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STATISTICAL ANALYSIS FOR TANDEM AND BULK SERVICE QUEUEING SYSTEMS

Paul Savariappan[1], **P. Chandrasekhar**[2] and **V. Mani**[2]
[1] Department of Mathematics, Luther College, Decorah, Iowa, 52101, USA.
Email: paul@luther.edu
[2] Department of Statistics, Loyola College, Chennai-600 034, India.

ABSTRACT

A maximum likelihood estimator (MLE), a consistent asymptotically normal (CAN) estimator and asymptotic confidence limits for the expected number of customers of (i) a k station tandem queue and (ii) a bulk service queue are obtained.

KEYWORDS

Bulk service queue-CAN estimator; multivariate central limit theorem; slutsky theorem-tandem queue.

1. INTRODUCTION

Most of the studies on several queueing models are confined to only obtaining expressions for transient or stationary (steady state) solutions and do not consider the associated statistical inference problems. Nowadays, statistical analysis of queueing systems is an important area of research in queueing theory. Parametric estimation and interval estimation are some of the statistical tools to understand any random phenomena using stochastic models. Analysis of queueing systems in this direction has not received due attention in the past. Whenever the systems are fully observable in terms of their random components such as interarrival times and service times, standard parametric techniques of statistical theory are quite appropriate. Recently, Narayan Bhat (2003) has provided an overview of methods available for estimation, when the information is restricted to the number of customers in the system at some discrete points in time. Bhat (2003) has also described how maximum likelihood estimation is applied directly to the underlying Markov chain in the queue length process in M/G/1 and GI/M/1 queues. Table 1 indicates the present state of work of queueing systems, wherein the asymptotic confidence limits for measures of system performance are obtained.

An attempt is made in this paper to obtain a MLE, CAN and asymptotic confidence limits for the expected number of customers of (i) a k station tandem queue and (ii) a bulk service queue. In the following section, these two models and the expected number of customers in the system for each model are explained briefly.

Table 1

S#	System Description	Authors	Confidence limits obtained
1	$M/M/1/\infty$ and $M/M/1/N$	Yadavalli et al. (2004)	W_Q
2	M/M/c/∞ and M/M/c/N	Yadavalli et al. (2006)	W_Q
3	Tandem queue with blocking and dependent structure for service times	Chandrasekhar et al. (2006)	L_S, W_S
4	Tandem queue with blocking and dependent structure for interarrival and service times	Chandrasekar et al. (2005)	L_S, W_S
5	Tandem queue with blocking and with at most one customer between the stations	Chandrasekhar and Yadavalli(2006)	L_S
6	Tandem queue with blocking	Chandrasekhar et al. (2007)	W_S
7	Bulk-arrival queueing system $M^{[X]}/M/1$	Paul Savariappan and Chandrasekhar (2006)	L_S

2. SYSTEM DESCRIPTION

2.1 Model I: k Station Tandem Queue

Generally speaking, the queueing models assume that each service channel consists of only one station. Situations do exist, where each service channel may consist of several stations in series. In this situation, a customer must pass through successively all these stations before completing his service. Such situations are known as queues in series or tandem queues. e.g., (a) In a manufacturing process, units must pass through a series of service channels (work stations), where each service channel performs a given task or job. (b) In a university registration process, each registrant must pass through a series of counters such as advisor, department chairman (Head of the Department), cashier etc. (c) In a clinical physical examination procedure, a patient must go through a series of stages such as lab tests, ECG, chest X-ray etc. In all these models it is not only sufficient to know how many persons are there in the system but also where they are.

