

**ON CONSTRUCTION OF ONE DIMENSIONAL ALL ORDER
NEIGHBOR BALANCED DESIGNS BY CYCLIC SHIFTS**

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ABSTRACT

Iqbal *et al.* (2006) presented a theorem to construct second order neighbor designs which is generalized here for all order neighbor balanced designs. By using method of cyclic shifts, two algorithms are developed to construct all order neighbor balanced designs for v odd prime when (i) $k = v$ and (ii) k is relatively prime to $v(v-1)/2$.

KEY WORDS

Neighbor designs; second order neighbor designs; all order neighbor balanced designs; equineighbored designs.

1. INTRODUCTION

1.1 Neighbor Balanced Designs

Neighbor designs were initially used in serology. Rees (1967) presented a technique used in virus research, which requires the arrangement in circles of samples from a number of virus preparations in such a way that over the whole set, a sample from each virus preparation appears next to a sample from every other virus preparation. Experiments in agriculture, horticulture and forestry often show neighbor effects. Neighbor balanced designs ensure that treatment comparisons will be as little affected by competition / neighbor effects as possible. Ahmed and Akhtar (2008) developed several algorithms to construct neighbor designs.

1.2 Second Order Neighbor Balanced Designs

Keedwell (1984) considered 2-fold perfect circuit designs, these being balanced circuit designs whose neighbor properties apply not only to immediate neighbors but also to neighbors that are two places apart. Lindner et al. (1991) considered 2-perfect k -cycle systems of order v , i.e. balanced circuit RND's (Rees neighbor designs) whose neighbor properties hold both for immediate neighbors and for 2-places-apart neighbors. Iqbal *et al.* (2006) constructed second order neighbor designs for $3 \leq k \leq 7$ in circular blocks using method of cyclic shifts. They explained the logic behind their method of cyclic shifts as: If treatment '0' is on plot i in block j , then the treatment, a , say, on plot $(i+1) \bmod (k+1)$ is the right- neighbour of treatment '0' in that block. The treatment on plot $(i-1) \bmod (k+1)$ in block j , is the left- neighbour of treatment '0' in block j . The

cyclic method of construction ensures that if any treatment, a , say, is the right- neighbour of treatment '0' in block j then treatment $v-a \pmod v$ will be the left- neighbour of treatment '0' in some other block j' , ($j \neq j'$). This property makes the counting up of the right- and left- neighbors of treatment '0' quite easy. If a shift, q , say, is applied to treatment '0' then treatment a will be a right- neighbour of treatment '0'. In some other block treatment $v-a$ (named the complement of treatment a) will be a left- neighbor of treatment '0'. Therefore, given a set of shifts \mathbf{Q} , we can determine all right- and left- neighbors of treatment '0' from the shifts contained in \mathbf{Q} . The shifts q_1, q_2, \dots, q_{k-1} define the right- neighbors of treatment '0' and $q_1 + q_2 + \dots + q_{k-1} \pmod v$ defines the left- neighbour of treatment '0' in the first block. By taking the complements of these k shifts, we get all the remaining neighbors of treatment '0'. They also presented a theorem which is: A design is second order neighbor balanced if each shift appears an equal number (λ') of times in a new set of shifts where this new set of shifts consists of (i) the sum of every two and $(k-2)$ successive shifts, (ii) the complements of the shifts in (i). This theorem is generalized for ANBD in section 2. Method to generate the design through set of shifts is explained in section 1.4.

1.3 All Order Neighbor Balanced Designs

The designs in which every unordered pair of treatments appears as neighbors equally often at every order, are called all order neighbor balanced designs (AND). Ipinoyomi (1985) constructed all order neighbor balanced designs with minimum blocks for $v=2m+1$ and prime. He also constructed equineighbored designs using complete sets of orthogonal Latin squares and suggested that where a complete set of orthogonal Latin squares is not available, the concept of equineighbored difference set ($\pmod v$) can be employed in general to construct equineighbored designs. Ipinoyomi (1985) developed the methods to construct incomplete block equineighbored designs of size

- i) (v, mkv, k) , for $v = 3k + 1$, v is a prime power and k is not even.
- ii) $(v, mkv/2, k)$, for $v = 3k + 1$, v is a prime power and k is even.
- iii) (v, kv, k) , for $v = 2k + 1$ is prime.
- iv) $(v, mv, 3)$, for $v = 2m + 1$.
- v) (v, mv, k) , for $v = 2m + 1$, v is prime, $k < v$.
- vi) $(v, v(v-1), k)$, for $k \leq v$ by the columns of the k rows of a complete set of orthogonal Latin squares of order v .

Mingyao *et al.* (2007) constructed all order neighbor balanced designs for odd prime $v=2m+1$ in $2m$ circular blocks through the initial block. $B_i = \{(0, i, 2i, \dots, (v-1)i) \pmod v : i = 1, 2, \dots, v-1\}$. They also developed the methods of construction for all order balanced circular block neighbor designs when $k < v$ and showed that the CNBD's (circular neighbor balanced design) exists under the following conditions.

