

**RELIABILITY EQUIVALENCE FACTORS FOR SOME SYSTEMS
WITH MIXTURE LINEAR INCREASING FAILURE RATES**

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ABSTRACT

In this article, the failure rates of the system's components are functions of time t . We study two cases (i) the life time distribution of a components has two stages with increasing failure rates, (ii) the failure rates of the components have the two stages. The reliability equivalence factors of some systems with identical components are obtained. Two different methods are used to improve the given systems. Numerical examples are presented to interpret how one can utilize the obtained results. Some special cases are obtained from our results.

KEY WORDS

Linear increasing failure rate distribution; rayleigh distribution; exponential distribution; mixture distributions; mixture failure rates; hot duplication; reduction method; reliability equivalence factors.

1. INTRODUCTION

In reliability analysis, sometimes different system designs should be compared based on a reliability characteristic such as the reliability function or mean time to failure in case of no repair. The concept of reliability equivalence factors has been introduced by Råde (1993a). Råde (1993a, 1993b), Sarhan (2000, 2002, 2004 and 2005), Mustafa (2002, 2008a), Sarhan et al. (2004), Sarhan and Mustafa (2007), Mustafa et al. (2007), Sarhan and et al. (2008) and Mustafa (2008) have applied such concept to various systems.

Råde (1993) has calculated the reliability equivalence factors for a single component and for two independent and identical component series and parallel systems. He assumed that the reliability function of the system can be improved by three different methods as: (1) improving the quality of one or several components by decreasing their failure rates; (2) adding some hot redundant components to the system; and (3) adding some cold redundant component to the system.

Råde (1993) and Sarhan (2000) have used the survival function as the performance measure of the reliability system. Råde (1993) has calculated the reliability equivalence

factors for single components and for two independent and identical component series and parallel systems.

Sarhan(2000) has obtained the reliability equivalence factors of n independent and non-identical components series system. He used the survival function and mean time to failure as characteristics to compare different system designs. Sarhan (2002, 2004, and 2005), Mustafa (2002), Sarhan et al. (2004), Sarhan and Mustafa (2007), Mustafa et al. (2007), Mustafa (2008) and Sarhan et al. (2008), have applied the concept of reliability equivalence on a parallel(series) system with independent and identical(non-identical) components.

The previous articles (1989-2008) in reliability equivalence technique assumed that the system components have one type of constant failure rate. Mustafa et al. (2007) assumed a system components have three types of constant failure rates and made a mixture of these types; see Everitt and Hand (1981), Elsayed (1996).

All articles mentioned above are about components of exponential distribution. But Xia and Zhang (2007) applied the reliability equivalence factor of a parallel system with n independent and identical components with Gamma life time distribution. Gamma distribution has a failure rate of function of time. Mustafa (2008b) has applied the concept of reliability equivalence factor on a series system with non-constant failure rates. We hope to discuss more life time distributions, failure rates and apply the mixture approach on them to obtain the general system.

In the current study, we shall use the survival function and mean time to failure to calculate the reliability equivalence factors for some systems, consisting of independent and identical components. These components are assumed to be having two stages of failure rates or failure life times. The reliability of the system can be improved according to one of the following two different methods:

1. Reducing the failure rates of some of the system components. This method will be refer by the reduction method
2. Assuming hot duplication of some of the system components. This means that each component is duplicated by a hot redundant standby component. This method will be called the hot duplication method.

The methods of cold and imperfect switch duplication contain some problems in the integrations and Maple program can not compute it.

This paper is organized as follows. Section 2 presents the one component system with mixture of life time distribution. Section 3 gives n -components series system with mixture of two stages of linear increasing failure rates. Some special cases of our work are discussed in the end of sections 2 and 3.

2. MIXTURE OF LIFE TIME DISTRIBUTIONS

We consider a system whose components fail if they enter either of two stages of failure mechanisms. The first mechanism is due to excessive voltage, and the second is due to excessive temperature. Suppose that the failure mechanism enters either the first

stage with probability θ_1 or the second stage with probability θ_2 . Let the probability density function (p.d.f.) of the first stage is $f_1(t)$ and the p.d.f. of the second stage is $f_2(t)$. Hence the failure of a component occurs at the end of either the first or the second stage. Therefore, the p.d.f. of the failure time for a component is

$$f(t) = \theta_1 f_1(t) + \theta_2 f_2(t). \quad (2.1)$$

The reliability function of the component is

$$R(t) = \int_t^{\infty} f(u) du = \theta_1 R_1(t) + \theta_2 R_2(t). \quad (2.2)$$

The hazard (failure) rate function of the component is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\theta_1 f_1(t) + \theta_2 f_2(t)}{\theta_1 R_1(t) + \theta_2 R_2(t)}, \quad (2.3)$$

where $\theta_1 + \theta_2 = 1$.

