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GENERALIZED MULTIVARIATE RATIO ESTIMATORS USING MULTI-AUXILIARY VARIABLES FOR MULTI-PHASE SAMPLING

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ABSTRACT

In this paper we propose a number of generalized multivariate ratio estimators for two-phase and multi-phase sampling in the presence of multi-auxiliary variables for estimating population mean for a single variable and a vector of variables of interest(s). The expressions for mean square errors are also derived. The suggested estimators are theoretically compared and an empirical study has also been conducted.

KEY WORDS

Multi-Phase Sampling; Multivariate Ratio Estimator; Multi-Auxiliary Variables.

1. INTRODUCTION

The estimation of the population mean is an unrelenting issue in sampling theory and several efforts have been made to improve the precision of the estimators in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas of ratio, regression and product estimators.

Olkin (1958) was the first author to deal with the problem of estimating the mean of a survey variable when auxiliary variables are made available. John (1969) proposed two multivariate generalizations of ratio and product estimators which actually reduce to the Olkin's (1958) and Singh's (1967a) estimators. Srivastava (1971) proposed a general ratio-type estimator which generates a large class of estimators including most of the estimators up to that time proposed. Sen (1972) developed a multivariate ratio estimator under two-phase sampling using multi-auxiliary variables. Singh and Namjoshi (1988) discussed a class of multivariate regression estimators of population mean of study variable in two-phase sampling.

Ceccon and Diana (1996) provided a multivariate extension of the Naik and Gupta (1991) univariate class of estimators. Ahmed (2003) put forward chain based general estimators using multivariate auxiliary information under multiphase sampling. In the same situation, Perri (2005) recommended some new estimators obtained from Singh's (1965, 1967b) estimators.

In multipurpose surveys, the problem is to estimate population means of several variables simultaneously [Swain (2000)]. Tripathi and Khattree (1989) estimated means of several variables of interest, using multi-auxiliary variables, under simple random sampling. Further Tripathi (1989) extended the results to the case of two phase sampling.

we suggest general classes of ratio estimators for estimating the population mean of study variable for two-phase and multi-phase sampling using multi-auxiliary variables when information on all multi-auxiliary variables (Full Information Case) or not on all auxiliary variables (No Information Case) is available for population (see Samiuddin and Hanif, 2007).

Before suggesting the estimators we provide Multi-phase sampling scheme and some useful notations and results in the following section.

2. MULTI-PHASE SAMPLING USING MULTI-AUXILIARY VARIABLES

Consider a population of N units. Let Y be the variable of interest and $X_1, X_2, ...X_q$ are q auxiliary variables. For multi-phase sampling design let n_h and n_k $\left(n_h < n_k\right)$ be sample sizes for h^{th} and k^{th} phase respectively. $x_{(h)i}$ and $x_{(k)i}$ denote the i^{th} auxiliary variables from h^{th} and k^{th} phase samples respectively and y_k denote the variable of interest from the k^{th} phase. Let, $\bar{X}_i$, C_{x_i} and ρ_{yx_i} denote the population mean, coefficient of variation of i^{th} auxiliary variables respectively and the population correlation coefficient of Y and X_i. Further let $\theta_h = \frac{1}{n_k} - \frac{1}{N}$, $\theta_k = \frac{1}{n_k} - \frac{1}{N}$. Also $y_{i(k)} = Y + e_{y_{i(k)}}$, $x_{(h)i} = X_i + e_{x_{(h)i}}$ and $x_{(k)i} = X_i + e_{x_{(k)i}}$.; $\left(i = 1, 2, ..., k\right)$ where $e_{y_{(k)}}$, $e_{x_{(h)i}}$ and $e_{x_{(k)i}}$ are sampling errors. We assume that $E_k\left(e_{y_{(k)}}\right) = E_h\left(e_{x_{(h)i}}\right) = E_k\left(e_{x_{(k)i}}\right) = 0$ where E_h and E_k denote the expectations of errors of h^{th} and k^{th} phase sampling. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

$$E_k\left(e_{y_k}\right)^2 = \left(1 - \frac{n_k}{N}\right)\sigma_y^2,\ E_k\left(\bar{e}_{y_k}\right)^2 = \left(1 - \frac{n_k}{N}\right)\frac{\sigma_y^2}{n_k} = \theta_k \bar{Y}^2 C_y^2,$$

$$E_k\left(e_{y_k} e_{x_{(k)i}}\right) = \left(1 - \frac{n_k}{N}\right)\sigma_{yx_i} = \theta_k \overline{Y}\overline{X}_i C_y C_{x_i} \rho_{yx_i}$$

$$E_k\left(\bar{e}_{y_k} \bar{e}_{x_{(k)i}}\right) = \left(1 - \frac{n_k}{N}\right)\frac{\sigma_{yx_i}}{n_k} = \theta_k \overline{Y}\overline{X}_i C_y C_{x_i} \rho_{yx_i},$$

$$E_h E_{k|h}\left[e_{y_k}\left(e_{x_{(h)i}} - e_{x_{(k)i}}\right)\right] = E_h E_{k|h}\left(e_{y_k} e_{x_{(h)i}}\right) - E_k\left(e_{y_k} e_{x_{(k)i}}\right) = \frac{1}{N}\left(n_k - n_h\right)\sigma_{yx_i},$$

$$E_h E_{k|h}\left[\bar{e}_{y_k}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\right]=\left(1-\frac{n_h}{N}\right)\frac{\sigma_{yx_i}}{n_h}-\left(1-\frac{n_k}{N}\right)\frac{\sigma_{yx_i}}{n_k}=\left(\theta_h-\theta_k\right)\overline{YX}_i C_y C_{x_i}\rho_{yx_i}.$$

