Pak. J. Statist.
2009 Vol. 25(2), 135-140

ON NEGATIVE MOMENTS OF CERTAIN DISCRETE DISTRIBUTIONS

Masood Anwar[1] and **Munir Ahmad**[2]
[1] Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan. Email: masoodanwar@comsats.edu.pk
[2] National College of Business Administration & Economics, Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

Negative moments of certain discrete probability distributions in terms of hypergeometric power series functions are obtained.

KEY WORDS

Negative moments; discrete distributions.

1. INTRODUCTION

Recently negative moments have been studied by Roohi (2003) who obtained negative moments of some discrete distributions in terms of hypergeometric series functions. In this paper we have extended her work by considering further discrete probability distributions and expressed the moments in terms of newly defined generalized hypergeometric series function.

2. NEGATIVE MOMENTS OF SOME DISCRETE DISTRIBUTIONS

Theorem 2.1

Let X be a geometric-compound random variable, with parameters α and β having probability mass function (pmf)

$$P(X=x)=\frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+x-1)\,\Gamma(\beta+1)}{\Gamma\alpha\;\;\Gamma\beta\;\;\Gamma(\alpha+\beta+x)},\ \alpha>0,\ \ \beta>0,\ \ x=1,2,\ldots . \tag{2.1}$$

The negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{\beta}{(A+1)^{k}\,(\alpha+\beta)}\,{}_{3}H_{2}\left[(A+1,k),\alpha,1;(A+2,k),(\alpha+\beta+1);1\right], \tag{2.2}$$

where $A\geq 0$.

Proof:

Since X is a geometric-compound random variable with parameters α and β then

$$E(X+A)^{-k} = \frac{\Gamma(\alpha+\beta)\Gamma(\beta+1)}{\Gamma\alpha\Gamma\beta}\sum_{x=1}^{\infty}\frac{1}{(x+A)^{k}}\frac{\Gamma(\alpha+x-1)}{\Gamma(\alpha+\beta+x)},$$

$$= \frac{\beta}{(A+1)^{k}(\alpha+\beta)}\left[1+\frac{(A+1)^{k}\alpha\ \ 1}{(A+2)^{k}(\alpha+\beta+1)}\right.$$

$$\left. + \frac{(A+1)^{k}(A+2)^{k}\alpha(\alpha+1)\ \ 1.2}{(A+2)^{k}(A+3)^{k}(\alpha+\beta+1)(\alpha+\beta+2)}\frac{1}{2!}+\ldots\right]$$

$$= \frac{\beta}{(A+1)^{k}(\alpha+\beta)}\,{}_{3}H_{2}\left[(A+1,k),\alpha,1;(A+2,k),(\alpha+\beta+1);1\right].$$

where

$${}_{p}H_{q}\left[(a_1,k),(a_2,k),\ldots,(a_p,k);(b_1,k)(b_2,k),\ldots,(b_q,k);z\right]$$

$$= 1+\frac{a_1^{k}.a_2^{k}\ldots a_p^{k}}{b_1^{k}.b_2^{k}\ldots b_q^{k}}z+\frac{\left[a_1(a_1+1)\right]^{k}\left[a_2(a_2+1)\right]^{k}\ldots.\left[a_p(a_p+1)\right]^{k}}{\left[b_1(b_1+1)\right]^{k}\left[b_2(b_2+1)\right]^{k}\ldots.\left[b_q(b_q+1)\right]^{k}}\frac{z^{2}}{2!}+\ldots$$

is a generalized hypergeometric series function with usual conditions (Ahmad, 2008).

If $k=1$, then ${}_{p}H_{q} = {}_{p}F_{q}$.

If $k=2$, then ${}_{p}H_{q}$ is ${}_{2p}F_{2q}\left[a_1,a_1,a_2,a_2,\ldots,a_p,a_p;b_1,b_1,b_2,b_2,\ldots,b_q,b_q;z\right]$,

In general ${}_{p}H_{q} = {}_{kp}F_{kq}$.

If $k's$ are different say k_i , then ${}_{p}H_{q} = {}_{p\sum_{i=1}^{p}k_i}F_{q\sum_{i=1}^{q}k_i}$.