2.1 The Original System

In this section, we consider a simple system consists of one component has two stages with linear increasing failure rates as

$$\left. \begin{aligned} h_i(t) &= \alpha_i t + \beta_i, \\ R_i(t) &= \exp \left\{ - \left(\frac{\alpha_i}{2} t^2 + \beta_i t \right) \right\}, \end{aligned} \right\} \quad (2.4)$$

where $\alpha_i, \beta_i \geq 0, i = 1, 2$. The life time of the stage i , has the linear increasing failure rate distribution, from equation (2.2), the reliability function is given as follows

$$R(t) = \sum_{i=1}^2 \theta_i \exp \left\{ - \left(\frac{\alpha_i}{2} t^2 + \beta_i t \right) \right\}. \quad (2.5)$$

Let MTTF be the system mean time to failure, which is given by

$$\text{MTTF} = \sqrt{\frac{\pi}{2}} \sum_{i=1}^2 \left\{ \frac{\theta_i}{\sqrt{\alpha_i}} \left[1 - \text{erf} \left(\frac{\beta_i}{\sqrt{2\alpha_i}} \right) \right] \exp \left\{ \frac{\beta_i^2}{2\alpha_i} \right\} \right\}, \quad (2.6)$$

where $\text{erf}(x)$ is the error function that defined as follows

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp \{-t^2\} dt, \quad x \in [0, \infty],$$

with $\text{erf}(x) \in [0, 1]$.

2.2 The Improved Systems

2.2.1 Reduction Method

In this method, we can reduce the failure rate of the stage i by the factor ρ_i , $0 < \rho_i < 1$, $i = 1, 2$. Let $R_p(t)$ be the reliability function of the improved system when we reduce the failure rate of the stage i by the factor ρ_i . One can obtain the function $R_p(t)$ as follows

$$R_p(t) = \sum_{i=1}^2 \theta_i \exp \left\{ -\rho_i \left(\frac{\alpha_i}{2} t^2 + \beta_i t \right) \right\}. \quad (2.7)$$

From equation (2.7), the mean time to failure of the improved system say $MTTF_p$ becomes

$$MTTF_p = \sqrt{\frac{\pi}{2}} \sum_{i=1}^2 \left\{ \frac{\theta_i}{\sqrt{\rho_i \alpha_i}} \left[1 - \operatorname{erf} \left(\beta_i \sqrt{\frac{\rho_i}{2\alpha_i}} \right) \right] \exp \left\{ \frac{\rho_i \beta_i^2}{2\alpha_i} \right\} \right\}. \quad (2.8)$$

2.2.2 Hot Duplication Method

Let $R^H(t)$ be the reliability function of the improved system assuming hot duplication. The function $R^H(t)$ can be obtained as

$$R^H(t) = R(t)[2 - R(t)], \quad (2.9)$$

from equation (2.5), one can easily find $R^H(t)$.

Let $MTTF^H$ denotes the mean time to failure of the system improved in this case. Using equation (2.9), one can obtain $MTTF^H$ as

$$MTTF^H = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left\{ \frac{4\theta_1 \exp \left\{ \frac{\alpha_1^2}{2\beta_1} \right\}}{\sqrt{\beta_1}} \left[1 - \operatorname{erf} \left(\frac{\alpha_1}{\sqrt{2\beta_1}} \right) \right] - \frac{2\theta_1^2 \exp \left\{ \frac{\alpha_1^2}{\beta_1} \right\}}{\sqrt{\beta_1}} \right. \\ \times \left[1 - \operatorname{erf} \left(\frac{\alpha_1}{\sqrt{\beta_1}} \right) \right] - \frac{4\theta_1 \theta_2 \exp \left\{ \frac{(\alpha_1 + \alpha_2)^2}{2(\beta_1 + \beta_2)} \right\}}{\sqrt{\beta_1 + \beta_2}} \left[1 - \operatorname{erf} \left(\frac{\alpha_1 + \alpha_2}{\sqrt{2(\beta_1 + \beta_2)}} \right) \right] \\ \left. + \frac{4\theta_2 \exp \left\{ \frac{\alpha_2^2}{2\beta_2} \right\}}{\sqrt{\beta_2}} \left[1 - \operatorname{erf} \left(\frac{\alpha_2}{\sqrt{2\beta_2}} \right) \right] - \frac{\sqrt{2} \theta_2^2 \exp \left\{ \frac{\alpha_2^2}{\beta_2} \right\}}{\sqrt{\beta_1}} \left[1 - \operatorname{erf} \left(\frac{\alpha_2}{\sqrt{\beta_2}} \right) \right] \right\}. \quad (2.10)$$

