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ON SOME CHARACTERIZATIONS OF UNIVARIATE DISTRIBUTIONS BASED ON TRUNCATED MOMENTS OF ORDER STATISTICS

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ABSTRACT

Some characterizations of well known univariate continuous distributions based on a truncated moment of the ith order statistics are presented. It is shown that some of the known results are special cases of the results presented in this paper.

1. INTRODUCTION

The problem of characterizations of distributions which has attracted the interest of many researchers. Suppose $X_1, X_2, \dots, X_n$ be a random sample of size n from an absolutely continuous distribution with cumulative distribution function (cdf) $F(x)$ and the corresponding probability density function (pdf) $f(x)$.

Let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$ be the corresponding order statistics. Ahsanullah and Hamedani (2007) have characterized beta of the first kind and the power function distribution using 1st order and n^{th} order statistics respectively. Hamedani et al. (2008) characterized certain univariate distributions using truncated moment of $X_{1,n}$. We like to mention here the works of Galambos and Kotz (1978), Kotz and Shanbag (1980), Ahsanullah (1973, 1989), Glanzel (1987), Oncel et al. (2005) and Wesolowski and Ahsanullah (2004). In this paper characterizations of some univariate distributions based on the truncated moments of the i^{th} $(i \geq 1)$ order statistic are given. It is shown that some of the known results are special cases of the results of this paper.

2. MAIN RESULTS

It can be shown easily that for a positive random variable with $F(0) = 0$ and $F(b) = 1$ with $\mathrm{E}(X^{\alpha})$, as finite, then

$$E\left(x_{i,n}^{\alpha} \middle| x_{i,n} > t\right) = t^{\alpha} + \frac{\int_t^b \alpha x^{\alpha-1}(1-F(x))^{n-i+1}dx}{(1-F(t))^{n-i+1}}.$$

We present here a theorem based on the truncated moment of the i^{th} order statistic.

Theorem 2.1

Let $X: \Omega \to (a,b), a \geq 0$ be a continuous random variable with cdf $F(x)$ and $\lim_{x\to b} x^{\alpha}(1-F(x)) = 0$ for some $\alpha > 0$. We assume $g(x, i.n)$ is differentiable with respect to x and $\int_a^b \frac{\alpha x^{\alpha-1}}{g(x,i,n)} dx = \infty$. Then

$$\mathrm{E}\left(x_{i,n}^{\alpha} \middle| x_{i,n} > t\right) = t^{\alpha} + g(t,i,n), a < t < b, \tag{2.1}$$

implies that

$$F(x) = 1 - \left(\frac{g(a,i,n}{g(x,i,n)}\right)^{\frac{1}{n-i+1}} e^{-\int_a^x \frac{\alpha t^{\alpha-1}}{(n-i+1)g(t,i,n)} dt}. \tag{2.2}$$

Proof:

The condition (2.1) and the assumption $\lim_{x \to b} x^{\alpha} (1 - F(x)) = 0$ for some $\alpha > 0$ imply that

$$\int_t^b \alpha x^{\alpha-1} (1 - F(x))^{n-i+1} dx = g(t,i,n)(1 - F(t))^{n-i+1}. \tag{2.3}$$

Differentiating both sides of (2.3) with respect to t, we obtain

$$-\alpha t^{\alpha-1}(1 - F(t)^{n-i+1}$$
$$= \left[\frac{d}{dt} g(t,i,n)\left(1 - F(t)\right) - (n - i + 1) g(t,i,n) f(t)\right] \left(1 - F(t)\right)^{n-i}.$$

On simplification, we have

$$\frac{f(t)}{1-F(t)} = \frac{\frac{\partial}{\partial t} g(t,i,n)}{(n-i+1) g(t,i,n)} + \frac{\alpha t^{\alpha-1}}{(n-i+1) g(t,i,n)} . \tag{2.4}$$

Integrating (2.4) with respect to t from a to x results in

$$F(x) = 1 - \left(\frac{g(a,i,n)}{g(x,i,n)}\right)^{\frac{1}{n-i+1}} \quad e^{-\int_a^x \frac{\alpha t^{\alpha-1}}{(n-i+1)g(t,i,n)} dt}. \tag{2.5}$$

Remark 2.1.