If $k=1$, then negative moment is

$$E(X+A)^{-1} = \frac{\beta}{(A+1)(\alpha+\beta)}\,{}_{3}H_{2}\left[(A+1,1),\alpha,1;(A+2,1),(\alpha+\beta+1);1\right].$$

Theorem 2.2

Let X be a beta-binomial random variable with parameters α , $\alpha>0$, β , $\beta>0$ and pmf

$$P(X=x) = \binom{n}{x}\frac{\Gamma(\alpha+\beta)\ \ \Gamma(x+\alpha)\ \ \Gamma(n+\beta-x)}{\Gamma\alpha\ \ \Gamma\beta\ \ \Gamma(n+\alpha+\beta)},\ x=0,1,2,\ldots.,n, \qquad (2.3)$$

then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k}\;{}_3H_2\left[(A,k),\alpha,-n;(A+1,k),-n-\beta+1;1\right],\; A>0\,, \tag{2.4}$$

where $P_0 = P(X=0) = \dfrac{\Gamma(\alpha+\beta)\Gamma(n+\beta)}{\Gamma\beta\,\Gamma(n+\alpha+\beta)}$.

Proof:

Suppose X is a beta-binomial random variable with parameters α and β then

$$E(X+A)^{-k} = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta\,\Gamma(n+\alpha+\beta)}\sum_{x=0}^{n}\binom{n}{x}\frac{\Gamma(\alpha+x)\Gamma(n+\beta-x)}{(x+A)^k}\,,$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(n+\beta)}{A\Gamma\beta\,\Gamma(n+\alpha+\beta)}\left[1+\frac{A^k\,\alpha\,(-n)}{(A+1)^k\,(-n-\beta+1)}\right.$$

$$\left.+\frac{A^k\,(A+1)^k\,\alpha(\alpha+1)\,(-n)(-n+1)}{(A+1)^k\,(A+2)^k\,\Gamma(-n-\beta+1)\Gamma(-n-\beta+2)}\frac{1}{2!}+\ldots\right],$$

$$= \frac{P_0}{A^k}\;{}_3H_2\left[(A,k),\alpha,-n;(A+1,k),-n-\beta+1;1\right].$$

If $k=1,$ then negative moment is

$$E(X+A)^{-1} = \frac{P_0}{A}\;{}_3H_2\left[(A,1),\alpha,-n;(A+1,1),-n-\beta+1;1\right].$$

Theorem 2.3

Let X be a hypergeometric random variable, with parameters $a,\ a>0$, $b,\ b>0$ and pmf

$$P(X=x) = \binom{a}{x}\binom{b}{n-x}\Bigg/\binom{a+b}{n},\; x = 0,1,2,\ldots.,\min(n,a)\,, \tag{2.5}$$

then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k}\;{}_3H_2\left[(A,k),-a,-n;(A+1,k),b-n+1;1\right],\; A>0 \tag{2.6}$$

where $P_0 = P(X=0) = \dfrac{b!(a+b-n)!}{(b-n)!(a+b)!}$.

Proof:

Suppose X is a hypergeometric random variable with parameters a and b then

$$E(X+A)^{-k} = \sum_{x=0}^{n} \frac{1}{(x+A)^k} \binom{a}{x}\binom{b}{n-x} \Big/ \binom{a+b}{n}$$

$$= \frac{b!(a+b-n)!}{A^k (b-n)!(a+b)!}\left[1+\frac{A^k\,(-a)(-n)}{(A+1)^k\,(b-n+1)}\right.$$

$$\left. + \frac{A^k\,(A+1)^k\,(-a)(-a+1)(-n)(-n+1)}{(A+1)^k\,(A+2)^k\,(b-n+1)(b-n+2)}\frac{1}{2!}+\cdots\right],$$

$$E(X+A)^{-k} = \frac{P_0}{A^k}\ {}_3H_2\left[(A,k),-a,-n;(A+1,k),b-n+1;1\right].$$

If $k=1$, then negative moment is

$$E(X+A)^{-1} = \frac{P_0}{A}\ {}_3H_2\left[A,-a,-n;A+1,b-n+1;1\right].$$

Theorem 2.4

Let X be a Waring random variable, with parameters $a,\ a \ge 2$, $c, c > a$ and pmf

$$P(X=x) = \frac{(c-a)(a+x-1)!(c)!}{c(a-1)!(c+x)!}, \quad c > a \ge 2,\ x = 0,1,2,.... \tag{2.7}$$

then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k}\ {}_3H_2\left[(A,k),a,1;(A+1,k),c+1;1\right],\ A > 0, \tag{2.8}$$

where $P_0 = P(X=0) = \dfrac{(c-a)}{c}$.