2.3 γ – Fractiles

This section presents the γ – fractiles of the original and improved systems. Let $L(\gamma)$ be the γ – fractiles of the original system and $L^H(\gamma)$, the γ – fractiles of the improved system assuming hot duplication method.

The γ – fractiles $L(\gamma)$ and $L^H(\gamma)$ are defined as the solution of the two following equations, respectively,

$$R\left(\frac{L(\gamma)}{\alpha_1 + \alpha_2}\right) = \gamma, \quad R^H\left(\frac{L(\gamma)}{\alpha_1 + \alpha_2}\right) = \gamma. \quad (2.11)$$

It follows from equation (2.5) and the first equation of (2.11) that $L=L(\gamma)$ satisfies the following equation

$$\sum_{i=1}^2 \theta_i \exp\left\{-\left(\frac{\alpha_i L^2}{2(\alpha_1 + \alpha_2)^2} + \frac{\beta_i L}{\alpha_1 + \alpha_2}\right)\right\} = \gamma. \quad (2.12)$$

From the second equation of (2.11), and equation (2.9), one can verify that $L=L^H(\gamma)$ satisfies the following equation

$$\left[2 - \sum_{i=1}^2 \theta_i \exp\left\{-\left(\frac{\alpha_i L^2}{2(\alpha_1 + \alpha_2)^2} + \frac{\beta_i L}{\alpha_1 + \alpha_2}\right)\right\}\right] \sum_{i=1}^2 \theta_i \exp\left\{-\left(\frac{\alpha_i L^2}{2(\alpha_1 + \alpha_2)^2} + \frac{\beta_i L}{\alpha_1 + \alpha_2}\right)\right\} = \gamma. \quad (3.13)$$

Equations (2.12) and (2.13) have no closed form solution and can be solved using numerical method technique.

2.4 Reliability Equivalence Factors

Now we are ready to derive the reliability equivalence factors of the system. We will deduce the survival reliability equivalence factor, say SREF and mean reliability equivalence factor, say MREF of the underlying system as follows.

The first type of reliability equivalence factor, SREF say $\rho^H(\gamma)$, can be obtained by equating the reliability function of the improved system that obtained by improving the system according to reduction method with the reliability function of the system improved by improving the system according to hot duplication method at the level γ . Hence from equations (2.7) and (2.9), $\rho^H(\gamma)$, can be obtained by solving the following system of equations

$$R_p(t) = R^H(t) = \gamma. \quad (2.14)$$

Let us now explain how one can deduce the second type of reliability equivalence factor. This type is MREF say ξ^H that can be obtained by equating the mean time of the

improved system that obtained by improving the system according to reduction method with the mean time to failure of the system improved by improving the system according to hot duplication method. It means that, ξ^H can be derived from equations (2.8) and (2.10) as follows.

$$\text{MTTF}^H = \text{MTTF}_p. \quad (2.15)$$

Equations (2.14), (2.15) can be solved numerically by using numerical method technique.

2.5 Numerical Results and Conclusions

To explain how one can apply theoretical results obtained in the previous subsections, we introduce a numerical example. In this example, we assume $\alpha_1=0.07$, $\beta_1=0.04$, $\alpha_2=0.06$, $\beta_2=0.03$, $\theta_1=0.45$, and $\theta_2=0.55$ then the failure rates are $h_1(t)=0.07 t+0.04$, $h_2(t)=0.06 t + 0.03$ respectively.

The mean time to failure of the original system is $\text{MTTF}= 4.456$ and $\text{MTTF}^H=7.295$ then $\text{MTTF} < \text{MTTF}^H$.

The γ -fractiles, $L(\gamma)$, $L^H(\gamma)$ and the values of $\rho^H(\gamma)$ are calculated using Mathematica Program system according to the previous theoretical formulae. In these calculations the level γ is chosen to be 0.1, 0.3, ..., 0.9 and we find ρ_i when $\rho_j=1$, $i, j =1, 2$, $i \neq j$.

