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PROPERTIES OF A NEWLY DEFINED HYPERGEOMETRIC POWER SERIES FUNCTION

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ABSTRACT

Ahmad (2007b) has recently defined a generalized hypergeometric series function and referred to it as a hypergeometric power series function or ${}_rH_s$-function which is an alternative notation for the ${}_rF_s$-function, the ${}_rH_s$ notation has advantages when the arguments are large and parameters are repeated and discussed the some basic properties. In this paper further properties of the hypergeometric power series functions have been developed.

KEY WORDS

Generalized hypergeometric series function; hypergeometric power series function; poission distribution; hyper poisson; negative moments.

1. INTRODUCTION

Ahmad (2007a) has discussed the Conway-Maxwell Poisson distribution and Conway-Maxwell Hyper Poisson (CMHP) distribution. The structure of CMP and CMHP distributions shows that a more general definition of the hypergeometric series function is needed. When a large number of identical parameters are introduced in the generalized hypergeometric series function, its notation becomes cumbersome. Use of generalized hypergeometric series function is sometimes difficult and time consuming especially when we have a large number of parameters. We take powers on those parameters which are repeated and as such Ahmad (2007b) has introduced an alternative form of a hypergeometric series function called hypergeometric power series function or ${}_rH_s$-function. (See also Saboor, 2007).

1.1 Definitions

The hypergeometric power series function or ${}_rH_s$-function is defined as

$$ {}_rH_s\left[(a_1,m_1),(a_2,m_2),\ldots,(a_r,m_r);(b_1,n_1),(b_2,n_2),\ldots,(b_s,n_s);z\right] = \sum_{i=0}^{\infty}\frac{\prod_{k=1}^{r}\left[(a_k)_i\right]^{m_k}}{\prod_{j=1}^{s}\left[(b_j)_i\right]^{n_j}}\frac{z^i}{i!}, $$

where $a_k \in R$, $|z|<1$, $b_j \neq 0,-1,-2,...$, n_j, m_k are positive integers.

1. If $\sum_{j=1}^{r} m_j \leq \sum_{k=1}^{s} n_k$, the ${}_rH_s$-function converges for all finite z;
2. If $\sum_{j=1}^{r} m_j = \sum_{k=1}^{s} n_k + 1$, the ${}_rH_s$-function converges for $|z|<1$ and diverges for $|z|>1$;
3. If $\sum_{j=1}^{r} m_j > \sum_{k=1}^{s} n_k + 1$, the ${}_rH_s$-function diverges for $z \neq 0$.

When all m_r and n_s are equal to 1, then ${}_rH_s$ becomes

$$ {}_rF_s\left[a_1,a_2,....,a_r;b_1,b_2,...,b_s;z\right] = \sum_{i=0}^{\infty} \frac{\prod_{k=1}^{r}\left[(a_k)_i\right]}{\prod_{j=1}^{s}\left[(b_j)_i\right]} \frac{z^i}{i!}, $$

where $a_k \in R$, $1 \leq k \leq r, 1 \leq j \leq s$ and $b_j \neq 0,-1,-2,.......$, $|z|<1$.

2. PROPERTIES OF ${}_rH_s$-FUNCTION

We have discussed some basic properties of the ${}_rH_s$-function and developed different types of recurrence relationships, which are as follows:

Theorem 2.1:

Let $\mathrm{Re}(m)>0$, $\mathrm{Re}(n)>0$ and $b \neq 0,-1,-2,...$. If $|z|<1$ and $\beta(m,n)$ is the beta function, then

$$ \int_0^1 t^{m-1}(1-t)^{n-1}\,{}_1H_1\left[(a,r);(b,s);t\,z)\right]dt $$

$$ = \beta(m,n)\,{}_2H_2\left[(a,r),(m,1);(b,s);(m+n,1),z\right]. \tag{2.1} $$

Proof:

Expanding ${}_1H_1\left[(a,r);(b,s);t\,z\right]$ in (2.1), we have

$$ \int_0^1 t^{m-1}(1-t)^{n-1} \sum_{k=0}^{\infty} \frac{\left[(a)_k\right]^r}{\left[(b)_k\right]^s} \frac{t^k z^k}{k!} dt = \sum_{k=0}^{\infty} \frac{\left[(a)_k\right]^r}{\left[(b)_k\right]^s} \frac{z^k}{k!} \frac{\Gamma(m+k)\Gamma(n)}{\Gamma(m+n+k)}, \quad |z|<1. \tag{2.2} $$

After a simple algebra we obtain R.H.S of (2.1).

