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A DIRECT JUSTIFICATION OF THE BINOMIAL PRICING MODEL AS AN APPROXIMATION OF THE BLACK-SCHOLES FORMULA

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ABSTRACT

This paper pedagogically presents a proof of the binomial option pricing model as an approximation of the Black-Scholes formula. The proof only requires basic calculus and a direct approximation of the binomial probability by the normal distribution is used.

KEYWORDS

Binomial option pricing model, Black-Scholes formula, European option.

2000 Mathematics Subject Classification: 91B28.

1 INTRODUCTION

Options are contracts of buying (call options) or selling (put options) a particular asset (e.g., a stock) for a fixed price at or before a specific date in the future. The fixed price is called the exercise price or striking price and the specific date is the expiration date or maturity date. European options can only be exercised at the expiration date while American options can be sold or bought at any time before the maturity date. Hull [5] has a detailed discussion on options.

There are two basic pricing models for European call options: the binomial model and the Black-Scholes formula. The binomial option pricing model is attractively simple and easy to understand. When no cash dividends are paid before the expiration date, the binomial option pricing model is as follows.

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Suppose there are n periods to the expiration date with a stock moving up and down respectively by fixed rates u and d in each period, where u ("up") is 1 plus the percentage change in the stock price if the stock price increases during the period and d ("down") stands for one plus the (negative) percentage change in the stock price if the stock price decreases. Let S_0 be the present (time 0) stock price, r be the unit rate of return of the riskless security, and m be the number of periods with rising price such that the stock price exceeds the exercise price E for the first time (That is, $S_0u^{m-1}d^{n-(m-1)} < E \leq S_0u^md^{n-m}$). Then, the present value of an European call option is

$$C_0(n) = S_0B^*(m;n,p^*) - E(1+r)^{-n}B^*(m;n,p),$$

where $B^*(x;y,w) = \sum_{j=x}^{y} \left(\begin{array}{c} y \\ j \end{array} \right) w^j(1-w)^{n-j}$; $p^* = \frac{u}{1+r}p$; and $p = \frac{(1+r)-d}{u-d}$ (See details of Levy and Sarnat [6], P596).

Black and Scholes [1] proposed a pricing model (the Black-Scholes formula) based on conditions such as stock prices follow a geometric Brownian motion with a constant drift and volatility, there are no transaction costs, taxes, cash dividends, and no arbitrage etc. According to the Black-Scholes formula, the current European call price, say, C_0, is

$$C_0 = S_0\Phi(d_1) - Ee^{rt}\Phi(d_2), \tag{1.1}$$

where S_0 is the current (time 0) stock price, E is the exercise price, r is the continuously compounded unit (e.g., annual) rate of the riskless security, t is the remaining time to the expiration of the call expressed as a fraction of unit time (e.g., a year), σ is the volatility of the stock price or the standard deviation of the continuously compounded annual rate of return of the stock price, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}w^2}dw$ is the cumulative distribution function of the standard normal,

$$d_1 = \frac{\ln\frac{S_0}{E} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

and

$$d_2 = \frac{\ln\frac{S_0}{E} + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}.$$

Note that the Black-Scholes formula was derived from a partial-differential equation (PDE) of option prices as a function of time and asset prices. Merton [8] extended the Black-Scholes theory for European options to American options where the Black-Scholes PDE still holds in the continuous region but with a free-boundary condition. Unlike the explicit

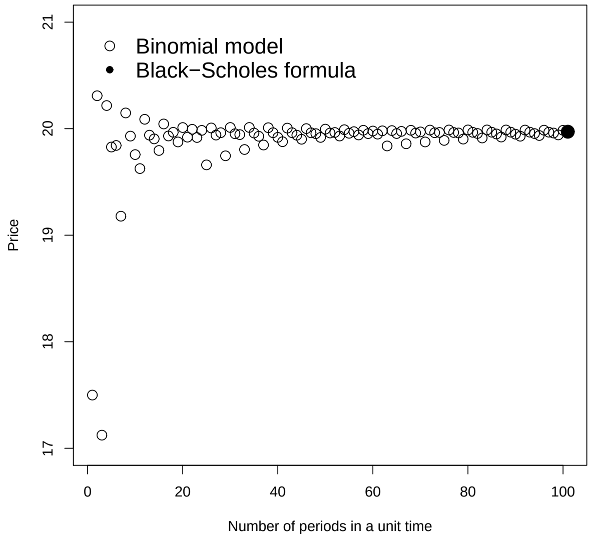

Figure 1: Binomial approximation to the Black-Scholes formula

formula in equation (1.1) for European options, there is no closed-form solution of the free-boundary PDE and numerical methods such as finite differences are needed to compute American option prices. More details on options are given in Pliska [9] and Luenberger [7].

