Pak. J. Statist.
2009 Vol. 25(2), 71-82

ON BAYESIAN ANALYSIS OF THE RAYLEIGH SURVIVAL TIME ASSUMING THE RANDOM CENSOR TIME

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ABSTRACT

Random censoring is an important kind of right censoring in lifetime data analysis. The Rayleigh distributed survival time with a Rayleigh distributed censor time is considered to derive the Maximum Likelihood and the Bayes estimators for the unknown parameters and their corresponding variances. The Uniform and the Square Root Inverted Gamma priors are assumed to find the Bayes estimators under the squared error loss function. The posterior predictive distribution of the future observation, the predictive intervals, the credible intervals and the highest posterior intervals are derived and evaluated. The Inverse transform method of simulation is used to generate data so that the performance of the derived point and interval estimators can be described in terms of numbers. The study is performed for different sample sizes ranging from small to large for various combinations of parameters covering the parameter space to explore and compare properties of the said estimators.

KEY WORDS

Information matrix, predictive interval, credible intervals, highest posterior density intervals, inverse transform method, random censoring

1. INTRODUCTION

Censoring is an important feature of the lifetime data because most of the times it is not feasible to continue the experiment till the last observation. A complete data set is one with life times of all the objects are observed. A censored data has only a known bound on the failure time of at least one object. A censored observation may be left, right or interval censored. Among three types of the right censoring, type I and type II are encountered more frequently in literature as compared to the random censoring. An account of random censoring can be found in Leemis (1995) and Kalbfleisch and Prentice (2002). Censoring is said to be random if the objects on the test can be withdrawn randomly from the experiment at any random time. Rayleigh distribution is especially suitable to analyse life times of the objects that age with time and consequently have a increasing hazard rate. The lifetimes with constant hazard rates are studied using exponential distribution that is famous for its memory less property. Mostert et al. (1998) discussed Bayesian analysis of survival data using the Rayleigh model and Linex Loss. Raqab and Ahsanullah (2001) dealt with generalized exponential distribution. Wang and Li (2005) are among the others who focused estimators for survival function when censoring times are known. Ahmed et al. (2005) considered robust weighted likelihood estimation of a special case of the Weibull distribution. Saleem and Aslam (2008) worked on type I right censored data with fixed censor time. But Abu-Taleb et al. (2007) considered the exponentially distributed survival time with an exponentially distributed censor time and presented some interesting expressions with no focus on computational aspects. In this paper, the ML estimators, the Bayes estimators assuming the Uniform prior and the Bayes estimators assuming the Squared Root Inverted Gamma (SRIG) prior are presented along with their respective variances. The posterior predictive distribution, the predictive intervals, the credible intervals and the Highest Posterior Density (HPD) intervals are discussed as well. A simulation study is conducted to highlight performance of the estimators.

2. THE MAXIMUM LIKELIHOOD ESTIMATORS

The construction of likelihood for random censoring as given in Smith (2002) is made here. It is assumed that the survival time, X, and the censor time, T, independently follow the Rayleigh distributions with unknown parameters θ and λ respectively. Their densities are:

$$f(x|\theta) = \frac{2x}{\theta} e^{-\frac{1}{\theta^2}x^2}, \quad x>0, \quad \theta>0$$

and

$$f(t|\lambda) = \frac{2t}{\lambda} e^{-\frac{1}{\lambda^2}t^2}, \quad t>0, \quad \lambda>0 . \tag{2.1}$$

A sample of n randomly censored, independently and identically distributed observations $\left(\mathrm{Y_i}, \mathrm{D_i}\right)$, i=1,2,...,n are considered with $\mathrm{Y_i} = \mathrm{Min}\left(\mathrm{X_i}, \mathrm{T_i}\right)$ and

$$\mathrm{D_i} = \begin{cases} 1, \text{ if } \mathrm{X_i} \le \mathrm{T_i} \\ 0, \text{ otherwise} \end{cases}$$