2.2 Assumptions

Consider a simplified queueing system consisting of k stations (service facilities) arranged in series as below

$$\underset{1}{00\ldots..000\boxed{0}} \rightarrow \underset{2}{00\ldots..000\boxed{0}} \rightarrow \underset{k}{00\ldots..000\boxed{0}} \rightarrow$$

Fig 1. System configuration

A customer arriving for service must pass through station 1, station 2,..., station k before completing his service. Precisely the assumptions of the model are as follows:

i) Customers arrive according to a Poisson process with mean rate λ for service at station 1. After completing service at station 1, the customers join another queue for service at station 2. The same procedure is continued until the customers join the queue for service at station k before completing the service.

ii) Service times at stations $1,2,3,...,k$ are all exponentially distributed with service rates $\mu_1,\mu_2,...,\mu_k$ respectively. Thus the system consists of k distinct queueing systems, which we label as sub systems $1,2,3,...,k$.

iii) There is no queue limit at any station.

2.3 Analysis of the System

Let $p_{n_1,n_2,...,n_k}(t)$ be the probability that there are n_1 customers in the first subsystem (in queue or in service), n_2 customers in the second subsystem (in queue or in service),…, n_k customers in the k^{th} subsystem at time t. In the steady state, it can be shown that,

$$p_{n_1,n_2,...,n_k} = \prod_{i=1}^{k} \rho_i^{n_i}(1-\rho_i), \qquad n_i = 0,1,2,... ;\ i = 1,2,...,k \tag{2.1}$$

where $\rho_i = \left(\frac{\lambda}{\mu_i}\right)$ and the steady state results exist provided $\rho_i < 1$, i = 1, 2, …,k. see Medhi (2006). Clearly, (2.1) corresponds to the product of probability mass functions of k independent but not identically distributed Geometric variates. The marginal probability of n_1 customers in the first subsystem whatever may be the number in other subsystems is given by

$$p_{n_1,.,.,...,.} = \sum_{n_2,n_3,...,n_k=0}^{\infty} p_{n_1,n_2,...,n_k}$$

$$= (1-\rho_1)\rho_1^{n_1}$$

In general, the marginal probability of n_i customers in i^{th} subsystem whatever may be the number in other subsystems is given by

$$p_{.,.,.,.,n_i,.,.,...,.} = (1-\rho_i)\rho_i^{n_i}$$

The expected number of customers in the entire system is given by

$$_1L_s = \sum_{n_1,n_2,...,n_k=0}^{\infty} \left(\sum_{i=1}^{k} n_i\right) p_{n_1,n_2,...,n_k}.$$

$$= \sum_{i=1}^{k} \frac{\rho_i}{(1-\rho_i)}, \qquad where \quad \rho_i = \frac{\lambda}{\mu_i}, \quad i = 1,2,...,k \tag{2.2}$$

2.4 Model II: Bulk Service $(M/M^{[Y]}/1)$ Queue

Bulk service queues are the queueing situations in which arrivals occur singly but service is rendered in bulk. In fact, they are the queueing models that allow the arrivals and (or) service of various customers simultaneously. A typical bulk queue model is described in terms of interarrival times of groups of customers, group sizes, service times of customer batches and batch sizes. Many transportation processes such as buses, airplanes, trains, ships, elevators all have common feature of bulk service. Bulk service queues provide very useful tools for analyzing many applications in a variety of fields such as

a) Communication systems (each message consisting of different packets),
b) Transportation systems (air and maintenance of traffic, etc.),
c) Manufacturing systems (in mechanical, electrical and electronic industries making products such as cars, computers, etc.).

Situations do arise, where we consider places like an amusement park, a museum or an art gallery, where there are guided tours. Suppose that a tour is not started until there are at least k prospective customers (visitors). Thus introduces what we call a "quorum", which occurs in other service systems as well. Precisely, the assumptions of the model are as follows:

2.5 Assumptions

i) The arrivals occur at a single channel facility as Poisson process and they are served FCFS.
ii) There is no waiting capacity constraint.
iii) The batch size for service is exactly k and if less than k are in service, new arrivals immediately enter service up to the limit k and finish with others, regardless of the time into service after service begins.
iv) The service time of any batch is an exponential random variable with the parameter μ, whether the batch is of full size k or not.