The γ -fractile and the survival reliability equivalence factor are given in table 2.1 at some values of γ .

Table 2.1: γ -Fractiles, $\rho^H(\gamma)$

γ	$L(\gamma)$	$L^H(\gamma)$	$\rho_1^H(\gamma)$	$\rho_2^H(\gamma)$
0.1	2.5744	2.9887	0.5870	0.6818
0.2	2.0778	2.5384	0.4942	0.5833
0.3	1.7389	2.2318	0.4117	0.4956
0.4	1.4648	1.9825	0.3301	0.4086
0.5	1.2334	1.7603	0.2442	0.3169
0.6	0.9987	1.5491	0.1491	0.2154
0.7	0.7794	1.3357	0.0381	0.0967
0.8	0.5546	1.1035	NA	NA
0.9	0.3084	0.8150	NA	NA

Based on the results presented in table 2.1,

1. One can recognize that $L(\gamma) < L^H(\gamma)$ for all studied cases, which confirms the results obtained for MTTF.
2. The hot duplication of the component increase $L(0.1)$ from $\frac{2.5744}{\alpha_1 + \alpha_2}$ to $\frac{2.9887}{\alpha_1 + \alpha_2}$.
3. The same effect on $L(0.1)$ can occur by reducing the failure rates of:

- (i) the first stage of the component, by the factor $\rho_1=0.5870$,
 - (ii) the second stage of the component, by the factor $\rho_2=0.6818$, see table 2.1.
4. In the same manner one can read the rest results.
 5. The notation NA in table 2.1 means that the value of SREF is not available and therefore there is possible equivalence between the system improved by reduction method and that system improved by using the redundancy method at this level.

And the mean reliability equivalence factors are $\xi_1^H = 0.1626, \xi_2^H = 0.1064$ then one can conclude that, the improved system that can be obtained by improving the system according to hot duplication method, has the same mean time to failure of that system which can be obtained by doing one of the following:

- (i) reducing the failure rate of the first stage by the factor $\xi=0.1626$,
- (ii) reducing the failure rate of the second stage by the factor $\xi=0.1064$.

2.6 Special Cases

We can calculate the reliability equivalence factors for the special cases from the present system as follows

1. if $\alpha_i=0$, equation (2.4), can be reduced to

$$\left. \begin{aligned} h_i(t) &= \beta_i \\ R_i(t) &= \exp\{-\beta_i t\} \end{aligned} \right\} \quad (2.16)$$

in this case we have the Exponential distribution with parameter β_i .

2. if $\beta_i=0$, equation (2.4), can be reduced to

$$\left. \begin{aligned} h_i(t) &= \alpha_i t \\ R_i(t) &= \exp\left\{-\frac{\alpha_i}{2} t^2\right\} \end{aligned} \right\} \quad (2.17)$$

in this case we have the Rayleigh distribution with parameter α_i .

Therefore, the reliability equivalence factor for the system with two stages of the life time distributions, when the life time of the stages is exponential or Rayleigh distribution can be obtained as special cases from the studied system in this section.

3. MIXTURE OF LINEAR INCREASING FAILURE RATES

In this section, we consider a system whose components fail if they enter either of two stages of failure mechanisms. The first mechanism is due to excessive voltage, and the second is due to excessive temperature. Suppose that the failure mechanism enters the first stage with probability θ_1 , and the failure rate is of the failure time is $h_1(t)$. It enters the second stage with probability θ_2 and the failure rate of its failure time $h_2(t)$. The

failure of a component occurs at the end of either first stage or the second stage. Hence the failure rate of the component i is

$$h_i(t) = \theta_1 h_1(t) + \theta_2 h_2(t) \quad (3.1)$$

The reliability function of the component i is given as follows

$$\begin{aligned} R_i(t) &= \exp \left\{ - \int_0^t h_i(u) du \right\} \\ &= \exp \left\{ - (\theta_1 H_1(t) + \theta_2 H_2(t)) \right\} \end{aligned} \quad (3.2)$$

where $H_i(t) = \int_0^t h_i(u) du$ be the cumulative failure rate of the stage i , $i=1, 2$.

In this section, we consider a simple system with n -independent and identical components in series system. Each component has two stages of the hazard (failure) rate functions.