If $i = 1$ and $b = \infty$, then we obtain the Theorem 2.1 of Hamedani et al. (2006).

2.1. Burr Type XII Distribution

The pdf of this distribution is given by

$$f(x, \alpha, \beta) = \alpha \beta x^{\alpha-1} (1 + x^{\alpha})^{-\beta+1}, x > 0. \tag{2.6}$$

where $\alpha > 0$ and $\beta > 0$ are parameters.

Proposition 2.1 Let $X \colon \Omega \to R^{+}$ be a continuous random variable with cdf $F(x)$ such that

$$\lim_{x \to \infty} x^{\alpha} (1 - F(x))^{n-i+1} = 0.$$

Then X has the pdf (2.6) for $(n-i+1)\beta > 1$ if and only if

$$E\left(X_{i,n}^{\alpha} \middle| X_{i,n} > t\right) = \frac{(n-i+1)\beta t^{\alpha}+1}{(n-i+1)\beta-1}. \tag{2.7}$$

Proof:

Suppose that $F(x,\alpha,\beta) = 1-(1+x^{\alpha})^{-\beta}$, then

$$E(X_{i,n}^{\alpha} \mid X_{i,n} > t) = t^{\alpha} + \frac{\int_t^{\infty} \alpha x^{\alpha-1}(1-F(x))^{n-i+1}\,dx}{(1-F(t))^{n-i+1}}$$

and

$$g(t,i,n) = \frac{\int_t^{\infty} \alpha x^{\alpha-1}(1-F(x))^{n-i+1}\,dx}{(1-F(t))^{n-i+1}}$$

$$= \frac{\int_t^{\infty} \alpha x^{\alpha-1}(1+x^{\alpha})^{-(n-i+1)\beta}\,dx}{(1+t^{\alpha})^{-(n-i+1)\beta}}$$

$$= \frac{1+t^{\alpha}}{(n-i+1)\beta-1}.$$

Thus

$$E\left(X_{i,n}^{\alpha} \middle| X_{i,n} > t\right) = \frac{(n-i+1)\beta t^{\alpha}+1}{(n-i+1)\beta-1}.$$

Suppose $g(t,i,n) = \frac{1+t^{\alpha}}{(n-i+1)\beta-1}$, then using Theorem 2.1, we obtain

$$F(x) = 1-\left(\frac{g(a,i,n}{g(x,i,n)}\right)^{\frac{1}{n-i+1}} e^{-\int_a^x \frac{\alpha t^{\alpha-1}}{(n-i+1)g(t,i,n)}dt}$$

$$= 1-\left(\frac{1}{1+x^{\alpha}}\right)^{\frac{1}{n-i+1}} e^{-\int_0^x \frac{dt^{\alpha-1}(ni+1)\beta-1}{(n-i+1)(1+\epsilon^{\alpha})}\mathrm{dt}}$$

$$= 1-(1+x^{\alpha})^{-\beta}.$$

Remark 2.2.

If $\alpha = 1$ and $i = 1$, then we get the result given in Proposition 2.1 in Hamedani et al. (2008).

2.2. F Distribution with 2 and m Degrees of Freedom

The pdf of this family is as follows

$$f(x,2,m) = m^{\frac{m}{2}}(m+2x)^{\left(\frac{m}{2}+1\right)}, x > 0 \tag{2.8}$$

where $m > 0$ is a parameter.

Proposition 2.2. Let $X\colon \Omega \to (0,\infty)$ be a continuous random variable with cdf F (x). The random variable X has the pdf (2.8) if and only if

$$E(X_{i,n} \mid X_{i,n} > t) = \frac{(n-i+1)mt+m}{(n-i+1)m-2}, (n-i+1)m > 2 . \tag{2.9}$$

Proof:

Suppose

$$\mathrm{F(x)} = 1 - \frac{m^{m/2}}{(m+2x)^{m/2}} \text{ , then}$$

$$E(X_{i,n} \mid X_{i,n} > t) = t + \frac{\int_t^{\infty} \left(\frac{m}{m+2x}\right)^{\frac{m(n-i+1)}{2}}}{\left(\frac{m}{m+2t}\right)^{\frac{m(n-i+1)}{2}}}$$

and

$$\mathrm{g(t,i,n)} = \frac{\int_t^{\infty} \left(\frac{m}{m+2x}\right)^{\frac{m(n-i+1)}{2}}}{\left(\frac{m}{m+2t}\right)^{\frac{m(n-i+1)}{2}}}$$

$$= \frac{m+2t}{(n-i+1)m-2} .$$

Suppose $g(t,i,n) = \frac{m+2t}{(n-i+1)m-2}$, then using Theorem 2.1, we obtain

$$F(x) = 1 - \left(\frac{m}{m+2x}\right)^{\frac{1}{n-i+1}} e^{-\int_0^x \frac{(n-i+1)m-2}{(n-i+1)(m+2t)} \mathrm{dt}} \text{ , m(n-i+1)> 2}$$

$$= 1 - \left(\frac{m}{m+2x}\right)^{\frac{1}{n-i+1}} \left(\frac{m}{m+2x}\right)^{\frac{(n-i+1)m-2}{2(n-i+1)}}$$

$$= 1 - \left(\frac{m}{m+2x}\right)^{m/2}.$$

2.3. Generalized Beta 2 Distribution

The pdf of this family is given by

$$f(x,\alpha,\beta,\mu) = \alpha\gamma\beta^{-\alpha}x^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right]^{-(\gamma+1)}, x > 0 \tag{2.10}$$

where $\alpha > 0, \beta > 0$ and $\gamma > 0$ are parameters.

Proposition 2.3. Let $X{:}\,\Omega \to R^{+}$be a continuous random variable with cdf F(x) such that $\lim_{x\to\infty} x^{\alpha}\,(1-F(x))^{n-i+1} = 0$.

The random variable X has the pdf (2.10) for $(n-i+1)\gamma > 1$ if and only if

$$E\left(X_{i,n}^{\alpha} \middle| X_{i,n} > t\right) = \frac{\beta^{\alpha}+(n-i+1)\gamma t^{\alpha}}{(n-i+1)\gamma-1} .$$

Proof:

Suppose

$$F(x,\alpha,\beta)=1-\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-\gamma}, x>0, \text{ then}$$

$$E\left(X_{i,n}^{\alpha}\middle|X_{i,n}>t\right)=t^{\alpha}+\frac{\int_t^{\infty}\alpha x^{\alpha-1}\left[1+\left(\frac{x}{\beta}\right)^{\alpha}\right]^{-\gamma(n-i+1)}}{\left[\left(1+\frac{t}{\beta}\right)^{\alpha}\right]^{(n-i+1)\gamma}}$$

and

$$g(t,i,n)=\frac{\int_t^{\infty}\alpha x^{\alpha-1}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{-\gamma(n-i+1)}}{\left[1+\left(\frac{t}{\beta}\right)^{\alpha}\right]^{-\gamma(n-i+1)}}$$

$$=\frac{\beta^{\alpha}+t^{\alpha}}{(n-i+1)\gamma-1}$$

Suppose

$$g(t,i,n)=\frac{\beta^{\alpha}+t^{\alpha}}{(n-i+1)\gamma-1}$$

then using Theorem 2.1, we obtain

$$F(x)=1-\left(\frac{\beta^{\alpha}}{x^{\alpha}+\beta^{\alpha}}\right)^{\frac{1}{n-i+1}}\ e^{-\int_o^x \alpha t^{\alpha-1}\frac{(n-i+1)\gamma-1}{(n-i+1)(\beta^{\alpha}+t^{\alpha})}dt}$$

$$=1-\left(\frac{\beta^{\alpha}+x^{\alpha}}{\beta^{\alpha}}\right)^{\gamma}$$

2.4. Generalized Pareto Distribution

The pdf of this family is as follows

$$f(x,\alpha,\beta)=\frac{\beta+1}{\alpha}\left(1+\frac{\beta}{\alpha}x\right)^{-(2+\frac{1}{\beta})}, x>0, \tag{2.11}$$

where $\alpha>0$ and $\beta>0$ are parameters.