Proof:

Suppose X is a Waring random variable with parameters a and c then

$$E(X+A)^{-k} = \frac{c!(c-a)}{c(a-1)!}\sum_{x=0}^{\infty}\frac{1}{(x+A)^k}\frac{(a+x-1)!}{(c+x)!},$$

$$= \frac{(c-a)}{A^k\,c}\left[1+\frac{A^k\,(a).1}{(A+1)^k\,(c+1)}+\frac{A^k\,(A+1)^k\,(a)(a+1).1.2}{(A+1)^k\,(A+2)^k\,(c+1)(c+2)}\frac{1}{2!}+\cdots\right],$$

$$E(X+A)^{-k} = \frac{P_0}{A^k}\ {}_3H_2\left[(A,k),a,1;(A+1,k),c+1;1\right].$$

If $k=1$, then negative moment is

$$E(X+A)^{-1} = \frac{P_0}{A}\ {}_3H_2\left[A, a, 1; A+1, c+1; 1\right].$$

Corollary 2.1

If $a = 1$, the Waring function reduces to Yule probability function and Waring results holds for Yule function.

Theorem 2.5

Let X be a random variable having Poisson-binomial distribution with parameters n and $p, 0 \le p \le 1$, $a, a > 0$ and pmf

$$P(X = x) = e^{-a} \sum_{m=0}^{\infty} \frac{a^m}{m!} \binom{nm}{x} p^x (1-p)^{nm-x}\ , (n,m) \in \mathbb{Z}^+\ , x = 0,1,2,....,nm\ . \quad (2.9)$$

The negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{e^{-a}}{A^k} \sum_{m=0}^{\infty} \frac{a^m}{m!} (1-p)^{nm}\ {}_2H_1\left((A,k), -nm; (A+1,k); \frac{-p}{1-p} \right), A > 0\ , \quad (2.10)$$

Proof:

Suppose X is a Poisson-binomial random variable and negative moment of first order is given by

$$E(X+A)^{-k} = e^{-a} \sum_{x=0}^{nm} \frac{1}{(x+A)^k} \sum_{m=0}^{\infty} \frac{a^m}{m!} \binom{nm}{x} p^x (1-p)^{nm-x}\ ,$$

$$= \frac{e^{-a}}{A^k} \sum_{m=0}^{\infty} \frac{a^m}{m!} (1-p)^{nm} \left[1 + \frac{A^k(-nm)}{(A+1)^k} \left(\frac{-p}{1-p} \right) \right.$$

$$\left. + \frac{A^k (A+1)^k (-nm)(-nm+1)}{(A+1)^k (A+2)^k\, 2!} \left(\frac{-p}{1-p} \right)^2 + \cdots \right],$$

$$E(X+A)^{-k} = \frac{e^{-a}}{A^k} \sum_{m=0}^{\infty} \frac{a^m}{m!} (1-p)^{nm}\ {}_2H_1\left((A,k), -nm; (A+1,k); \frac{-p}{1-p} \right).$$

If $k = 1,$ then negative moment is

$$E(X+A)^{-1} = \frac{e^{-a}}{A} \sum_{m=0}^{\infty} \frac{a^m}{m!} (1-p)^{nm}\ {}_2H_1\left((A,1), -nm; (A+1,1); \frac{-p}{1-p} \right).$$

Corollary 2.2

Let X be a random variable having Hermite distribution with parameters $p, 0 \le p \le 1$ and $a,\ a > 0$, having pmf

$$P(X = x) = e^{-a} \sum_{m=0}^{\infty} \frac{a^m}{m!} \binom{2m}{x} p^x (1-p)^{2m-x},\ m \in \mathbb{Z}^+,\ x = 0,1,2,....,2m\,. \qquad (2.11)$$

The negative moment of k^{th} order is given by (2.10) when $n = 2$.

REFERENCES

1. Ahmad, M. and Saboor, A. (2009). Properties of a newly defined hypergeometric power series function. *Pak. J. Statist.*, (In press).
2. Rainville, E.D. (1960). *Special Functions.* Chelsea Publication Co. Bornmx. USA`.
3. Roohi, A. (2003). *Negative and Factorial Moments of Discrete Distributions Involving Hyper-Geometric Series Functions*. Unpublished Ph.D. dissertation, National College of Business Administration and Economics, Lahore.