Compared to the binomial model, the derivation of the Black-Scholes formula is quite involved. However, it is well known statistically that when the number of periods n is large, the binomial distribution approximates to the normal distribution. Numerically, Figure 1 shows how the call price calculated from the Black-Scholes formula is approached by prices from the binomial model of $n = 1$ to $n = 100$ for a stock with an initial price of \$100, exercise price \$ 110, riskless annual rate of 5%, and volatility of \$0.3, where the solid circle is the price from the Black-Scholes formula while empty circles are prices from the binomial model.

Cox et al. [3] suggested the first justification of the convergence of the binomial option price to the Black-Scholes price and Hsia [4] proposed a more general proof. However, none of them make use of the direct approximation of the binomial probability by the normal distribution, which is presented in this paper.

2 Main Results

The essential tool of the proof in this paper is the following result from Uspensky ([10], P129). Its proof is lengthy but rudimentary.

Lemma. For a fixed N, $0 < p < 1$, $q = 1 - p$, and $0 \le k \le N$, let $x_k = \frac{k-Np}{\sqrt{Npq}}$,

$$\pi_{\lambda,\nu} = \Phi\left(x_\nu + \frac{1}{2\sqrt{Npq}}\right) - \Phi\left(x_\lambda - \frac{1}{2\sqrt{Npq}}\right),$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}w^2}dw$ is the probability distribution function of the standard normal and $P_{\lambda,\nu} = \sum_{j=\lambda}^{\nu}\begin{pmatrix}\nu\\ j\end{pmatrix} p^j q^{N-j}$. Then,

$$P_{\lambda,\nu} = \pi_{\lambda,\nu} + \frac{q-p}{6\sqrt{2\pi Npq}}\left[(1-x^2)e^{-\frac{x^2}{2}}\right]\Bigg|_{x=x_\lambda-\frac{1}{2\sqrt{Npq}}}^{x=x_\nu+\frac{1}{2\sqrt{Npq}}} + \varepsilon$$

with $|\varepsilon| < \frac{0.13+0.18|p-q|}{Npq} + e^{-\frac{3\sqrt{Npq}}{2}}$ provided $Npq \ge 25$.

Theorem. Let t be the remaining time to the expiration of the call expressed as a fraction of the unit time. Consider partitioning the unit time into n periods and n is sufficient large such that the total number of periods nt is an integer. Let $u = e^{\frac{\sigma}{\sqrt{n}}}$ and $d = e^{-\frac{\sigma}{\sqrt{n}}}$, where u and d are fixed up and down rate in each period, respectively. Then,

$$\lim_{n\to\infty} B^*(m; nt, p) = \Phi(d_2), \tag{2.1}$$

where $B^*(x; y, w) = \sum_{j=x}^{y}\begin{pmatrix}y\\ j\end{pmatrix} w^j(1-w)^{n-j}$, and $p = \frac{(1+r)-d}{u-d}$, $d_2 = \frac{\ln\frac{S_0}{E}+(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}w^2}dw$ is the probability distribution function of the standard normal and

$$\lim_{n\to\infty} B^*(m; nt, p^*) = \Phi(d_1), \tag{2.2}$$

where $d_1 = \frac{\ln\frac{S_0}{E}+(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$ and $p^* = \frac{u}{1+r}p$.

Proof. Since the unit time is partitioned into n periods, the return rate of the riskless security at each period is $\frac{r}{n}$ and $\lim\limits_{n\to\infty}\left(1+\frac{r}{n}\right)^{-nt}=e^{-rt}$. Let m be the smallest number of upward moves for the call to exceed the exercise price, i.e., $m\leq nt$ and $S_0u^md^{nt-m}\geq E$. Then,

$$m\geq\frac{\ln\frac{E}{S_0}-nt\ln d}{\ln u-\ln d}=\frac{\sqrt{n}\ln\frac{E}{S_0}+nt\sigma}{2\sigma}.$$

Let

$$m=\left\lceil\frac{\sqrt{n}\ln\frac{E}{S_0}+nt\sigma}{2\sigma}\right\rceil,$$

where $\lceil x\rceil$ is the smallest integer that greater than x.