Similarly

$$E_h E_{k|h}\left[\bar{e}_{x_{(k)i}}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\right]=\left(\theta_h-\theta_k\right)\sigma_{x_i}^2=\left(\theta_h-\theta_k\right)\bar{X}_i^2 C_{x_i}^2,$$

$$E_h E_{k|h}\left[\bar{e}_{x_{(h)i}}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\right]=0,$$

$$E_h E_{k/h}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)^2=\left(\theta_k-\theta_h\right)\sigma_{x_i}^2=\left(\theta_k-\theta_h\right)\bar{X}_i^2 C_{x_i}^2,$$

$$E_h E_{k|h}\left[\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\left(\bar{e}_{x_{(h)j}}-\bar{e}_{x_{(k)j}}\right)\right]=\left(\theta_k-\theta_h\right)\sigma_{x_i x_j}$$

$$=\left(\theta_k-\theta_h\right)\bar{X}_i\bar{X}_j C_{x_i}C_{x_j}\rho_{x_i x_j};\left(i\neq j\right),$$

and

$$E_h E_{k|h}\left[\left(\bar{e}_{x_{(k)i}}\right)\left(\bar{e}_{x_{(h)j}}-\bar{e}_{x_{(k)j}}\right)\right]=\left(\theta_h-\theta_k\right)\sigma_{x_i x_j}=\left(\theta_h-\theta_k\right)\bar{X}_i\bar{X}_j C_{x_i}C_{x_j}\rho_{x_i x_j},\left(i\neq j\right).$$

The following notations will be used in deriving the mean square errors of proposed estimators

$|\mathrm{R}|_{y\underline{x}_q}$ Determinant of population correlation matrix of variables $y,\ x_1,\ x_2,...,x_{q-1}$ *and* x_q.

$\left|R_{yx_i}\right|_{y\underline{x}_q}$ Determinant of i^{th} minor of $|\mathrm{R}|_{y\underline{x}_q}$ corresponding to the i^{th} element of ρ_{yx_i}.

$\rho^2_{y.\underline{x}_s}$ Denotes the multiple coefficient of determination of y on $x_1,\ x_2,...,x_{r-1}$ and x_r.

$\rho^2_{y.\underline{x}_q}$ Denotes the multiple coefficient of determination of y on $x_1,\ x_2,...,x_{q-1}$ and x_q.

$|R|_{\underline{x}_s}$ Determinant of population correlation matrix of variables $x_1,\ x_2,...,x_{r-1}$ and x_r.

$|R|_{\underline{x}_q}$ Determinant of population correlation matrix of variables $x_1,\ x_2,...,x_{q-1}$ and x_q.

$|R|_{y_i\underline{x}_s}$ Determinant of the correlation matrix of $y_i,\ x_1,\ x_2,...,x_{r-1}$ and x_r.

$|R|_{y_i\underline{x}_q}$ Determinant of the correlation matrix of $y_i,\ x_1,\ x_2,...,x_{q-1}$ and x_q.

$|R|_{y_i y_j\underline{x}_s}$ Determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i,\ y_j,x_1,\ x_2,...,x_{r-1}$ and x_r, for $\left(i\neq j\right)$.

$|R|_{y_i y_j\underline{x}_q}$ Determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i,\ y_j,x_1,\ x_2,...,x_{q-1}$ and x_q, for $\left(i\neq j\right)$.

2.1 Result: 1

The following result will help in deriving the mean square errors of suggested estimators

$$\frac{|\mathrm{R}|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} = \left(1-\rho^2_{y.\underset{\sim}{x}_q}\right), \quad \text{[Arora and Lal (1989)].}$$

3. GENERALIZED MULTIVARIATE RATIO ESTIMATOR FOR MULTI-PHASE SAMPLING

We propose a more general multivariate ratio estimator when information on all auxiliary variables is not available for population (No Information Case) and we obtain information on variables of interests $(y_1, y_2, ..., y_p)$ and for auxiliary variables $(x_1, x_2, ..., x_q)$ form k^{th} phase and also for auxiliary variables from h^{th} phase. The proposed estimator is

$$T_{hk(1\times p)} = \left[\bar{y}_{(k)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}}\right)^{\alpha_{i1}} \quad \bar{y}_{(k)2}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}}\right)^{\alpha_{i2}} \quad \cdots \quad \bar{y}_{(k)p}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}}\right)^{\alpha_{ip}}\right]. \quad (3.1)$$

$$= \left[\left(\bar{Y}_1 + \bar{e}_{y_{(k)1}}\right)\left(1+\sum_{i=1}^{q}\frac{\alpha_{i1}}{\bar{\bar{X}}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right)\right.$$

$$\left(\bar{Y}_2 + \bar{e}_{y_{(k)2}}\right)\left(1+\sum_{i=1}^{q}\frac{\alpha_{i2}}{\bar{\bar{X}}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right)$$