2.6 Analysis of the System

Clearly, the model is a non-birth and death Markovian problem and the stochastic balance equations are given by (see Gross and Harris (2002))

$$-\lambda p_0 + \mu p_k = 0 \tag{2.3}$$

$$-\lambda p_n + \mu p_{n+k} + \lambda p_{n-1} = 0\,, \quad n = 1, 2, ..., k-1. \tag{2.4}$$

$$-(\lambda + \mu)\, p_n + \mu p_{n+k} + \lambda p_{n-1} = 0\,, \quad n = k, k+1, ..., \tag{2.5}$$

The solution is given by

$$p_n = \begin{cases} \dfrac{\left(1 - r_0^{n+1}\right)}{\left(1 - r_0\right)}\, p_o, & n = 1, 2, ..., k-1. \\ \dfrac{\lambda}{\mu} r_0^{n-k}\, p_0, & n = k, k+1, ..., \end{cases} \tag{2.6}$$

where $r_0 \in (0,1)$ and $p_o = \frac{(1-r_0)}{k}$.

The expected number of customers in the system is given by

$$
\begin{aligned}
{}_2L_s &= \sum_{n=0}^{\infty} np_n \\
&= \sum_{n=1}^{k-1} np_n + \sum_{n=k}^{\infty} np_n \\
&= \frac{(k-1)}{2} + \frac{\lambda}{k\mu}\left[\frac{r_0}{(1-r_0)} + k\right] - \frac{r_0}{k}\left[\sum_{j=1}^{k-1} jr_0^j\right]
\end{aligned}
\tag{2.7}
$$

In the next section, we obtain MLE and CAN for the expected number of customers in the system for each model.

3. MLE AND CAN ESTIMATOR FOR THE EXPECTED NUMBER OF CUSTOMERS IN THE SYSTEM

3.1 ML Estimator

Let $X_{i1}, X_{i2}, ..., X_{in}$ (with $i = 1,2$ representing Models I and II) be random samples of size n, each drawn from different exponential interarrival time populations with the parameter λ. Also, let $Y_{11}, Y_{12}, ..., Y_{1n}$; $Y_{21}, Y_{22}, ..., Y_{2n}$; $,...,; Y_{k1}, Y_{k2}, ..., Y_{kn}$ be k random samples each of size n drawn from k different exponential service time populations with the parameters $\mu_1, \mu_2, ..., \mu_k$ respectively. (Model I). Similarly, let $Y_1, Y_2, ..., Y_n$ be a random sample of size n drawn from an exponential service time population of any batch, whether or not the batch is of full size k or not (Model II). It is clear that $E(\overline{X}_i) = \frac{1}{\lambda}$, where $\overline{X}_i$, $i = 1,2$ are the sample means of interarrival times corresponding to Model I and Model II and $E(\overline{Y_i}) = \frac{1}{\mu_i}$ $\forall\ i = 1,2,...,k$, where $\overline{Y_i} = \frac{\sum_{j=1}^{n} Y_{ij}}{n}$, $i = 1,2,...,k$; Note that, $\overline{X}_1$ is the sample mean of interarrival times, whereas $\overline{Y}_1, \overline{Y}_2, ..., \overline{Y}_n$ are the sample means of service times of station 1, station 2,…, station k respectively corresponding to Model I. Similarly, for Model II, we have $E(\overline{X}_2) = \frac{1}{\lambda}$ and $E(\overline{Y}_2) = \frac{1}{\mu}$, where $\overline{X}_2$ and $\overline{Y}_2$ are the sample means of interarrival times and service times respectively. It can be shown that $\overline{X}_i$ (with $i = 1,2$ representing the Models I and II) and $\overline{Y}_i$, $i = 1,2,...,k$ (for Model I) are the MLEs of $\frac{1}{\lambda}$ and $\frac{1}{\mu_i}$ respectively. Similarly, $\overline{Y}_2$ is the MLE of $\frac{1}{\mu}$, where $\overline{Y_2} = \frac{\sum_{j=1}^{n} Y_j}{n}$ corresponding to Model II.