The discrete probability distribution of D_i and the joint density of $\left(\mathrm{Y_i}, \mathrm{D_i}\right)$ are

$$P\left(D_i = d_i\right) = \left(\frac{\lambda^2}{\theta^2+\lambda^2}\right)^{d_i} \left(\frac{\theta^2}{\theta^2+\lambda^2}\right)^{1-d_i}, d_i = 0,1$$

and

$$f\left(y_i, d_i \middle|_{\theta,\lambda}\right) = 2y\left(\frac{1}{\theta^2}+\frac{1}{\lambda^2}\right) e^{-\left(\frac{1}{\theta^2}+\frac{1}{\lambda^2}\right)y_i^2} \left(\frac{\lambda^2}{\theta^2+\lambda^2}\right)^{d_i} \left(\frac{\theta^2}{\theta^2+\lambda^2}\right)^{1-d_i}, y_i>0, \theta, \lambda>0$$

respectively. Consequently, the marginal distribution of Y_i is

$$f\left(y_i \middle|_{\theta,\lambda}\right) = \sum_{d_i=0}^{1} f\left(y_i, d_i \middle|_{\theta,\lambda}\right) = 2y\left(\frac{1}{\theta^2}+\frac{1}{\lambda^2}\right) e^{-\left(\frac{1}{\theta^2}+\frac{1}{\lambda^2}\right)y_i^2}, y_i>0, \ \theta, \lambda>0 \tag{2.2}$$

and the expectations of Y_i and D_i are

$$E(Y_i)=\int_0^\infty y\ f\left(y\,\middle|_{\theta,\,\lambda}\right)dy=\frac{\theta^2\lambda^2}{\lambda^2+\theta^2}\ \text{ and } E(D_i)=\sum_{d_i=0}^{1} d_i\ \mathrm{P}(\mathrm{D_i}=d_i)=\frac{\lambda^2}{\lambda^2+\theta^2}$$

respectively. It is interesting to note that $P(D_i=d_i)=\int_0^\infty f\left(y_i,d_i\,\middle|_{\theta,\,\lambda}\right)dy$

The likelihood function can be expressed as

$$L\left(\theta,\lambda\middle|_{\mathbf{z}}\right)\propto\left(\theta^2\right)^{-\sum d_i}e^{-\frac{1}{\theta^2}\sum y_i^2}\left(\lambda^2\right)^{-\left(n-\sum d_i\right)}e^{-\frac{1}{\lambda^2}\sum y_i^2}$$

where

$$\mathbf{z}=\left[z_1,z_2,\ldots,z_n\right]\ \text{ and }\ z_i=\left(Y_i,D_i\right),\ i=1,2,\ldots,n$$

and the natural log of the likelihood function is

$$l=\ln L\left(\theta,\lambda\middle|_{\mathbf{z}}\right)\propto\left(-\sum d_i\right)\ln\theta^2-\left(n-\sum d_i\right)\ln\lambda^2-\frac{1}{\theta^2}\sum y_i^2-\frac{1}{\lambda^2}\sum y_i^2 \qquad (2.3)$$

The ML estimators are obtained by solving the two equations $\frac{\delta l}{\delta\theta}=0=\frac{\delta l}{\delta\lambda}$, obtained by setting the first order derivatives of the log likelihood to zero, and they are $\hat{\theta}=\sqrt{\frac{\sum y_i^2}{\sum d_i}}$ and $\hat{\lambda}=\sqrt{\frac{\sum y_i^2}{n-\sum d_i}}$. These ML estimators can be viewed as the Bayes estimators under the Zero-One Loss and the posterior assuming the Uniform prior.

The second order derivatives of the log likelihood are

$$\frac{\delta^2 l}{\delta\theta^2}=\frac{-6\left(\sum y_i^2\right)}{\theta^4}+\frac{2\sum d_i}{\theta^2},\ \frac{\delta^2 l}{\delta\lambda^2}=\frac{2(n-\sum d_i)}{\lambda^2}-\frac{6(\sum y_i^2)}{\lambda^4},$$

and

$$\frac{\delta^2 l}{\delta\theta\ \delta\lambda}=0=\frac{\delta^2 l}{\delta\lambda\ \delta\theta},$$