Note that

$$e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{6}+O(x^4)$$

where $\lim\limits_{x\to 0}\frac{O(x^4)}{x^4}=\text{constant}\neq 0$.

$$e^{\frac{\sigma}{\sqrt{n}}}-e^{-\frac{\sigma}{\sqrt{n}}}=\frac{2\sigma}{\sqrt{n}}+\frac{\sigma^3}{3n\sqrt{n}}+O(n^{-\frac{5}{2}}),$$

$$1+\frac{r}{n}-e^{-\frac{\sigma}{\sqrt{n}}}=\frac{\sigma}{\sqrt{n}}+\frac{r-\frac{1}{2}\sigma^2}{n}+\frac{\sigma^3}{6n\sqrt{n}}+O(n^{-2}).$$

It follows that,

$$p=\frac{1+\frac{r}{n}-e^{-\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}}-e^{-\frac{\sigma}{\sqrt{n}}}}=\frac{1}{2}+\frac{r-\frac{1}{2}\sigma^2}{2\sigma\sqrt{n}}+O(n^{-2}).$$

Therefore,

$$q=1-p=\frac{1}{2}-\frac{r-\frac{1}{2}\sigma^2}{2\sigma\sqrt{n}}+O(n^{-2}).$$

$$ntpq=\frac{nt}{4}-\frac{(r-\frac{1}{2}\sigma^2)^2t}{4\sigma^2}+O(n^{-1}).$$

Since

$$x_{nt}+\frac{1}{2\sqrt{ntpq}}=\frac{2nt-2ntp+1}{\sqrt{2ntpq}}=\frac{nt-\frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{nt}+O(n^{-1})}{\sqrt{nt-\frac{(r-\frac{1}{2}\sigma^2)^2}{\sigma^2}t+O(n^{-1})}}=\sqrt{t}O(n^{\frac{1}{2}}),$$

$$\lim_{n\to\infty}\Phi(x_{nt}+\frac{1}{2\sqrt{ntpq}})=1.$$

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty}\left(x_m - \frac{1}{2\sqrt{ntpq}}\right) &= \lim_{n\to\infty}\frac{\left\lceil \frac{\sqrt{n}\ln\frac{E}{S_0}+nt\sigma}{\sigma}\right\rceil - nt - \frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{n}t - 1 + O(n^{-1})}{\sqrt{nt - \frac{(r-\frac{1}{2}\sigma^2)^2}{\sigma^2}t + O(n^{-1})}} \\
&= \frac{1}{\sigma\sqrt{t}}\ln\frac{E}{S_0} - \frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{t} \\
&= -\frac{\ln\frac{S_0}{E} + (r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}.
\end{aligned}$$

Therefore, there exists a constant M such that $|\left(x_m - \frac{1}{2\sqrt{ntpq}}\right)| < M$ and

$$1 - \lim_{n\to\infty}\Phi\left(x_m - \frac{1}{2\sqrt{ntpq}}\right) = \Phi\left(\frac{\ln\frac{S_0}{E} + (r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) = \Phi(d_2).$$

Moreover, $\lim\limits_{x\to\infty}(1-x^2)e^{-\frac{x^2}{2}} = 0$ and for $|x| < M$,

$$\lim_{n\to\infty}\frac{q-p}{6\sqrt{2\pi Npq}}\left[(1-x^2)e^{-\frac{x^2}{2}}\right] = 0,$$

and $\lim\limits_{n\to\infty}|\varepsilon| = 0$. Equation (2.1) follows from the Lemma directly.

Similarly, it can be shown that

$$p^* = \frac{1}{2} + \frac{r+\frac{1}{2}\sigma^2}{2\sigma\sqrt{n}} + O(n^{-2}),$$

$$q^* = 1 - p^* = \frac{1}{2} - \frac{r+\frac{1}{2}\sigma^2}{2\sigma\sqrt{n}} + O(n^{-2}),$$

and

$$ntp^*q^* = \frac{nt}{4} - \frac{(r+\frac{1}{2}\sigma^2)^2}{4\sigma^2}t + O(n^{-1}).$$

Repeat all the steps in the proof of equation (2.1) with $r - \frac{1}{2}\sigma^2$ taken place by $r - \frac{1}{2}\sigma^2$ for $B^*(m; nt, p^*)$ and equation (2.2) is obtained straightforwardly. Hence,

$$C_0 = \lim_{n\to\infty}C_0(nt) = S_0\Phi(d_1) + Ee^{-rt}\Phi(d_2).$$

□

3 Conclusions

A direct proof of the binomial option pricing model as an approximation of the Black-Scholes formula is presented in this paper. Unlike most justifications where the De Moivre-Laplace central limit theorem is used without an explicit error estimation, our proof uses the direct approximation of binomial probability from the normal distribution and explicitly highlights the order of the approximation. Chang and Palmer [2] extended such an approximation to reveal the convergence speed of the binomial pricing model to the Black-Scholes formula.

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