3.1 The Original System

In this section, we consider the stages for component i , have the linear increasing failure rates as follows

$$h_i(t) = \alpha_i t + \beta_i, \quad i = 1, 2. \quad (3.3)$$

and the reliability function for the component i becomes

$$R_i(t) = \exp \left\{ - \left[(\theta_1 \alpha_1 + \theta_2 \alpha_2) \frac{t^2}{2} + (\theta_1 \beta_1 + \theta_2 \beta_2) t \right] \right\}. \quad (3.4)$$

Then component i has linear increasing failure rate distribution with parameters ϑ, v where $\vartheta = \theta_1 \alpha_1 + \theta_2 \alpha_2$ and $v = \theta_1 \beta_1 + \theta_2 \beta_2$.

Therefore, the reliability function of n -independent and identical components in series system is given as

$$R(t) = \exp \left\{ -n \left[(\theta_1 \alpha_1 + \theta_2 \alpha_2) \frac{t^2}{2} + (\theta_1 \beta_1 + \theta_2 \beta_2) t \right] \right\}. \quad (3.5)$$

Let MTTF be the system mean time to failure, which is given by

$$\text{MTTF} = \sqrt{\frac{\pi}{2n\vartheta}} \exp \left\{ \frac{nv^2}{2\vartheta} \right\} \left[1 - \text{erf} \left(v \sqrt{\frac{n}{2\vartheta}} \right) \right] \quad (3.6)$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp \{-t^2\} dt$ be the error function.

3.2 The Improved Systems

3.2.1 Reduction Method

In this method, we can reduce the mixture failure rate by reducing the stages of the failure rates by the factor $\rho_i, 0 < \rho_i < 1, i = 1, 2$. Let $R_{\rho,r}(t)$ be the reliability function of the system improved when reducing the failure rates of r components. One can obtain the function as follows

$$R_{\rho,r}(t) = \exp \left\{ - \left[(r\theta_1\rho_1 + n - r) \left(\frac{\alpha_1}{2} t^2 + \beta_1 t \right) + (r\theta_2\rho_2 + n - r) \left(\frac{\alpha_2}{2} t^2 + \beta_2 t \right) \right] \right\}. \quad (3.7)$$

From equation (3.7), the mean time to failure of the improved system, say $MTTF_{\rho,r}$ becomes

$$MTTF_{\rho,r} = \sqrt{\frac{\pi}{2(a\alpha_1 + b\alpha_2)}} \left[1 - \operatorname{erf} \left(\frac{a\beta_1 + b\beta_2}{\sqrt{2(a\alpha_1 + b\alpha_2)}} \right) \right] \exp \left\{ \frac{(a\beta_1 + b\beta_2)^2}{2(a\alpha_1 + b\alpha_2)} \right\} \quad (3.8)$$

where $a = r\theta_1\rho_1 + n - r$, $b = r\theta_2\rho_2 + n - r$.

3.2.2 Hot Duplication Method

Let $R_m^H(t)$ be the reliability function of the improved system assuming hot duplication of m system components. The function $R_m^H(t)$ can be obtained as

$$R_m^H(t) = \sum_{k=0}^m \left\{ \binom{m}{k} 2^k (-1)^{m-k} \exp \left\{ -(m-k+n) \left[(\theta_1\alpha_1 + \theta_2\alpha_2) \frac{t^2}{2} + (\theta_1\beta_1 + \theta_2\beta_2) t \right] \right\} \right\}. \quad (3.9)$$

Let $MTTF_m^H$ denote the mean time to failure of the system improved in this case.

Using equation (3.9), one can obtain $MTTF_m^H$ as

$$MTTF_m^H(t) = \sum_{k=0}^m \left\{ \binom{m}{k} 2^k (-1)^{m-k} \sqrt{\frac{\pi}{2(m-k+n)(\theta_1\alpha_1 + \theta_2\alpha_2)}} \exp \left\{ \frac{(m-k+n)(\theta_1\beta_1 + \theta_2\beta_2)^2}{2(\theta_1\alpha_1 + \theta_2\alpha_2)} \right\} \left[1 - \operatorname{erf} \left((\theta_1\beta_1 + \theta_2\beta_2) \sqrt{\frac{m-k+n}{2(\theta_1\alpha_1 + \theta_2\alpha_2)}} \right) \right] \right\}. \quad (3.10)$$

