \beta \partial p} = A_{23} = 0\,, \\
-\frac{\partial^2 \mathcal{L}(\theta,p)}{\partial p^2} &= A_{33} = \frac{\sum_{i=1}^{m-1} r_i}{p^2} + \frac{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i}{(1-p)^2}.
\end{aligned}
$$

So that the variance-covariance matrix may be approximated as

$$
\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} & 0 \\ V_{21} & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}^{-1}.
$$

It is known that the asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta}, \hat{p})$ is given by, see Miller (1981),

$$
\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{p} \end{pmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \beta \\ p \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & 0 \\ V_{21} & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix} \right] \tag{3.10}
$$

Since $\mathbf{V}$ involves the parameters α, β and p, we replace the parameters by the corresponding MLE's in order to obtain an estimate of $\mathbf{V}$, which is denoted by $\hat{\mathbf{V}}$. By using (3.10), approximate $100(1-\alpha)\%$ confidence intervals for α, β and p are determined, respectively, as

$$
\hat{\alpha} \pm z_{\vartheta/2} \sqrt{\hat{\mathbf{V}}_{11}}, \quad \hat{\beta} \pm z_{\vartheta/2} \sqrt{\hat{\mathbf{V}}_{22}} \quad \text{and} \quad \hat{p} \pm z_{\vartheta/2} \sqrt{\hat{\mathbf{V}}_{33}} \tag{3.11}
$$

where z_{ϑ} is the upper $100\vartheta-$th percentile of the standard normal distribution.

4 Expected Test Time

It is often useful, in applications, to have an idea of the termination time of the whole test. The termination point for the experiment, for progressively type II censoring sampling plan with random or binomial removals, is given by the expectation of the mth order statistic $X_{(m)}$. The conditional expectation of $X_{(m)}$ for a fixed set of $\mathbf{R} = (R_1 = r_1, \ldots, R_{m-1} = r_{m-1})$ is, see Balakrishan and Aggrawla (2000),

$$E\left[X_{(m)}|\mathbf{R}=\mathbf{r}\right] = C(\mathbf{r}) \sum_{\ell_1=1}^{r_1} \cdots \sum_{\ell_m=1}^{r_m} (-1)^{\sum_{i=1}^{m} \ell_i} \frac{\binom{r_1}{\ell_1} \cdots \binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1} h(\ell_i)} \int_0^\infty x f(x) F^{h(\ell_i)-1}(x) dx \quad (4.1)$$

where $C(\mathbf{r}) = n(n-r_1-1)(n-r_1-r_2-2)\ldots(n-\sum_{i=1}^{m-1}(r_i+1))$, $h(\ell_i) = \ell_1 + \ldots + \ell_i + i$ and i is the number of live units removed from experimentation (or number of failure units).

Substituting (2.1) and (2.2) into (4.1), we get

$$E\left[X_{(m)}|\mathbf{R}=\mathbf{r}\right] = C(\mathbf{r}) \sum_{\ell_1=1}^{r_1} \cdots \sum_{\ell_m=1}^{r_m} (-1)^{\sum_{i=1}^{m} \ell_i} \frac{\binom{r_1}{\ell_1} \cdots \binom{r_m}{\ell_m}}{\prod_{i=1}^{m-1} h(\ell_i)} \sum_{k=0}^{h(\ell_i)-1} \frac{\binom{h(\ell_i)-1}{k} \Gamma(1/\beta+1)}{(-1)^k \alpha^{\frac{1}{\beta}} (k+1)^{1+\frac{1}{\beta}}}. \quad (4.2)$$

The expected termination time of a type-II censored sample without removal is defined as the expected value of the mth failure time, denoted $X^*_{(m)}$,

$$E\left[X^*_{(m)}\right] = m\binom{n}{m} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} \Gamma(1/\beta+1)}{(-1)^k \alpha^{\frac{1}{\beta}} (k+1)^{1+\frac{1}{\beta}}}. \quad (4.3)$$

The relation (4.3) can be derived from (4.2) by setting $r_i = 0$ for all $i = 1, \ldots, m-1$.

Similarly, the expected termination point time of the complete sample can be derived from (4.3) by setting $n = m$ as

$$E\left[X^{**}_{(m)}\right] = n \sum_{k=0}^{n-1} \frac{\binom{n-1}{k} \Gamma(1/\beta+1)}{(-1)^k \alpha^{\frac{1}{\beta}} (k+1)^{1+\frac{1}{\beta}}}. \quad (4.4)$$

For the type-II progressive censoring with random removals, the expected termination point, is defined by

$$E[X_{(m)}] = E_{\mathbf{R}}\left[E\left[X_{(m)}|\mathbf{R}\right]\right]$$

For binomial removals we have

$$E[X_{(m)}] = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} P(\mathbf{R}) E\left[E\left[X_{(m)}|\mathbf{R}=\mathbf{r}\right]\right] \quad (4.5)$$

where $g(r_i) = n - m - r_1 - r_2 - \ldots - r_{i-1}$, and $P(\mathbf{R})$ is given by (2.9).