- i) Let v be an odd prime and $r \geq 1$ be an integer. Then there exists a CNBD $(v^r, v^{r-1} (v^{r-1}-1), v, v-1)$.
- ii) If $v = p^r$ be any prime power and $k > 2$ be such that k is a divisor of $v-1$, there exists a CNBD $(v, v(v-1)/k, k; k-1)$.

- iii) If q is a prime power and f is the greatest common divisor of $q-1$ and k , there exists a CNBD $(q, q(q-1)/f, k; k-1)$.
- iv) If $q = kn+1$ is a prime power, there exists a CNBD $(n, n(n-1), k; k-1)$.

1.4 Method of Cyclic Shifts

The method of cyclic shift given by Iqbal *et al.* (2006) is a particular way of constructing a cyclic block design, here, v treatments are labeled as $0, 1, 2, \dots, v-1$. To illustrate the method, let us consider the construction of a design for $v = 7$ and $k = 5$ in $b = 21$. The set of shifts defined by

$$Q_1 = [1, 1, 1, 1], Q_2 = [2, 2, 2, 2] \text{ and } Q_3 = [3, 3, 3, 3].$$

First row from each set of shifts will be $0, 1, 2, 3, 4, 5, 6$. Obtain second row by adding the first element of $Q_1 \pmod v$. Similarly third, fourth and fifth rows will be obtained by adding second, third and fourth element of Q_1 respectively. Following are the five rows developed from Q_1 . Each column will be considered as the block of the design.

| B1 | B2 | B3 | B4 | B5 | B6 | B7 |
|----|----|----|----|----|----|----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |

Fourteen more blocks will be obtained from Q_2 and Q_3 in the similar way.

To illustrate partial sets of blocks, we construct another design for $v = k = 5$ through the two sets of cyclic shifts $Q_1 = [1, 1, 1, 1](1/5)$ and $Q_2 = [2, 2, 2, 2](1/5)$. Rows will be obtained by adding shifts mode v as we have done above but here only first $(1/5)$ columns from each set will be taken as block of design.

- B1:** (0, 1, 2, 3, 4)
- B2:** (0, 2, 4, 1, 3)

2. GENERALIZATION OF THEOREM

Theorem 1:

A design is ℓ th- order neighbor balanced if all shifts along with their complements contain all elements $1, 2, \dots, v-1$ and each element appears an equal number (λ') of times in a new set of shifts where this new set of shifts consists of (i) the sum of every ℓ and $(k-\ell)$ successive shifts, (ii) the complements of the shifts in (i).

Proof:

If $i < k - \ell$, treatment '0' in row i is the left ℓ th - order neighbor of the treatment in row $(i + \ell) \bmod k$ and can be obtained by adding the corresponding ℓ successive shifts (i.e. the shifts used to construct the $(i + 1)$ th, $(i + 2)$ th, ..., $(i + \ell)$ th row).

If $i = k$, the left ℓ th - order neighbor can be found as the complement of the sum of $(k - \ell)$ successive shifts $q_\ell, q_{\ell+1}, \dots, q_{k-1}$.

If $i = k - 1$, the left ℓ th - order neighbour can be found as the complement of the sum of $(k - \ell)$ successive shifts $q_{\ell-1}, q_\ell, \dots, q_{k-2}$, and so on.

If $i = k - \ell$, the left ℓ th - order neighbor can be found as the complement of the sum of $(k - \ell)$ successive shifts $q_1, q_2, \dots, q_{k-\ell-1}$.

Similarly, if $i > \ell$ treatment '0' in row i is the right ℓ th - order neighbor of the treatment in row $(i - \ell)$ and can be obtained by taking the complement of the sum of every ℓ successive shifts.

If $i \leq \ell$ treatment '0' is the right ℓ th - order neighbor of the treatment in row $(k - \ell + i)$ and can be obtained by taking the sum of every $k - \ell$ successive shifts.

Since these two contain all the values of list (i) & (ii) and each element appears an equal number (λ') of times, therefore, the design is ℓ th - order neighbor balanced with $\lambda' = 2r / (v - 1)$. In the next section, ANBD are constructed by method of cyclic shifts.