$$\left.\cdots \quad \left(\bar{Y}_p + \bar{e}_{y_{(k)p}}\right)\left(1+\sum_{i=1}^{q}\frac{\alpha_{ip}}{\bar{\bar{X}}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right)\right]$$

$$= \left[\left(\bar{Y}_1 + \bar{e}_{y_{(k)1}}\right) \quad \left(\bar{Y}_2 + \bar{e}_{y_{(k)2}}\right) \quad \cdots \quad \left(\bar{Y}_p + \bar{e}_{y_{(k)p}}\right)\right]$$

$$+\left[\sum_{i=1}^{q}\alpha_{i1}\frac{\bar{Y}_1}{\bar{\bar{X}}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right) \quad \sum_{i=1}^{q}\alpha_{i2}\frac{\bar{Y}_2}{\bar{\bar{X}}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right.$$

$$\left.\cdots \quad \sum_{i=1}^{q}\alpha_{ip}\frac{\bar{Y}_p}{\bar{\bar{X}}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right]$$

$$= \left[\left(\bar{Y}_1 + \bar{e}_{y_{(k)1}}\right) \quad \left(\bar{Y}_2 + \bar{e}_{y_{(k)2}}\right) \quad \cdots \quad \left(\bar{Y}_p + \bar{e}_{y_{(k)p}}\right)\right]$$

$$+\left[\left(\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}\right) \quad \left(\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}\right) \quad \cdots \quad \left(\bar{e}_{x_{(h)q}} - \bar{e}_{x_{(k)q}}\right)\right]\left[\frac{\bar{Y}_j}{\bar{\bar{X}}_i}\alpha_{ij}\right]_{(q\times p)}$$

$$\left(T_{hk(1\times p)}-\bar{Y}\right)=\left[\bar{e}_{y_{(k)1}} \quad \bar{e}_{y_{(k)2}} \quad \ldots \quad \bar{e}_{y_{(k)p}}\right]$$
$$+\left[\left(\bar{e}_{x_{(h)1}}-\bar{e}_{x_{(k)1}}\right) \quad \left(\bar{e}_{x_{(h)2}}-\bar{e}_{x_{(k)2}}\right) \quad \ldots \quad \left(\bar{e}_{x_{(h)q}}-\bar{e}_{x_{(k)q}}\right)\right]A^{*}$$

or

$$T_{hk(1\times p)}-\bar{Y}=\bar{D}_y+\bar{D}_x A^{*}.$$

where

$$T_{hk(1\times p)}=\left[\bar{y}_{(k)1} \quad \bar{y}_{(k)2} \quad \ldots \quad \bar{y}_{(k)p}\right], \bar{Y}=\left[\bar{Y}_1 \quad \bar{Y}_2 \quad \ldots \quad \bar{Y}_p\right],$$
$$\bar{D}_y=\left[\bar{e}_{y_{(k)1}} \quad \bar{e}_{y_{(k)2}} \quad \ldots \quad \bar{e}_{y_{(k)p}}\right],$$
$$\bar{D}_x=\left[\left(\bar{e}_{x_{(h)1}}-\bar{e}_{x_{(k)1}}\right) \quad \left(\bar{e}_{x_{(h)2}}-\bar{e}_{x_{(k)2}}\right) \quad \ldots \quad \left(\bar{e}_{x_{(h)q}}-\bar{e}_{x_{(k)q}}\right)\right] \text{ and}$$
$$A^{*}=\left[\alpha^{*}_{ij}\right]_{(q\times p)}=\left[\frac{\bar{Y}_j}{\bar{X}_i}\alpha_{ij}\right]_{(q\times p)} ;\left(i=1,2,...,q\ ,\ j=1,2,...,p\right).$$

We use information related to auxiliary variables from h^{th} and k^{th} phase both then the variance covariance matrix of $t_{1m(1\times p)}$ is written as:

$$\Sigma_{T_{hk}(p\times p)}=E_h E_{k/h}\left(T_{hk(1\times p)}-\bar{Y}\right)^{'}\left(T_{hk(1\times p)}-\bar{Y}\right)$$
$$=E_h E_{k/h}\left[\left(\bar{D}_y+\bar{D}_x A^{*}\right)^{'}\left(\bar{D}_y+\bar{D}_x A^{*}\right)\right].$$

We can write

$$E_h E_{k/h}\left(\bar{D}_y'\bar{D}_y\right)=\theta_k\Sigma_y=\theta_k[\sigma_{y_i y_j}]_{(p\times p)}, \text{ for } i=j, \sigma_{y_i y_j}=\sigma^2_{y_i}$$
$$E_h E_{k/h}\left(\bar{D}_y'\bar{D}_x\right)=\left(\theta_h-\theta_k\right)\Sigma_{yx}=\left(\theta_h-\theta_k\right)[\sigma_{y_i x_j}]_{(p\times q)}$$

and

$$E_h E_{k/h}\left(\bar{D}_x'\bar{D}_x\right)=\left(\theta_k-\theta_h\right)\Sigma_x=\left(\theta_k-\theta_h\right)[\sigma_{x_i x_j}]_{(q\times q)}, \text{for } i=j, \sigma_{x_i x_j}=\sigma^2_{x_i}$$