Theorem 2.2:

Let neither c nor d be zero or a negative integer. If $\delta > 1$ and $|t\,z| < 1, \left|\frac{z}{\lambda}\right| < 1,\ \lambda > 0$, then

$$\int_0^\infty e^{-\lambda t} t^{\delta-1}\ {}_2\mathrm{H}_2\left[(\mathrm{a},\mathrm{m}_1),(\mathrm{b},\mathrm{m}_2);(\mathrm{c},\mathrm{n}_1),(\mathrm{d},\mathrm{n}_2);z\,t\right]dt$$

$$= \frac{\Gamma(\delta)}{(\lambda)^{\delta}}\ {}_3H_2\left[(a,m_1),(b,m_2),(\delta,1);(c,n_1),(d,n_2);z/\lambda\right]. \tag{2.3}$$

Proof:

$$L.H.S. = \int_0^\infty e^{-\lambda t} t^{\delta-1} \sum_{i=0}^{\infty} \frac{\left[(a)_i\right]^{m_1}\left[(b)_i\right]^{m_2}(z\,t)^i}{\left[(c)_i\right]^{n_1}\left[(d)_i\right]^{n_2}\, i!} dt. \tag{2.4}$$

Let $\lambda t = v$ and after taking integral, we get R.H.S of (2.6).

Similarly we obtain

$$\int_0^\infty e^{-t/z}\ {}_1H_1[(a,r);(b,s);t]\,t^{a-1}dt = z^a\Gamma(a)\ {}_1H_1[(a,r+1);(b,s);z], \tag{2.5}$$

when $b \neq 0,-1,-2,....$. If $|z| < 1$.

Theorem 2.3:

i) Let $a \neq \frac{1}{2},0,-\frac{1}{2},-\frac{3}{2},....$. If $|z| < 1$, then

$${}_1H_1\left[(a/2,2);(a\text{-}1/2,1);z\right]\,{}_1H_1\left[(a/2,2);(a+1/2,1);z\right]$$

$$= {}_1H_2\left[(a,3);(2a\text{-}1,1),(a+1/2,1);z\right] \tag{2.6}$$

Similarly we find

ii) $$\left({}_1H_1\left[(a/2,2);(a+1/2,1);z\right]\right)^2 = {}_1H_2\left[(a,3);(2a,1),(a+1/2,1);z\right], \tag{2.7}$$

where $a \neq 0,-\frac{1}{2},-1,-\frac{3}{2},-2....$. If $|z| < 1$. (see Clausian, 1828).

iii) $${}_1H_0[(a,2);\text{-};z]\,{}_1H_0[(a,2);\text{-};\text{-}z] = {}_3H_1\left[(a,2),(a,1),((2a+1)/2,1);(2a,1);4z^2\right], \tag{2.8}$$

where $a \neq 0,-1/2,-1,-3/2,....$ If $|z| < 1$.

Proofs are trivial.

3. NEGATIVE MOMENTS OF SOME DISCRETE PROBABILITY FUNCTIONS

We now consider the negative moments of the form $E(X+A)^{-k}, k>0$, for some discrete distributions.

Theorem 3.1:

Let X be a geometric-compound random variable, with parameters α and β having probability mass function (pmf)

$$P(X=x)=\frac{\Gamma(\alpha+\beta)\ \ \Gamma(\alpha+x-1)\ \ \Gamma(\beta+1)}{\Gamma(\alpha)\ \ \Gamma(\beta)\ \ \Gamma(\alpha+\beta+x)},\ \alpha>0,\ \beta>0,\ x=1,2,\dots. \quad (3.1)$$

Then the negative moment of k^{th} order is given by

$$E\left(X+A\right)^{-k}=\frac{\beta}{(A+1)^{k}(\alpha+\beta)}\ {}_3H_2\left((A+1,k),(\alpha,1),(1,1);(A+2,k),(\alpha+\beta+1,1);\ 1\right). \quad (3.2)$$

Proof:

The k^{th} negative moment of (3.1) is

$$\mathrm{E(X+A)^{-k}}=\frac{\Gamma(\alpha+\beta)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{x=1}^{\infty}\frac{1}{(\mathrm{x+A})^{k}}\frac{\Gamma(\alpha+\mathrm{x}-1)}{\Gamma(\alpha+\beta+\mathrm{x})}$$

After some algebraic computations, we get (3.2).

Theorem 3.2:

Let X be a beta-binomial random variable, with parameter $\alpha>0$, $\beta>0$ and probability mass function

$$P(X=x)=\binom{n}{x}\frac{\Gamma(\alpha+\beta)\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)},\ x=0,1,2,....,n. \quad (3.3)$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{P_0}{A^k}\ {}_3H_2\left((A,k),(\alpha,1),(-n,1);(A+1,1),(-n-\beta+1,1);\ 1\right), \quad (3.4)$$

where $P_0=P(x=0)=\dfrac{\Gamma(\alpha+\beta)\Gamma(n+\beta)}{\Gamma(\beta)\Gamma(n+\alpha+\beta)}$.

Following the procedure of theorem 3.1, we obtain (3.4).

Theorem 3.3:

Let X be a Waring random variable, with parameters a, $a\geq 2$, $c, c>a$ and pmf

$$P(X=x)=\frac{(c-a)(a+x-1)!(c)!}{c(a-1)!(c+x)!}, \quad c>a\geq 2,\ x=0,1,2,\ldots.$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{P_0}{A^k}\,{}_3H_2\left((A,k),\ (a,1),(1,1);\ (A+1,k),(c+1,1);\ 1\right), \tag{3.5}$$

where $P_0=P(X=0)=\dfrac{(c-a)}{c}$, $A>0$.