3.3 The γ -Fractiles

This section presents the γ -fractiles of the original and improved systems. Let $L(\gamma)$, $L_m^H(\gamma)$ be the γ -fractiles of the original and improved system assuming hot duplication

method. The γ -fractiles $L(\gamma)$ and $L_m^H(\gamma)$, are defined as the solution of the following equations, respectively,

$$R\left(\frac{L(\gamma)}{n(\alpha_1 + \alpha_2)}\right) = \gamma, \quad R_m^H\left(\frac{L(\gamma)}{n(\alpha_1 + \alpha_2)}\right) = \gamma. \quad (3.11)$$

It follows from equation (3.5) and the first equation of (3.6) that $L=L(\gamma)$ satisfies the following equation

$$\frac{\theta_1\alpha_1 + \theta_2\alpha_2}{2(\alpha_1 + \alpha_2)^2} L^2 + \frac{\theta_1\beta_1 + \theta_2\beta_2}{\alpha_1 + \alpha_2} L + \frac{\ln \gamma}{n} = 0. \quad (3.12)$$

By using $L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then L can be obtained as

$$L = \frac{1}{2} \left[-1 + \sqrt{1 - \frac{8(\alpha_1 + \alpha_2)^2}{n(\theta_1\alpha_1 + \theta_2\alpha_2)} \ln \gamma} \right]. \quad (3.13)$$

From the second equation of (3.11), and equation (3.9), one can verify that $L = L_m^H(\gamma)$ satisfies the following equation

$$\sum_{k=0}^m \left\{ \binom{m}{k} 2^k (-1)^{m-k} \exp \left\{ -(m-k+n) \left[\frac{\theta_1\alpha_1 + \theta_2\alpha_2}{2(\alpha_1 + \alpha_2)^2} L^2 + \frac{\theta_1\beta_1 + \theta_2\beta_2}{\alpha_1 + \alpha_2} L \right] \right\} \right\} = \gamma. \quad (3.14)$$

Equation (3.14) has no closed form solution and can be solved using numerical method technique.

3.4 Reliability Equivalence Factors

Now we are ready to derive the reliability equivalence factors of the system. We will deduce survival reliability equivalence factor, say SREF and mean reliability equivalence factor, say MREF of the underlying system as follows.

The first type of reliability equivalence factor, SREF say $\rho_{m,r}^H(\gamma)$, can be obtained by solving the following system of equations

$$R_m^H(t) = R_{r,\rho}(t) = \gamma. \quad (3.15)$$

Let us now explain how one can deduce the second type of reliability equivalence factor. This type is MREF say $\xi_{m,r}^H(\gamma)$ that can be obtained by solving the following system of equations

$$\text{MTTF}_m^H = \text{MTTF}_{r,\rho}. \quad (3.16)$$

Equations (3.15) and (3.16) have no closed form, which can be solved numerical by using numerical method technique.

3.5 Numerical Results and Conclusions

To explain how one can apply theoretical results obtained in the previous subsections, we introduce a numerical example. In this example, we assume three components series system with identical components, $h_1(t)=2t + 0.09$, $h_2(t)= t + 0.08$, $\theta_1=0.45$ and $\theta_2=0.55$. The mean time to failure of the original system is $MTTF=0.54681$ and are given in table 3.1 as follows

Table 3.1: The $MTTF_m^H$

M	1	2	3
$MTTF_m^H$	0.626718	0.732113	0.874151

From table 3.1, one can conclude that $MTTF < MTTF_m^H$ for all $m=1, 2, 3$, and increases with m .

The γ -fractiles, $L(\gamma)$, $L_m^H(\gamma)$ and the values of $\rho_m^H(\gamma)$ are calculated using Mathematica Program system according to the previous theoretical formulae. In these calculations the level γ is chosen to be 0.1, 0.2, 0.3, ..., 0.9.

The γ -fractile of the original and improved system are given in table 3.2 at some values of γ

Table 3.2: The γ -Fractiles

γ	L	$L_m^H(\gamma)$		
		m=1	m=2	m=3
0.1	2.6269	3.2181	3.5405	3.8721
0.2	2.1286	2.7121	3.0508	3.4119
0.3	1.7874	2.3570	2.7044	3.0888
0.4	1.5104	2.0617	2.4133	2.8189
0.5	1.2658	1.7939	2.1456	2.5724
0.6	1.0375	1.5359	1.8828	2.3318
0.7	0.8137	1.2732	1.6083	2.0816
0.8	0.5832	0.9879	1.2984	1.7997
0.9	0.3282	0.6430	0.8987	1.4319

Based on the results presented in table 3.2, one can recognize that $L(\gamma) < L_m^H(\gamma)$, for all studied cases, which confirms the results obtained for MTTF.