Using above substitutions in expression of variance covariance matrix, we write:

$$\Sigma_{T_{hk}(p\times p)}=\theta_k\Sigma_{y_{(p\times p)}}+\left(\theta_h-\theta_k\right)A^{*'}_{(p\times q)}\Sigma'_{yx_{(q\times p)}}$$
$$+\left(\theta_h-\theta_k\right)\Sigma_{yx_{(p\times q)}}A^{*}_{(q\times p)}+\left(\theta_k-\theta_h\right)A^{*'}_{(p\times q)}\Sigma_{x_{(q\times q)}}A^{*}_{(q\times p)}.$$

Given that $\Sigma^{-1}_{x(q\times q)}$ exist, the value $A^{'}$ of that minimizes the variance covariance matrix of $t_{(1\times p)}$ will be

$$A^{*}_{(q\times p)}=\Sigma^{-1}_{x_{(q\times q)}}\Sigma'_{yx_{(q\times p)}}. \tag{3.2}$$

The transpose of $A^*_{(q\times p)}$ is $\Sigma_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)} = \mathrm{B}_{\mathrm{yx}}$ it is actually the matrix of regression coefficients for a multivariate regression model in which $p\left(Y_1, Y_2, ..., Y_p\right)$ dependent variables are regressed on q $\left(X_1, X_2, ..., X_q\right)$ dependent variables. These regression coefficients are usually unknown and they are estimated from second phase sample. Then the variance covariance matrix after simplification can be written as:

$$\Sigma_{T_{hk}(p\times p)} = \theta_k \Sigma_{y(p\times p)} - \left(\theta_k - \theta_h\right)\Sigma'_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)}\Sigma_{yx(q\times p)}\,. \tag{3.3}$$

The variance covariance matrix form of variance of y_i, covariances and correlation coefficients of x_i and y_i is written as:

$$\Sigma_{T_{hk}(p\times p)} = \left[\sigma_{y_i}\sigma_{y_j}\left(\theta_k\left(\rho_{y_i y_j} - \rho_{y_i y_j \cdot \underset{\sim}{x}_q}\right) + \theta_h \rho_{y_i y_j \cdot \underset{\sim}{x}_q}\right)\right]_{p\times p};\left(i, j = 1, 2, ..., p\right), \tag{3.4}$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j}\left(\rho_{y_i y_j} - \rho_{y_i y_j \cdot \underset{\sim}{x}_q}\right) = \sigma^2_{y_i}\left(1 - \rho^2_{y_i \cdot \underset{\sim}{x}_q}\right)$.

For $\left|R\right|_{\underset{\sim}{x}_q} \neq 0$, we see that

$$\left(1 - \rho^2_{y_i \cdot \underset{\sim}{x}_q}\right) = \frac{\left|R\right|_{y_i \underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}};\left(i = 1, 2, ..., p\right) \text{ and } \left(\rho_{y_i y_j} - \rho_{y_i y_j \cdot \underset{\sim}{x}_q}\right) = \frac{\left|R\right|_{y_i y_j \underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}};\left(i, j = 1, 2, ..., p\right).$$

Now after simplification we write the variance covariance matrix as:

$$\Sigma_{T_{hk}(p\times p)} = \left[\sigma_{y_i}\sigma_{y_j}\left[\theta_k\left(\frac{\left|R\right|_{y_i y_j \underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}\right) + \theta_h\left(\rho_{y_i y_j} - \frac{\left|R\right|_{y_i y_j \underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}\right)\right]\right]_{p\times p};\ \left(i, j = 1, 2, ..., p\right), \tag{3.5}$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j}\dfrac{\left|R\right|_{y_i y_j \underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}} = \sigma^2_{y_i}\dfrac{\left|R\right|_{y_i \underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}$.

Remark-1:

To develop generalized multivariate ratio estimator for two-phase sampling using multi-auxiliary variables when information on all auxiliary variables is not available for population (No Information Case), replace h by 1 and k by 2 in (3.1), we get the following estimator

$$T_{12(1\times p)} = \left[\bar{y}_{(2)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{i1}} \quad \bar{y}_{(2)2}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{i2}} \quad \cdots \quad \bar{y}_{(2)p}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{ip}}\right] \tag{3.6}$$

The expression of unknown matrix for which the mean square error will be minimum is same as given in (3.2) and the expression for variance covariance matrix can be directly written from (3.3) just replacing h by 1 and k by 2 as:

$$\Sigma_{T_{12}(p\times p)}=\theta_2\Sigma_{y(p\times p)}-(\theta_2-\theta_1)\Sigma'_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)}\Sigma_{yx(q\times p)} \tag{3.7}$$

The variance covariance matrix in the form of variance of y_i, covariances and correlation coefficients of x_i and y_i is written as:

$$\Sigma_{T_{12}(p\times p)}=\left[\sigma_{y_i}\sigma_{y_j}\left(\theta_2\left(\rho_{y_iy_j}-\rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right)+\theta_1\rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right)\right]_{p\times p};(i,j=1,2,...,p), \tag{3.8}$$

for $i=j$, $\sigma_{y_i}\sigma_{y_j}\left(\rho_{y_iy_j}-\rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right)=\sigma^2_{y_i}\left(1-\rho^2_{y_i\cdot \underset{\sim}{x}_q}\right)$.