Model I:

Let $\theta_i = \frac{1}{\mu_i}$, $i = 1, 2, ..., k$ and $\theta_{k+1} = \frac{1}{\lambda}$. Clearly, the expected number of customers in the system given in (2.2) reduces to

$$ {}_1L_s = \sum_{i=1}^{k} \frac{\theta_i}{(\theta_{k+1} - \theta_i)} \tag{3.1}$$

and hence the MLE of ${}_1L_s$ is given by

$$ {}_1\hat{L}_s = \sum_{i=1}^{k} \frac{\overline{Y}_i}{\left(\overline{X}_1 - \overline{Y}_i\right)} \tag{3.2}$$

Model II:

By choosing $\theta_1 = \frac{1}{\lambda}$ and $\theta_2 = \frac{1}{\mu}$, it can be shown that the expected number of customers in the system given in (2.7) reduces as

$$ {}_2L_s = \frac{(k-1)}{2} + \frac{1}{k}\frac{\theta_2}{\theta_1}\left[\frac{r_0}{(1-r_0)} + k\right] - \frac{r_0}{k}\left[\sum_{j=1}^{k-1} j r_0^j\right] \tag{3.3}$$

and hence the MLE of ${}_2L_s$ is given by

$$ {}_2\hat{L}_s = \frac{(k-1)}{2} + \frac{\overline{Y}_2}{k\overline{X}_2}\left[\frac{r_0}{(1-r_0)} + k\right] - \frac{r_0}{k}\left[\sum_{j=1}^{k-1} j r_0^j\right] \tag{3.4}$$

It may be noted that, ${}_1\hat{L}_s$ given in (3.2) is a real valued function in $\overline{X}_1$, $\overline{Y}_i$, $i = 1, 2, ..., k$ whereas ${}_2\hat{L}_s$ given in (3.4) is a real valued function in $\overline{X}_2$ and $\overline{Y}_2$. Also, ${}_i\hat{L}_s$, $i = 1, 2$ is differentiable. Consider the following application of multivariate central limit theorem. see Rao (1974).

3.2 Application of Multivariate Central Limit Theorem

Suppose $T_1', T_2', T_3', ...$ are independent and identically distributed k dimensional random variables such that

$$T_n' = (T_{1n}, T_{2n}, T_{3n}, ..., T_{kn}), \quad n = 1, 2, 3, ...$$

Having the first and second order moments $E(T_n) = \mu$ and $\text{var}(T_n) = \sum$. Define the sequence of random variables

$$\overline{T}_n' = \left(\overline{T}_{1n}, \overline{T}_{2n}, \overline{T}_{3n}, ..., \overline{T}_{kn}\right), \quad n = 1, 2, 3, ...$$

where $\overline{T}_{in} = \dfrac{\sum_{j=1}^{n} T_{ij}}{n}, \quad i = 1, 2, 3, ..., k.$

Then, $\sqrt{n}\left(\overline{T}_n - \mu\right) \xrightarrow{d} N\left(0, \sum\right)$ as $n \to \infty$.

3.3 CAN Estimator

Model I:

By applying the multivariate central limit theorem given in section 3.2, it is readily seen that

$$\sqrt{n}\left[\left(\overline{Y}_1, \overline{Y}_2, ..., \overline{Y}_k, \overline{X}_1\right) - \left(\theta_1, \theta_2, ...\theta_k, \theta_{k+1}\right)\right] \xrightarrow{d} N\left(0, \sum\right) \text{ as } n \to \infty,$$

where the dispersion matrix $\sum = \left(\left(\sigma_{ij}\right)\right)$ is given by

$$\sum = diag\left(\theta_1^2, \theta_2^2, ...\theta_k^2, \theta_{k+1}^2\right).$$

Again from Rao (1974), we have

$$\sqrt{n}\left({}_1\hat{L}_s - {}_1L_s\right) \xrightarrow{d} N\left(0, {}_1\sigma^2(\theta)\right) \ as\ n \to \infty$$

where $\theta = (\theta_1, \theta_2, ..., \theta_k, \theta_{k+1})$ and

$$\begin{aligned} {}_1\sigma^2(\theta) &= \sum_{i=1}^{k}\left(\frac{\partial\, {}_1L_s}{\partial\theta_i}\right)^2 \sigma_{ii} \\ &= \theta_{k+1}^2 \sum_{i=1}^{k} \frac{\theta_i^2}{\left(\theta_{k+1} - \theta_i\right)^4} \end{aligned} \tag{3.5}$$

Hence, ${}_1\hat{L}_s$ is a CAN estimator of ${}_1L_s$. There are several methods for generating CAN estimators and the Method of moments and the method of Maximum likelihood are commonly used to generate such estimators. See Sinha(1986).