If the set of removals follow a discrete uniform distribution (PCR), the the expected test termination will be

$$E[X_{(m)}] = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} P(\mathbf{R}) E\left[E\left[X_{(m)} | \mathbf{R} = \mathbf{r}\right]\right] \tag{4.6}$$

where $P(\mathbf{R})$, in this case, is

$$\begin{aligned} P(\mathbf{R}) &= P(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \ldots, R_1 = r_1) \\ &\quad P(R_{m-2} = r_{m-2} | R_{m-3} = r_{m-3}, \ldots, R_1 = r_1) \ldots P(R_2 = r_2 | R_1 = r_1) \\ &= \frac{1}{\left(n - m + 1 - \sum_{i=1}^{i-1} r_\ell\right)} \frac{1}{\left(n - m + 1 - \sum_{i=1}^{i-2} r_\ell\right)} \cdots \frac{1}{(n - m + 1)}. \end{aligned} \tag{4.7}$$

To compare (4.4) and (4.5), we compute the ratio of expected experiment test (REET) under the Type II progressive censoring with binomial removals over the expected termination point for complete sample which is given by

$$\text{REET}_1 = \frac{E[X_{(m)}] \text{ under binomial removals for a sample with size } n}{E[X^{**}_{(m)}] \text{ under complete sampling for a sample with size } n}.$$

Replacing the numerator by the expected termination point under type II progressive censoring with random removal (PCR), we have the ratio REET_2. The ratios REET_1 and REET_2 provide important information in determining the shortest experiment time significantly if the sample size n is large. When REET_1 and REET_2 are closer to 1, the termination point will be closer to the complete sample. We can study the influence of the binomial probability removals p on the the expected termination point by analyzing REET_1 for various p's. The comparisons between the three expected times will be made in order to reward some information about n and m on the duration of the experiment. As it seems, analytical comparisons between these three expected times is difficult. Therefore, we will make these comparisons numerically for various values of n, m, α and β. The next section presents such numerical comparisons.

5 Numerical Study

In this section we compute the expected termination times of progressive censoring with random removals plan and a complete sample plan using equations (4.4) and (4.5), respectively. The expected termination times for type-II progressive censoring with binomial removals of censoring probability p are computed when $p = 0.1, 0.3, 0.5$, 0.7 and 0.9. The computations were made for different choices of n and m. In this study the values of α and β were 0.12 and 0.5, respectively. The results are listed in Table 1. The first five columns in the table show the expected termination times for type-II progressive censoring with binomial removals $p = 0.1(0.1), 0.9$ and the last column gives the values of the expected experiment times for type II progressive censoring with random removals (PCR).

From the results shown in Table 1, one can conclude that:

1. For binomial removals, the larger p its the larger expected termination point will be.
2. For any m, the expected termination point for binomial removal with $p = 0.5$ is very close to that of PCR.

Figure 1 gives the ratios REET_1, for the binomial removals with $p = 0.1, .3, .5, .7, .9$ when $n = 10, 15, 20$ and different values of m. It is observed from this figure that the expected termination time for type-II progressive censoring sample becomes close to the expected termination time of the complete sample when the number of censoring units is increased.

For fixed n and m, REET_1 increases with the binomial probability of removals and the expected termination point for type-II progressive censoring sample becomes close to the expected termination point for the complete sample at a faster speed. But this speed becomes slower when $p > 0.5$. When $p = .5$, the expected termination point becomes approximately as the expected termination point for PCR.

Figure 2 gives the ratios REET_2 for PCR. It seems from this figure that REET_2 becomes close to one when m increases.

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Table 1: The expected termination point for binomial removals.