In the form of determinates we can write

$$\Sigma_{T_{12}(p\times p)}=\left[\sigma_{y_i}\sigma_{y_j}\left[\theta_2\left(\frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right)+\theta_1\left(\rho_{y_iy_j}-\frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right)\right]\right]_{p\times p};\ (i,j=1,2,...,p), \tag{3.9}$$

for $i=j$, $\sigma_{y_i}\sigma_{y_j}\dfrac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}=\sigma^2_{y_i}\dfrac{|R|_{y_i\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}$.

Remark-2:

To develop a univariate generalized ratio estimator for multiphase sampling using multi auxiliary variable when information on all auxiliary variables is not known (No Information Case). This estimator can be made if we put $p=1$ in (3.1) as:

$$T_{hk}=\bar{y}_{(k)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}}\right)^{\alpha_{i1}}. \tag{3.10}$$

The expression for vector of unknown constants for which the mean square error will be minimum can be written from (3.2) as

$$A^*_{(q\times1)}=\Sigma^{-1}_{x_{(q\times q)}}\Sigma'_{yx_{(q\times1)}}. \tag{3.11}$$

It can be written in determinants form as:

$$\alpha^*_i=(-1)^{i+1}\frac{\bar{Y}}{\bar{X}_i}\frac{C_y}{C_{x_i}}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}=(-1)^{i+1}\beta_{yx_i\cdot \underset{\sim}{x}_q},\ (i=1,2,....q).$$

The expression for mean square error can be directly written from (3.3) as:

$$MSE\left(T_{hk}\right)=\theta_k\sigma_y^2-\left(\theta_k-\theta_h\right)\left(\sigma_x^2\right)^{-1}\Sigma'_{yx(1\times q)}\Sigma_{yx(q\times 1)} \tag{3.12}$$

It can be written the form of multiple coefficient of determination as:

$$MSE\left(T_{hk}\right)=\bar{Y}^2C_y^2\left[\theta_k\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right)+\theta_h\rho^2_{y.\underset{\sim}{x}_q}\right] \tag{3.13}$$

Remark-3:

To develop a generalized univariate ratio estimator for two sampling using multi-auxiliary variables when information on all auxiliary variables is not known (No Information Case) we put $h=1$ and $k=2$ in (3.10). The required estimator becomes

$$T_{12}=\bar{y}_{(2)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{i1}}. \tag{3.14}$$

The expression for vector of unknown constants for which the mean square error will be minimum is same as given in (3.10) and the expression for mean square error can be written from (3.12) just by replacing $h=1$ and $k=2$ as:

$$MSE\left(T_{12}\right)=\theta_2\sigma_y^2-\left(\theta_2-\theta_1\right)\left(\sigma_x^2\right)^{-1}\Sigma'_{yx(1\times q)}\Sigma_{yx(q\times 1)} \tag{3.15}$$

It can be written in the form of multiple coefficient of determination from (3.13) as:

$$MSE\left(T_{12}\right)=\bar{Y}^2C_y^2\left[\theta_2\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right)+\theta_1\rho^2_{y.\underset{\sim}{x}_q}\right] \tag{3.16}$$

Remark-4:

To develop generalized multivariate ratio estimator for multi-phase sampling using multi auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), replace $\bar{x}_{(h)i}$ by $\bar{X}_i$ in (3.1). For this replacement $n_h\rightarrow N$ and $\theta_h\rightarrow 0$. The estimator becomes

$$T_{k(1\times p)}=\left[\bar{y}_{(k)i}\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(k)i}}\right)^{\alpha_{i1}}\quad \bar{y}_{(k)2}\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(k)i}}\right)^{\alpha_{i2}}\quad\cdots\quad \bar{y}_{(k)p}\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(k)i}}\right)^{\alpha_{ip}}\right]. \tag{3.17}$$

The expression for the matrix of unknown constants is same as given in (3.2) and the expression for variance covariance matrix can be written from (3.3) as:

$$\Sigma_{T_k(p\times p)}=\theta_k\left(\Sigma_{y(p\times p)}-\Sigma_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)}\Sigma'_{yx(q\times p)}\right) \tag{3.18}$$

The variance covariance matrix in the form of variance of y_i, covariances and correlation coefficients of x_i and y_i can be written as:

$$\Sigma_{T_k(p\times p)} = \theta_k \left[\sigma_{y_i}\sigma_{y_j} \left(\rho_{y_i y_j} - \rho_{y_i y_j \cdot \underset{\sim}{x}_q} \right) \right]_{p\times p} ;\ \left(i, j = 1, 2, ..., p \right), \tag{3.19}$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j} \left(\rho_{y_i y_j} - \rho_{y_i y_j \cdot \underset{\sim}{x}_q} \right) = \sigma_{y_i}^2 \left(1 - \rho^2_{y_i \cdot \underset{\sim}{x}_q} \right)$.

The variance covariance matrix in the form of determinants can be written as:

$$\Sigma_{T_k(p\times p)} = \theta_k \left[\sigma_{y_i}\sigma_{y_j} \frac{|R|_{y_i y_j \underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} \right]_{p\times p},\ \left(i, j = 1, 2, ..., p \right), \tag{3.20}$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j} \dfrac{|R|_{y_i y_j \underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} = \sigma_{y_i}^2 \dfrac{|R|_{y_i \underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}$.