The values of the SREF are given in (i) table 3.3, when we reduce the first stage by the factor ρ_1 , (ii) table 3.4, when we reduce the second stage by the factor ρ_2 and (iii) table 3.5, when the first and second stages by the same factor ρ , at some values of γ .

Table 3.3: The SREF $\rho_1^H(\gamma)$

α	m=1			m=2			m=3		
	r=1	r=2	r=3	r=1	r=2	r=3	r=1	r=2	r=3
0.1	0.1974	0.5794	0.7196	NA	0.2395	0.4930	NA	NA	0.3150
0.2	NA	0.5088	0.6725	NA	0.1140	0.4093	NA	NA	0.2082
0.3	NA	0.4545	0.6586	NA	0.0685	0.3789	NA	NA	0.1661
0.4	NA	0.4068	0.6318	NA	NA	0.3272	NA	NA	0.0983
0.5	NA	0.4119	0.6079	NA	NA	0.2788	NA	NA	0.0336
0.6	NA	0.3795	0.5863	NA	NA	0.2316	NA	NA	NA
0.7	NA	0.3502	0.5668	NA	NA	0.1844	NA	NA	NA
0.8	NA	0.3263	0.5509	NA	NA	0.1371	NA	NA	NA
0.9	NA	0.3183	0.5455	NA	NA	0.0952	NA	NA	NA

Table 3.4: The SREF $\rho_2^H(\gamma)$

α	m=1			m=2			m=3		
	r=1	r=2	r=3	r=1	r=2	r=3	r=1	r=2	r=3
0.1	NA	0.3508	0.5672	NA	NA	0.2137	NA	NA	NA
0.2	NA	0.2491	0.4517	NA	NA	0.0909	NA	NA	NA
0.3	NA	0.1734	0.4489	NA	NA	NA	NA	NA	NA
0.4	NA	0.1095	0.4063	NA	NA	NA	NA	NA	NA
0.5	NA	0.0526	0.2775	NA	NA	NA	NA	NA	NA
0.6	NA	NA	0.2219	NA	NA	NA	NA	NA	NA
0.7	NA	NA	0.1609	NA	NA	NA	NA	NA	NA
0.8	NA	NA	0.0851	NA	NA	NA	NA	NA	NA
0.9	NA	NA	NA	NA	NA	NA	NA	NA	NA

Table 3.5: The SREF $\rho^H(\gamma)$

α	m=1			m=2			m=3		
	r=1	r=2	r=3	r=1	r=2	r=3	r=1	r=2	r=3
0.1	0.4895	0.7448	0.8298	0.0753	0.537645	0.691764	NA	0.3743	0.5829
0.2	0.4061	0.7031	0.8020	NA	0.462959	0.641972	NA	0.2784	0.5189
0.3	0.3427	0.6714	0.7809	NA	0.404783	0.603188	NA	0.2036	0.4691
0.4	0.2879	0.6439	0.7626	NA	0.352969	0.568646	NA	0.1367	0.4245
0.5	0.2376	0.6188	0.7459	NA	0.303598	0.535732	NA	0.0724	0.3816
0.6	0.1897	0.5949	0.7299	NA	0.254303	0.502869	NA	0.0071	0.3381
0.7	0.1429	0.5715	0.7143	NA	0.203032	0.468688	NA	NA	0.2916
0.8	0.0963	0.5481	0.6988	NA	0.14729	0.431527	NA	NA	0.2387
0.9	0.0489	0.5245	0.6828	NA	0.082929	0.388619	NA	NA	0.1709

Based on the results presented in tables 3.3, 3.4 and 3.5 one can conclude that:

1. The hot duplication of the one component, $m=1$, increases $L(0.1)$ from $\frac{2.6269}{n(\alpha_1 + \alpha_2)}$ to $\frac{3.2181}{n(\alpha_1 + \alpha_2)}$, see table 3. 2. The same effect on $L(0.1)$ can occur by reducing the failure rates of:

- (1.1) the first stage of
 - (i) one component, $r = 1$, by the factor $\rho = 0.1974$,
 - (ii) two components, $r = 2$, by the factor $\rho = 0.57948$,
 - (iii) three components, $r = 2$, by the factor $\rho = 0.7196$, see table 3.3.
- (1.2) the second stage of
 - (i) two components, $r = 2$, by the factor $\rho = 0.3508$,
 - (ii) three components, $r = 3$, by the factor $\rho = 0.5672$, see table 3.4
- (1.3) the first and second stages of
 - (i) one component, $r = 1$, by the same factor $\rho = 0.4895$,
 - (ii) two components, $r = 2$, by the same factor $\rho = 0.7448$,
 - (iii) three components, $r = 2$, by the same factor $\rho = 0.8298$, see table 3.5.
- 2. In the same manner one can read the rest results.
- 3. The notation NA in tables 3.3, 3.4 and 3.5 means that the value of SREF is not available and therefore there is possible equivalence between the system improved by HDM and that system improved by using the redundancy method at this level.