Proposition 2.4.

Let $X:\Omega\to[0,\infty)$ be a continuous random variable with cdf F(x) such that $\lim_{\to\infty} x^{\alpha}\,(1-F(x))^{n-i+1}=0$. The random variable X has pdf (2.11) if and only if

$$E(X_{i,n}\,|\,X_{i,n}>t)=\mathrm{t}+\frac{\alpha+\beta t}{(n-i)\beta+n-i+1} \tag{2.12}$$

Proof:

Suppose

$$\mathrm{F(x)}=1-\left(1+\frac{\beta}{\alpha}x\right)^{-\left(1+\frac{1}{\beta}\right)}, \text{ then}$$

$$\mathrm{E}\big(X_{i,n}\big|X_{i,n} > t\big) = t + \frac{\int_t^{\infty}\left(1+\frac{\beta}{\alpha}x\right)^{-(n-i+1)(1+\frac{1}{\beta})}dx}{(1+\frac{\beta}{\alpha}t)^{-(n-i+1)(1+\frac{1}{\beta})}} \text{ and}$$

$$g(t,i,n) = \frac{\int_t^{\infty}\left(1+\frac{\beta}{\alpha}x\right)^{-(n-i+1)(1+\frac{1}{\beta})}dx}{(1+\frac{\beta}{\alpha}t)^{-(n-i+1)(1+\frac{1}{\beta})}}$$

$$= \frac{\alpha}{\beta}\left[\frac{1+\frac{\beta}{\alpha}t}{(n-i+1)(1+\frac{1}{\beta})-1}\right]$$

$$= \frac{\alpha+\beta t}{(n-i)\beta+n-i+1}.$$

Suppose

$$g(t,i,n) = \frac{\alpha+\beta t}{(n-i)\beta+n-i+1},$$

then using Theorem 2.1, we obtain

$$F(x) = 1-\left(\frac{\alpha}{\alpha+\beta x}\right)^{\frac{1}{n-i+1}} e^{-\int_0^x \frac{(n-i)\beta+n-i+1}{(n-i+1)(\alpha+\beta t)}dt}$$

$$= 1-\left(\frac{1}{1+\frac{\beta}{\alpha}x}\right)^{\frac{1}{n-i+1}}\left(\frac{1}{1+\frac{\beta}{\alpha}x}\right)^{\frac{((n-i)\beta+n-i+i}{(n-i+1)\beta}}$$

$$= 1-(1+\frac{\beta}{\alpha}x)^{-(1+\frac{1}{\beta})}.$$

2.5. Pareto of the First Kind Distribution

This family has the pdf of the form

$$f(x,\alpha,\beta) = \alpha\beta^{\alpha}x^{-(a+1),}x \geq \beta, \tag{2.13}$$

where $\alpha > 0, \beta > 0$ are parameters.

Proposition 2.5.

Let $X\colon \Omega \rightarrow [\beta,\infty)$ be a continuous random variable with cdf F such that $\lim_{\rightarrow x} x\big(1-F(x)\big)^{n-i+1} = 0$. The random variable X has pdf (2.13) for $(n-i+1)$ $a > 1$ if and only if

$$E(X_{i,n} \mid X_{i,n} > t) = \frac{(n-i+1)\alpha t}{(n-i+1)\alpha-1} \tag{2.14}$$

Proof:

Suppose

$F(x, \alpha, \beta) = 1 - \left(\frac{x}{\beta}\right)^{-\alpha}$, then

$$E(X_{i,n} \,|\, X_{i,n} > t) = t + \frac{\int_t^{\infty} \left(\frac{x}{\beta}\right)^{-\alpha(n-i+1)} dx}{\left(\frac{t}{\beta}\right)^{-\alpha(n-i+1)}}$$

and

$$g(t, \alpha, i, n) = \frac{\int_t^{\infty} (x)^{-\alpha(n-i+1)} dx}{(t))^{-\alpha(n-i+1)}}$$
$$= \frac{t}{(n-i+1)\alpha - 1}.$$