Remark-5:

To develop generalized multivariate ratio estimator for two-phase sampling using multi auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), replace k by 2 in (3.17). The estimator becomes

$$T_{2(1\times p)} = \left[\bar{y}_{(2)i} \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_{i1}} \quad \bar{y}_{(k)2} \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_{i2}} \quad \cdots \quad \bar{y}_{(k)p} \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_{ip}} \right]. \tag{3.21}$$

The expression for the matrix of unknown constants will be same as given in (3.2) and the expression for variance covariance matrix can be written from (3.18) as:

$$\Sigma_{T_2(p\times p)} = \theta_2 \left(\Sigma_{y(p\times p)} - \Sigma_{yx(p\times q)} \Sigma^{-1}_{x(q\times q)} \Sigma'_{yx(q\times p)} \right) \tag{3.22}$$

The variance covariance matrix in the form of variance of y_i, covariances and correlation coefficients of x_i and y_i and in the form of determinants can obtained from (3.19) and (3.20) respectively just by replacing k by 2.

Remark-6:

To develop a generalized univariate ratio estimator for multi-sampling using multi-auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), put $p = 1$ in (3.17). The estimator becomes

$$T_k = \bar{y}_k \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(k)i}} \right)^{\alpha_i} \tag{3.23}$$

The expression for unknown constant for which the mean square error will be minimum of above estimator is same as given in (3.11). The expression of mean square error can be written from (3.18) by putting $p = 1$ as:

$$MSE\left(T_k\right) = \theta_k\left(\sigma_y^2 - \left(\sigma_x^2\right)^{-1}\Sigma'_{yx(1\times q)}\Sigma_{yx(q\times 1)}\right) \quad (3.24)$$

It can also be written as:

$$MSE\left(T_k\right) = \theta_k \bar{Y}^2 C_y^2\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right) \quad (3.25)$$

Remarks-7:

To develop a generalized univariate ratio estimator for two-phase sampling using multi-auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), replacing k by 2 in (3.23). The estimator becomes

$$T_2 = \bar{y}_2\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\alpha_i} \quad (3.26)$$

The expression for unknown constant for which the mean square error of above estimator will be minimum is same as given in (3.11). The expression of mean square error can be written from (3.24) just replacing k by 2 as:

$$MSE\left(T_2\right) = \theta_2\left(\sigma_y^2 - \left(\sigma_x^2\right)^{-1}\Sigma'_{yx(1\times q)}\Sigma_{yx(q\times 1)}\right) \quad (3.27)$$

It can also be written as:

$$MSE\left(T_2\right) = \theta_2 \bar{Y}^2 C_y^2\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right) \quad (3.28)$$

6. THEORETICAL COMPARISON OF NEWLY DEVELOPED ESTIMATORS

Obviously estimator for which the information on all auxiliary variables is available for population will be more efficient than that for which the information on all auxiliary variables is not available for population. It means in the case of two-phase sampling generalized ratio estimator developed for FIC T_2 is more efficient than or NIC T_{12}. It can be checked by considering the mean square errors of suggested estimators as:

$$MSE\left(T_2\right) - MSE\left(T_{12}\right) = -\theta_1\bar{Y}^2C_y^2\rho^2_{y.\underset{\sim}{x}_q} < 0$$

For multiphase sampling generalized ratio estimator developed for FIC T_k is more efficient than generalized ratio estimator for NIC t_{2m}. It can be checked by considering the mean square errors of these estimators as:

$$MSE\left(T_k\right) - MSE\left(T_{hk}\right) = -\theta_h\bar{Y}^2C_y^2\rho^2_{y.\underset{\sim}{x}_q} < 0$$

Also the estimators developed for multi-phase sampling will be less efficient than those which are developed for two-phase sampling because if we increase the phases the efficiency will decrease but cost will reduced. It can be checked for FIC and NIC as:

$$MSE\left(T_2\right)-MSE\left(T_k\right)=\left(\theta_2-\theta_k\right)\bar{Y}^2C_y^2\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right)<0;\qquad k>2,$$

and

$$MSE\left(T_{12}\right)-MSE\left(T_{hk}\right)=\bar{Y}^2C_y^2\left[\left(\theta_2-\theta_k\right)\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right)+\left(\theta_1-\theta_h\right)\rho_{y.\underset{\sim}{x}_q}^2\right]<0;$$
$$k>2 \text{ and } 1<h<k$$

Theoretical comparison on the basis of generalized MSE's can be made for all multivariate estimators. These comparisons give same results as discussed above for univariate case.

7. EMPIRICAL STUDY OF NEWLY DEVELOPED ESTIMATORS

For empirical comparison of newly developed multivariate and univariate estimators using multi-auxiliary variables for no and full information cases under two and multi-phase sampling we consider five natural populations. The data is used from five districts census reports of province Punjab, Pakistan. The detail of populations and variables description is given in Table A-1.1 and Table A-1.2 respectively of Appendix A. We consider three variables of interests denoted by Y's and five auxiliary variables denoted by X's for computing the determinants of matrices of MSE's of multivariate ratio estimators and for univariate we consider Y_2 as study variable and the same five auxiliary variables as considered in multivariate case. The necessary parameters of populations for computing MSE's are given in A-1.3. We calculate pair-wise (determinant of matrices of MSE's)/MSE's for no information case and for full information case we calculate (determinant of matrices of MSE's)/MSE's for each phase for first five phases. The determinant of matrices of mean square errors of multivariate ratio estimators for multiphase sampling using pair-wise phases for no information case are given in Table A-1.4.1 and for full information case using each phase for full information case are given in A-1.4.2. The mean square errors of univariate estimators for multiphase sampling using pair-wise phases for no information case are given in Table A-1.5.1 and for full information case using each phase are given in A-1.5.2.