3. CONSTRUCTION OF ALL ORDER NEIGHBOR BALANCED DESIGNS USING CYCLIC SHIFTS

3.1 Construction of all Order Neighbor Balanced Designs for v Prime and $k = v$

Proposition 1:

Let $v = 2m + 1$ and $k = v$, where v is prime, then following m sets of shifts, each of $k - 1$ elements, always give the ANBD in m circular blocks with $\lambda' = 1$.

$$Q_i = [i, i, \dots, i](1/v) : i = 1, 2, \dots, m.$$

Proof:

New set of shifts

$$i) S_i = [\ell \times i, \ell \times i, \dots, \ell \times i, (k - \ell) \times i, \dots, (k - \ell) \times i](1/v) \bmod v : i = 1, 2, \dots, m.$$

where in S_i , $\ell \times i$ appears $(k - \ell)$ times and $(k - \ell) \times i$ appears ℓ times.

$$ii) v - S_i : i = 1, 2, \dots, m.$$

Here, $\ell \times i \bmod v = k - (\ell \times i) \bmod v$, hence new sets of shifts (i) and (ii) contain all elements $1, 2, \dots, v - 1$ and each appears $k = v$ times. In each Q_i , there is a fraction v i.e. $(1/v)$, by this each element $1, 2, \dots, v - 1$ appears once, hence $\lambda' = 1$. \square

Example 1:

The ANBD for $v = k = 11$ can be constructed through the following five sets of cyclic shifts, each of ten elements

$$Q_1 = [1, 1, \dots, 1](1/11), Q_2 = [2, 2, \dots, 2](1/11), \dots, Q_5 = [5, 5, \dots, 5](1/11)$$

The required design is

- B1:** (0, 1, 2, 1 4, 5, 6, 7, 8, 9, 10)
- B2:** (0, 2, 4, 6, 8, 10, 1, 3, 5, 7, 9)
- B3:** (0, 3, 6, 9, 1, 4, 7, 10, 2, 5, 8)
- B4:** (0, 4, 8, 1, 5, 9, 2, 6, 10, 3, 7)
- B5:** (0, 5, 10, 4, 9, 3, 8, 2, 7, 1, 6)

3.2 Construction of all Order Neighbor Balanced Designs for v Prime and k Relatively Prime to $v(v-1)/2$

Proposition 2:

Let $v = 2m+1$, where v is prime and k is relatively prime to $v(v-1)/2$, then following m sets of shifts, each of $k-1$ elements, always give the ANBD in mv circular binary blocks with $\lambda' = k$.

$$Q_i = [i, i, \dots, i] : i = 1, 2, \dots, m.$$

Proof:

New set of shifts

- i) $S_i = [\ell \times i, \ell \times i, \dots, \ell \times i, (k-\ell) \times i, \dots, (k-\ell) \times i] \text{ mod } v : i = 1, 2, \dots, m,$
 where in S_i , $\ell \times i$ appears $(k-\ell)$ times and $(k-\ell) \times i$ appears ℓ times.
- ii) $v - S_i : i = 1, 2, \dots, m.$

Obviously, new sets of shifts (i) and (ii) contain all elements $1, 2, \dots, v-1$ and each appears k times, therefore, it is all order neighbor balanced design with $\lambda' = k$. \square

Example 2:

The ANBD for $v = 7$ and $k = 4$ can be constructed through the following three sets of cyclic shifts.

$$Q_1 = [1, 1, 1], Q_2 = [2, 2, 2] \text{ and } Q_3 = [3, 3, 3].$$

Required design is in 21 blocks.

- | | | |
|-------------------------|--------------------------|--------------------------|
| B1: (0, 1, 2, 3) | B8: (0, 2, 4, 6) | B15: (0, 3, 6, 2) |
| B2: (1, 2, 3, 4) | B9: (1, 3, 5, 0) | B16: (1, 4, 0, 3) |
| B3: (2, 3, 4, 5) | B10: (2, 4, 6, 1) | B17: (2, 5, 1, 4) |
| B4: (3, 4, 5, 6) | B11: (3, 5, 0, 2) | B18: (3, 6, 2, 5) |
| B5: (4, 5, 6, 0) | B12: (4, 6, 1, 3) | B19: (4, 0, 3, 6) |
| B6: (5, 6, 0, 1) | B13: (5, 0, 2, 4) | B20: (5, 1, 4, 0) |
| B7: (6, 0, 1, 2) | B14: (6, 1, 3, 5) | B21: (6, 2, 5, 1) |

4. ANALYSIS OF THE DESIGNS

For analyzing the all order neighbor balanced design, the appropriate model would be

$$Y = X_0\mu + X_1\tau + X_2\beta + X_3\alpha + X_4\gamma + \dots + X_a\eta + \varepsilon,$$

where Y is the $bk \times 1$ vector of response, μ is a general mean, X_0 is the $bk \times 1$ vector of 1's, X_1 is the $bk \times v$ incidence matrix for treatment effects X_2 is the $bk \times b$ incidence matrix for blocks effects. X_3, X_4, \dots, X_a are the $bk \times v$ incidence matrix for first order, second order, ..., all order neighbor effects respectively. β is $b \times 1$ vector of block effects. $\tau, \alpha, \gamma, \lambda, \dots, \eta$ are the $v \times 1$ vector of treatment effects, first order, second order, ..., all order neighbor effects respectively. ε is the $bk \times 1$ vector of random errors, after considering the neighbor effects it is assumed that $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2$.

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