Suppose $g(t, i, n) = \frac{t}{(n-i+1)\alpha-1}$, then using Theorem 2.1, we obtain

$$F(x) = 1 - \left(\frac{g(\beta, i, n)}{g(x, i, n)}\right)^{\frac{1}{n-i+1}} e^{-\int_{\beta}^{x} \frac{(n-i+1)\alpha-1}{(n-i+1)t}}$$
$$= 1 - \left(\frac{x}{\beta}\right)^{-\alpha}, \text{ x>}\beta.$$

2.6. Power Function Distribution

The pdf of this family is given by

$$f(x, \alpha) = \alpha(1-x)^{\alpha-1}, 0 < x < 1, \text{ where } \alpha > 0 \text{ is a parameter.} \qquad (2.15)$$

Proposition 2.6.

Let $X: \Omega \to [0,1]$ be a continuous random variable with cdf F(x). Then the random variable X has pdf (2.15) if and only if

$$E(X_{i,n} \,|\, X_{i,n} > t) = \frac{(n-i+1)\alpha t + 1}{(n-i+1)\alpha - 1} \qquad (2.16)$$

Proof:

Suppose

$F(x) = 1 - (1-x)^{\alpha}$, x$\geq$ 0, then

$$E(X_{i,n} \,|\, X_{i,n} > t) = t + \frac{\int_t^1 (1-x)^{(n-i+1)\alpha} dx}{(1-t)^{(n-i+1)\alpha}}$$

and

$$g(t, i, n) = \frac{\int_t^1 (1-x)^{(n-i+1)\alpha} dx}{(1-t)^{(n-i+1)\alpha}}$$

$$= \frac{1-t}{(n-i+1)\alpha+1}$$

Suppose $g(t,i,n) = \frac{1-t}{(n-i+1)\alpha+1}$, then using Theorem 2.1, we obtain

$$F(x) = 1-(1-\mathrm{x})^{-\frac{1}{\mathrm{n-i+1}}}\, e^{-\int_0^x \frac{(n-i+1)\alpha+1}{(n-i+1)(1-t)}dt}, x \ge 0,$$

$$= 1-\left(\frac{1}{1-x}\right)^{-\frac{1}{n-i+1}}(1-x)^{\frac{(n-i+1)\alpha+1}{n-i+1}}$$

$$= 1-(1-x)^{\alpha}$$

2.7. Weibull Distribution

The pdf of this family is given by

$$f(x,\alpha,\beta) = \alpha\beta x^{\beta-1}\, e^{-\alpha x^{\beta}} x > 0 \qquad (2.17)$$

where $\beta > 0$ is a parameter.

Proposition 2.7.

Let $\Omega \to [0,\infty)$ be a continuous random variable with cdf F such that $\lim_{\to\infty} x^{\alpha}(1-F(x))^{n-i+1} = 0,$ The random variable X has pdf (2.17) if and only if

$$E(X_{i,n}^{\beta} \mid X_{i,n} > t) = t^{\beta} + \frac{1}{(n-i+1)\alpha}. \qquad (2.18)$$

Proof:

Suppose $F(x) = 1 - e^{-\alpha x^{\beta}}$, then

$$E\left(X_{i,n}^{\beta} \middle| X_{i,n} > t\right) = t^{\beta} + \frac{\int_t^{\infty} \beta x^{\beta-1} e^{-(n-i+1)\alpha x^{\beta}} dx}{e^{-(n-i+1)\alpha t^{\beta}}}$$

and

$$g(t,i,n) = \frac{\int_t^{\infty} \beta x^{\beta-1} e^{-(n-i+1)\alpha x^{\beta}} dx}{e^{-(n-i+1)\alpha t^{\beta}}}$$

$$= \frac{1}{(n-i+1)\alpha}$$

Suppose $g(t,i,n) = \frac{1}{(n-i+1)\alpha}$, then using Theorem 2.1 we obtain

$$F(x) = 1 - e^{-\int_0^x \alpha\beta t^{\beta-1}dt}, x \ge 0,$$

$$= 1-e^{-\alpha x^{\beta}}.$$

Remark 2.3.

If $\beta = 1$, then we have the characterization of the exponential distribution. If $i = 1$, then we get the proposition 2.5 of Hamedani et al. (2008).

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