From Table A-1.4.1 and Table A-1.4.2, we can say that the multivariate ratio estimators for full information case are more efficient than no information case for each phase e.g. T_2 is more efficient than T_{12}, T_3 is more efficient than T_{13} & T_{23} etc. and the same is true for univariate ratio estimators (see Table A-1.5.1 and Table A-1.5.2). Furthermore we can say for no information case from Table A-1.4.1 that as we increase phase the efficiency decreases e.g. T_{12}, is more efficient than all others, T_{13} is more efficient than all others except T_{12}, T_{34} is more efficient than T_{35}, T_{45} but less efficient than all others and so on, similarly the same argument can be made for univariate case given in Table A-1.5.1. Also for full information case the estimators become less efficient as we increase phases because the sample size decreases by increasing phases, it can be seen from Table A-1.4.2 and A-1.5.2 for multivariate and univariate estimators respectively.

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Appendix A

Table A-1.1: Detail of Populations

S#	Source of Populations
1	Population census report of Jhang district (1998), Pakistan
2	Population census report of Faisalabad district (1998), Pakistan.
3	Population census report of Gujrat district (1998), Pakistan.
4	Population census report of Kasur (1998) Pakistan
5	Population census report of Sialkot district (1998), Pakistan.

Table A-1.2: Description of variables (Each variables is taken from Rural Locality)

Description of variables			
Y_1	Literacy ratio	X_2	Population of primary but below matric
Y_2	Population of currently married	X_3	Population of matric and above
Y_3	Total household	X_4	Population of 18 years old and above
X_1	Population of both sexes	X_5	Population of women 15-49 years old

Table A-1.3: Parameters of populations for calculating the Matrices of MSE's of multivariate estimators and MSE's of univariate estimators

Districts	N	n_1	n_2	n_3	n_4	n_5	$\bar{Y}_1$	$\bar{Y}_2$	$\bar{Y}_3$	C_{y_1}	C_{y_2}	C_{y_3}
Jhang	368	184	92	46	23	12	29.705	860.11	897.71	0.270	0.595	0.512
Faisalabad	283	142	71	35	18	9	51.394	1511.260	1540.530	3.210	0.522	0.478
Gujrat	204	102	51	26	13	6	57.535	1101.280	1102.540	0.145	0.484	0.487
Kasur	181	91	45	23	11	6	31.890	1393.200	1449.020	0.747	0.551	0.530
Sailkot	269	135	67	34	17	8	52.061	1058.740	998.220	0.147	0.647	0.646

Table A-1.3 (Contd…)

Districts	σ_{y_1}	σ_{y_2}	σ_{y_3}	σ_{x_1}	σ_{x_2}	σ_{x_3}	σ_{x_4}	σ_{x_5}	$\rho_{y_1y_2}$
Jhang	8.022	511.908	459.842	5626.450	455.060	170.670	2455.170	1064.480	.182
Faisalabad	164.950	788.380	736.395	5426.030	1677.920	525.670	6289.710	1482.170	.070
Gujrat	8.364	533.041	537.236	3507.160	940.480	381.690	8139.680	830.010	.055
Kasur	23.823	767.636	767.796	5515.420	1095.690	357.890	2719.210	1355.640	.295
Sailkot	7.641	685.019	644.886	4787.250	1172.710	603.220	2461.590	1151.320	.324

Table A-1.3 (Contd…)

Districts	$\rho_{y_1y_3}$	$\rho_{y_2y_3}$	$\rho_{y_1x_1}$	$\rho_{y_1x_2}$	$\rho_{y_1x_3}$	$\rho_{y_1x_4}$	$\rho_{y_1x_5}$	$\rho_{y_2x_1}$	$\rho_{y_2x_2}$	$\rho_{y_2x_3}$	$\rho_{y_2x_4}$	$\rho_{y_2x_5}$
Jhang	.164	.733	.131	.460	.548	.185	.129	.428	.912	.659	.484	.425
Faisalabad	.084	.943	.072	.025	.033	.039	.042	.943	.927	.599	.731	.501
Gujrat	.056	.988	.092	.334	.543	.069	.103	.995	.941	.764	.490	.996
Kasur	.301	.989	.299	.255	.352	.301	.250	.998	.758	.879	.989	.799
Sailkot	.316	.997	.323	.426	.461	.338	.313	.999	.983	.931	.996	.939

Table A-1.3 (Contd…)