n	m	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$	PCR
10	10	11.345	11.345	11.345	11.345	11.345	11.345
	9	10.902	11.177	11.236	11.249	11.253	11.236
	8	10.414	10.943	11.095	11.132	11.144	11.096
	7	9.884	10.613	10.901	10.984	11.010	10.905
	6	9.305	10.148	10.613	10.783	10.839	10.627
	5	8.665	9.507	10.157	10.482	10.605	10.200
15	15	10.891	10.891	10.891	10.891	10.891	10.891
	14	10.309	10.410	10.524	10.565	10.769	10.786
	13	10.110	10.236	10.325	10.436	10.623	10.653
	12	9.813	10.086	10.216	10.342	10.524	10.587
	11	9.533	9.818	10.121	10.245	10.412	10.471
	10	9.143	9.428	10.091	10.145	10.330	10.221
	9	9.043	9.228	9.959	10.052	10.213	10.064
	8	9.824	9.032	9.865	9.975	10.121	10.096
	8	9.824	9.032	9.865	9.975	10.121	9.916
	7	9.432	8.943	9.675	9.878	9.989	9.767
20	20	11.811	11.811	11.811	11.811	11.811	11.811
	19	11.665	11.771	11.776	11.778	11.780	11.776
	18	11.498	11.726	11.739	11.741	11.743	11.739
	17	11.307	11.605	11.697	11.702	11.704	11.697
	16	11.093	11.614	11.651	11.659	11.663	11.652
	15	10.857	11.541	11.600	11.612	11.617	11.601
	14	10.595	11.453	11.543	11.560	11.567	11.544
	13	10.311	11.343	11.477	11.501	11.511	11.479
	12	10.005	11.203	11.400	11.436	11.449	11.402
	11	9.676	11.022	11.307	11.358	11.378	11.312
	10	9.323	10.784	11.191	11.268	11.300	11.200

Fig. 1a

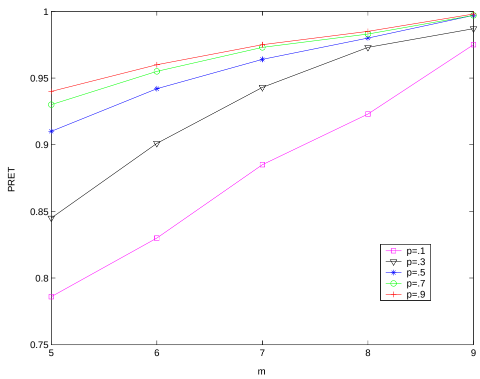

Fig. 1b

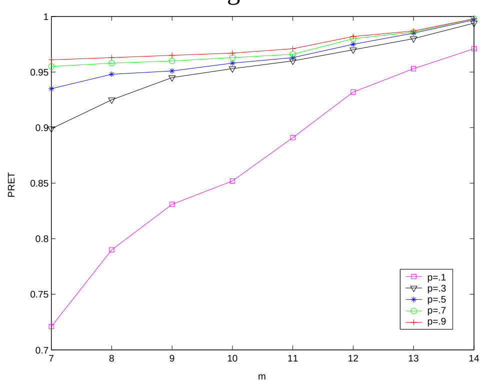

Fig. 1c

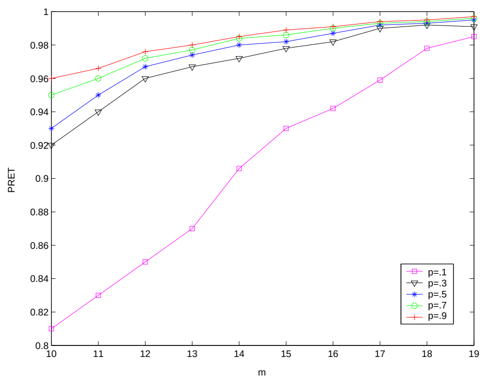

Figure 1: The ratios $REET_1$.

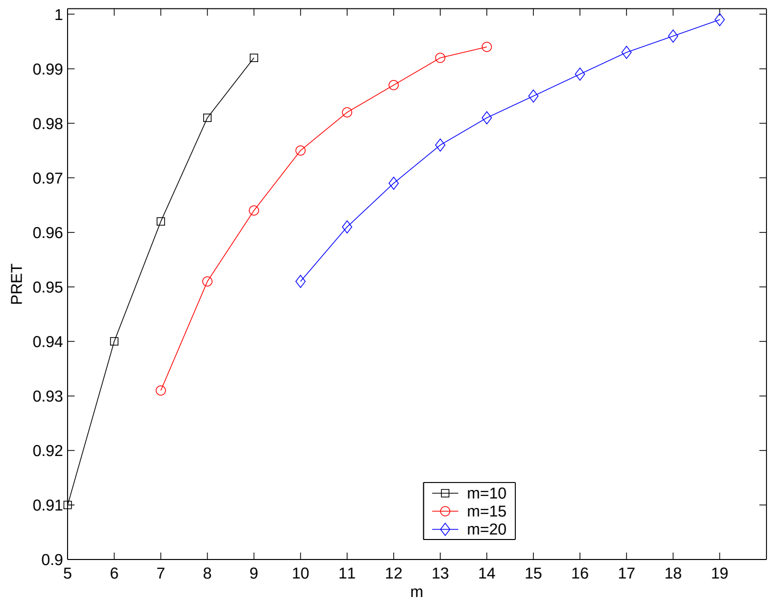

Figure 2: The ratios $REET_2$.