Proof is trivial.

Theorem 3.4:

Let *X* be a truncated Poisson random variable, with parameter λ, $\lambda>0$ and pmf

$$P(X=x)=\frac{\lambda^x}{(e^\lambda-1)x!}, \quad \lambda>0,\ X=1,2,\ldots.$$

Then the negative moment of the k^{th} order is given by

$$E\left(X+A\right)^{-k}=\frac{P_1}{\left(A+1\right)^k}\,{}_2H_2\left(\left(A+1,k\right),\left(1,1\right);\left(A+2,k\right),\left(2,1\right);1\right), \tag{3.6}$$

where $P_1=P(X=1)=\dfrac{\lambda}{(e^\lambda-1)}$, $A\geq 0$.

Theorem 3.5:

Let *X* be a truncated binomial random variable, with parameters n and *p*, $0\leq p\leq 1$, and pmf

$$P(X=x)=\frac{1}{(1-q^n)}\binom{n}{x}p^x q^{n-x}, \quad x=1,2,\ldots,\mathrm{n},\ q=1-p.$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{P_1}{(A+1)^k}\,{}_3H_2\left((A+1,k),(1,1),(-n+1,1);\ (A+2,k),\ (2,1);\ \frac{-p}{q}\right). \tag{3.7}$$

4. SUMMATION OF ${}_rH_s$-FUNCTIONS

Ahmad and Roohi (2004, 2005) have derived the sum of some combinations of hyoergeometric series functions using the binomial and logarithm probability functions. In this section, we have derived another set of sum of some combinations of hypergeometric power series function using Dacey (1972) probability function. Ahmad and Roohi (2004, 2005) summations of series become its special cases.

4.1 Lemma (Gould, 1972)

For $a,\ b_i\ ,\ c_j \neq 0,-1,-2,...$ and some k ,

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}{}_rF_s\left[a,\ b_1,\ b_2,...,\ b_{r-1};\ a+1,c_1,....,\ c_{s-1};z\right]$$
$$= {}_rF_s[1,b_1,\ b_2,...,\ b_{r-1};\ k+1,c_1,....,\ c_{s-1};z]\ ,$$

where ${}_rF_s$ is a hypergeometric series function with usual conditions for convergence.

Theorem 4.1:

Let $b \neq 0,-1,-2,...$ or $a \geq 0$. If $s \geq 1$, $k \geq 1$ and $0<\theta<1$. Then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}{}_2H_2\left[(s,1),(a,h);(s+1,1),(b,m);\theta\right]$$
$$= {}_2H_2\left[(1,1),(a,h);(k+1,1),(b,m);\theta\right] \quad (4.1)$$

Proof:

We have (4.1) as a repeated case of Lemma (4.1). Alternatively suppose X is a discrete random variable with probability function. (See Dacey, 1972).

$$P(X=x)=\frac{c\,\theta^x}{x!}\ \gamma_x\left[(a,h);(b,m)\right]\ ;\ x=0,1,2,...\ , \quad (4.2)$$

where $c=\dfrac{1}{{}_1H_1\left[(a,h);(b,m);\theta\right]}$ and $\gamma_x\left[(a,h);(b,m)\right]=\dfrac{\left[\Gamma(a+x)\right]^h\left[\Gamma(b)\right]^m}{\left[\Gamma(b+x)\right]^m\left[\Gamma(a)\right]^h}$.

It is known that

$$\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}\frac{1}{(x+s)},\ x\geq 0,\ k=1,2,3,...\ , \text{(See Jones, 1987).}$$

Using the definition of mathematical expectation $E(X)=\sum_{x=0}^{\infty}xf(x)$, we get

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}E\left(\frac{1}{X+s}\right). \quad (4.3)$$

Now

$$E\left(\frac{1}{X+s}\right)=c\sum_{x=0}^{\infty}\frac{\theta^x}{x!}\ \gamma_x\left[(a,\ h);(b,\ m)\right]\frac{1}{x+s}$$
$$=s^{-1}c\ {}_2H_2\left[(s,1),(a,h);(s+1,1),(b,m);\theta\right].$$

It thus follows from (4.3) that

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=c\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s)!}\ {}_2H_2\left[(s,1),(a,h);(s+1,1),(b,m);\theta\right]. \tag{4.4}$$

Also,

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=c\sum_{x=0}^{\infty}\frac{\theta^x}{x!}\ \gamma_x\left[(a,h);(b,m)\right]\prod_{s=1}^{k}\frac{1}{(x+s)}$$

$$=c\left(k!\right)^{-1}\ {}_2H_2\left[(1,1),(a,h);(k+1,1),(b,m);\theta\right]. \tag{4.5}$$

(4.4) and (4.5) imply (4.1).

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