District	$\rho_{y_3x_1}$	$\rho_{y_3x_2}$	$\rho_{y_3x_3}$	$\rho_{y_3x_4}$	$\rho_{y_3x_5}$	$\rho_{x_1x_2}$	$\rho_{x_1x_3}$	$\rho_{x_1x_4}$
Jhang	.474	.732	.748	.559	.489	.416	.421	.317
Faisalabad	.967	.615	.747	.520	.822	.641	.782	.513
Gujrat	.984	.933	.749	.487	.986	.954	.796	.509
Kasur	.991	.752	.878	.988	.792	.764	.889	.993
Sailkot	.996	.980	.933	.994	.938	.983	.931	.997

Table A-1.3 (Contd…)

District	$\rho_{x_1x_5}$	$\rho_{x_2x_3}$	$\rho_{x_2x_4}$	$\rho_{x_2x_5}$	$\rho_{x_3x_4}$	$\rho_{x_3x_5}$	$\rho_{x_4x_5}$	$\rho^2_{y.x_1...x_5}$
Jhang	.275	.824	.475	.432	.590	.464	.325	0.885
Faisalabad	.819	.708	.359	.559	.543	.685	.436	0.869
Gujrat	.996	.892	.500	.958	.420	.797	.505	0.996
Kasur	.802	.798	.764	.614	.896	.719	.797	0.995
Sailkot	.939	.959	.985	.928	.939	.887	.938	0.997

Table A-1.4.1 Determinants of matrices of MSE's of multivariate ratio estimators for pair-wise phases (No Information Case)

District	T_{12} (h=1,k=2)	T_{13} (h=1,k=3)	T_{14} (h=1,k=4)	T_{15} (h=1,k=5)	T_{23} (h=2,k=3)	T_{24} (h=2,k=4)
Jhang	212279.97	1223241.603	7221747.657	46128460.86	3572866.63	16011134.82
Faisalabad	212515.48	1190925.96	6725695.10	40958629.30	2777523.40	13258374.47
Gujrat	95363.18	312081.54	1102996.85	4211396.91	1123210.67	3372979.29
Kasur	203091.03	901801.71	3838230.04	16192403.14	2176260.22	8973735.68
Sialkot	9555.27	41464.31	173806.26	723929.58	126175.45	482925.70

Table A-1.4.1 (Contd…)

District	T_{25} (h=2,k=5)	T_{34} (h=3,k=4)	T_{35} (h=3,k=5)	T_{45} (h=4,k=5)
Jhang	79961159.68	38868782.02	158093587.7	2.60491E+11
Faisalabad	67439732.76	27555308.50	122925398.17	243999111.56
Gujrat	11251537.17	10702183.50	30944428.16	93069748.63
Kasur	36805031.49	19922737.86	79389873.19	170075469.89
Sialkot	1897366.87	1256890.23	4554991.24	11150157.88

Table A-1.4.2 Determinants of matrices of MSE's of multivariate ratio estimators for each phase (Full Information Case)

District	T_1 (k=1)	T_2 (k=2)	T_3 (k=3)	T_4 (k=4)	T_5 (k=5)
Jhang	1023.378901	27631.23032	351018.963	3453903.79	30487480.83
Faisalabad	1535.620269	27767.37297	311260.3873	2908814.863	25077800.89
Gujrat	27.15981853	367.7474988	3710.595857	33124.8694	279522.8025
Kasur	103.7803005	1306.215104	12804.97189	112839.0944	946342.2579
Sialkot	2.156156072	36.85754751	405.2293669	3755.836241	32256.39965

Table A-1.5.1 MSE's of univariate ratio estimators for pair-wise phases (No Information Case)

District	T_{12} (h=1,k=2)	T_{13} (h=1,k=3)	T_{14} (h=1,k=4)	T_{15} (h=1,k=5)	T_{23} (h=2,k=3)	T_{24} (h=2,k=4)
Jhang	212279.97	1223241.603	7221747.657	46128460.86	3572866.63	16011134.82
Faisalabad	212515.48	1190925.96	6725695.10	40958629.30	2777523.40	13258374.47
Gujrat	95363.18	312081.54	1102996.85	4211396.91	1123210.67	3372979.29
Kasur	203091.03	901801.71	3838230.04	16192403.14	2176260.22	8973735.68
Sialkot	9555.27	41464.31	173806.26	723929.58	126175.45	482925.70

Table A-1.5.1 (Contd…)

District	T_{25} (h=2,k=5)	T_{34} (h=3,k=4)	T_{35} (h=3,k=5)	T_{45} (h=4,k=5)
Jhang	79961159.68	38868782.02	158093587.7	2.60491E+11
Faisalabad	67439732.76	27555308.50	122925398.17	243999111.56
Gujrat	11251537.17	10702183.50	30944428.16	93069748.63
Kasur	36805031.49	19922737.86	79389873.19	170075469.89
Sialkot	1897366.87	1256890.23	4554991.24	11150157.88

Table A-1.5.2 MSE's of univariate ratio estimators for each-wise phase (Full Information Case)

District	T_1 (k=1)	T_2 (k=2)	T_3 (k=3)	T_4 (k=4)	T_5 (k=5)
Jhang	81.89064	245.6719	573.2344	1228.36	2538.61
Faisalabad	131.0829	344.0564	770.0035	1621.898	3325.686
Gujrat	213.5579	509.0065	1099.904	2281.698	4645.286
Kasur	251.1011	584.0929	1250.076	2582.043	5245.977
Sialkot	142.167	366.2247	814.34	1710.571	3503.032