Model II:

As in Model I, here too, we have

$$\sqrt{n}\left[\left(\overline{X}_2, \overline{Y}_2\right) - \left(\theta_1, \theta_2\right)\right] \xrightarrow{d} BVN\left(0, \sum\right) \ as\ n \to \infty,$$

where the dispersion matrix $\sum = \left(\left(\sigma_{ij}\right)\right)$ is given by $\sum = diag\left(\theta_1^2, \theta_2^2\right)$.

Further, $\sqrt{n}\left({}_2\hat{L}_s - {}_2L_s\right) \xrightarrow{d} N\left(0, {}_2\sigma^2\left(\theta\right)\right)$ *as* $n \to \infty$, where $\theta = \left(\theta_1, \theta_2\right)$ and

$$ {}_2\sigma^2\left(\theta\right) = \sum_{i=1}^{2}\left(\frac{\partial\, {}_2L_s}{\partial\theta_i}\right)^2 \sigma_{ii} = \frac{2\theta_2^2}{k^2\theta_1^2}\left[\frac{r_0}{1-r_0} + k\right]^2 \tag{3.6} $$

4. CONFIDENCE LIMITS FOR THE EXPECTED NUMBER OF CUSTOMERS

Let ${}_i\sigma^2\left(\hat{\theta}\right)$ be the estimator of ${}_i\sigma^2\left(\theta\right)$ (with $i = 1,2$ corresponding to Models I and II) obtained by replacing θ by a consistent estimator ${}_i\hat{\theta}$ namely ${}_1\hat{\theta} = \left(\overline{Y}_1, \overline{Y}_2, ..., \overline{Y}_k, \overline{X}_1\right)$ and ${}_2\hat{\theta} = \left(\overline{X}_2, \overline{Y}_2\right)$ respectively. Let ${}_i\hat{\sigma}^2 = {}_i\sigma^2\left(\hat{\theta}\right)$. Since ${}_i\sigma^2\left(\theta\right)$ is a continuous function of θ, ${}_i\hat{\sigma}^2$ is a consistent estimator of ${}_i\sigma^2\left(\theta\right)$.
i.e., ${}_i\hat{\sigma}^2 \xrightarrow{P} {}_i\sigma^2\left(\theta\right)$ $\quad$ *as* $n \to \infty,\ i = 1,2.$

By Slutsky theorem $\left(X_n \xrightarrow{d} X,\quad Y_n \xrightarrow{P} b \Rightarrow \dfrac{X_n}{Y_n} \xrightarrow{d} \dfrac{X}{b},\ b \neq 0\right)$, we have

$$ \sqrt{n}\left(\frac{{}_i\hat{L}_s - {}_iL_s}{{}_i\hat{\sigma}}\right) \xrightarrow{d} N\left(0,1\right). $$

$$ \text{i.e.,}\quad \Pr\left[-k_{\alpha/2} < \frac{\sqrt{n}\left({}_i\hat{L}_s - {}_iL_s\right)}{{}_i\hat{\sigma}} < k_{\alpha/2}\right] = \left(1-\alpha\right), $$

where $k_{\alpha/2}$ is obtained from normal tables. Hence, a $100\left(1-\alpha\right)\%$ asymptotic confidence interval for ${}_iL_s$ is given by

$$ {}_i\hat{L}_s \pm k_{\alpha/2}\frac{{}_i\hat{\sigma}}{\sqrt{n}}, \quad i = 1,2. \tag{4.1} $$

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