Table 3.6 introduces the values of the mean reliability equivalence factor, when we reduce the first stage by ξ_1^H , second stage by ξ_2^H , and the two stages by the same factor ξ^H .

Table 3.6: The MREF

r	ξ_1^H			ξ_2^H			ξ^H		
	m=2	m=3	m=1	m=2	m=3	m=1	m=2	m=3	m=1
1	NA	NA	NA	NA	NA	NA	0.3339	NA	NA
2	0.4444	NA	NA	0.1688	NA	NA	0.6669	0.3742	0.1267
3	0.6296	0.3069	0.0371	0.4458	NA	NA	0.7779	0.5828	0.4178

One can conclude that:

- 1. the improved system that can be obtained by improving the one component of the system components, $m=1$, according to HDM, has the same mean time to failure of that system which can be obtained by reducing
 - (1.1) the first stage of (i) two components, $r=2$ by the factor $\xi = 0.4444$, (ii) three components, $r=3$ by the factor, $\xi = 0.6296$,
 - (1.2) the second stage of (i) two components, $r = 2$ by the factor, $\xi = 0.1688$, (ii) three components, $r = 3$ by the factor, $\xi = 0.4458$,
 - (1.3) the two stages of (i) one component, $r = 1$ by the same factor, $\xi = 0.3339$, (ii) two components, $r = 2$ by the same factor, $\xi = 0.6669$, (iii) three components, $r = 3$ by the same factor, $\xi = 0.7779$.
- 2. In the same manner one can read the rest results.
- 3. The notation NA in table 3.6 means that the value of MREF is not available and therefore there is possible equivalence between the system improved by HDM and that system improved by using the redundancy method at this level.

3.6 Special Cases

We can calculate the reliability equivalence factors for the special cases from the present system as follows

1. if $\alpha_i = 0$, the failure rates of the stages are constant. We have the mixture of the constant failure rates, equation (20, 21), can be reduced to

$$\left. \begin{aligned} h_i(t) &= \beta_i, \\ R_i(t) &= \exp\{-(\theta_1\beta_1 + \theta_2\beta_2)t\}. \end{aligned} \right\} \quad (3.17)$$

in this, the component with mixture failure rate has the Exponential distribution with parameter $\theta_1\beta_1 + \theta_2\beta_2$.

2. if $\beta_i = 0$, the failure rates of the stages are increasing failure rate. We have the mixture of the increasing failure rates, equation (3.3, 3.4), can be reduced to

$$\left. \begin{aligned} h_i(t) &= \alpha_i t, \\ R_i(t) &= \exp\left\{-\left(\theta_1\alpha_1 + \theta_2\alpha_2\right)\frac{t^2}{2}\right\}. \end{aligned} \right\} \quad (3.18)$$

in this case, the component with mixture failure rate has the Rayleigh distribution with parameter $\theta_1\alpha_1 + \theta_2\alpha_2$.

3. if $\alpha_i = \beta_j = 0, i \neq j, i, j = 1, 2$, the failure rates of the stages i is constant and the failure rate of the stage j is increasing failure rate. we have the mixture of the constant and increasing failure rates, equation (3.3, 3.4), can be reduced to

$$\left. \begin{aligned} h_i(t) &= \beta_i, \quad h_j(t) = \alpha_j t, \\ R_i(t) &= \exp\left\{-\left(\theta_j\alpha_j\frac{t^2}{2} + \theta_i\beta_i t\right)\right\}. \end{aligned} \right\} \quad (3.19)$$

in this case, the component with mixture failure rate has the linear increasing failure rate distribution with parameters $\theta_j\alpha_j$, and $\theta_i\beta_i$

Therefore, the reliability equivalence factor for the system with two stages of the failure rates, when the mixture failure rate of the components are exponential or Rayleigh or linear increasing failure rate distribution can be obtained as a special cases from the studied system in this section.

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