which make

$$I(\boldsymbol{\theta})=-E\begin{bmatrix}\frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\theta\ \partial\lambda}\\ \frac{\partial^2 l}{\partial\lambda\ \partial\theta} & \frac{\partial^2 l}{\partial\lambda^2}\end{bmatrix}\ \text{ reduces to }\ I(\boldsymbol{\theta})=\begin{bmatrix}\frac{4\ n\ \lambda^2}{\theta^2\left(\lambda^2+\theta^2\right)} & 0\\ 0 & \frac{4\ n\ \theta^2}{\lambda^2\left(\lambda^2+\theta^2\right)}\end{bmatrix},$$

where $\boldsymbol{\theta}=(\theta,\lambda)$ is the vector of unknown parameters. $I(\boldsymbol{\theta})$ is a diagonal matrix that can easily be inverted by inverting the terms on the main diagonal. The terms on the main diagonal of the inverted information matrix, $I^{-1}(\boldsymbol{\theta})$, give the variances of ML estimators as $V\left(\hat{\theta}\right)=\dfrac{\theta^2\left(\lambda^2+\theta^2\right)}{4\,n\,\lambda^2}$ and $V\left(\hat{\lambda}\right)=\dfrac{\lambda^2\left(\lambda^2+\theta^2\right)}{4\,n\,\theta^2}$. These variances are in terms of unknown parameters θ and λ. Therefore, for practical purposes, estimated values of these variances are evaluated by replacing the parameters θ and λ by their respective ML estimators. The expressions for the estimated variances of the estimators $\hat{\theta}$ and $\hat{\lambda}$ turn out to be $v\left(\hat{\theta}\right)=\dfrac{\hat{\theta}^2\left(\hat{\lambda}^2+\hat{\theta}^2\right)}{4\,n\,\hat{\lambda}^2}=\dfrac{\left(\sum y_i^2\right)}{4\,\left(\sum d\right)^2}$ and $v\left(\hat{\lambda}\right)=\dfrac{\hat{\lambda}^2\left(\hat{\lambda}^2+\hat{\theta}^2\right)}{4\,n\,\hat{\theta}^2}=\dfrac{\left(\sum y_i^2\right)}{4\left(n-\sum d\right)^2}$. Here v represents estimate of V.

3. THE BAYES ESTIMATORS ASSUMING THE UNIFORM PRIOR

Let us assume a state of ignorance that θ and λ are Uniformly distributed over $(0,\infty)$. Hence $f_1(\theta)=k_1;\ 0<\theta<\infty$ and $f_2(\lambda)=k_2;\ 0<\lambda<\infty$. Assuming independence we have an improper joint prior that is proportional to a constant and is incorporated with the likelihood to yield a proper joint posterior distribution given by

$$p\left(\theta,\lambda\middle|\mathbf{z}\right)=\frac{\left(\sum y_i^2\right)^{n-1}}{\Gamma\left(\sum d_i-\frac{1}{2}\right)\,\Gamma\left(n-\sum d_i-\frac{1}{2}\right)}\left(\frac{1}{\theta^2}\right)^{\sum d_i}e^{-\frac{1}{\theta^2}\sum y_i^2}\left(\frac{1}{\lambda^2}\right)^{n-\sum d_i}e^{-\frac{1}{\lambda^2}\sum y_i^2},\ \theta,\lambda>0 \tag{3.1}$$

The marginal posterior distributions of θ and λ are

$$p_1\left(\theta\middle|\mathbf{z}\right)=\frac{\left(\sum y_i^2\right)^{\sum d_i-\frac{1}{2}}}{\Gamma\left(\sum d_i-\frac{1}{2}\right)}\left(\frac{1}{\theta^2}\right)^{\sum d_i}e^{-\frac{\sum y_i^2}{\theta^2}},\ \theta>0$$

and

$$p_2\left(\lambda\middle|\mathbf{z}\right)=\frac{\left(\sum y_i^2\right)^{n-\sum d_i-\frac{1}{2}}}{\Gamma\left(n-\sum d_i-\frac{1}{2}\right)}\left(\frac{1}{\lambda^2}\right)^{n-\sum d_i}e^{-\frac{\sum y_i^2}{\lambda^2}},\ \lambda>0 \tag{3.2}$$

respectively. Using the Squared Error Loss Function (SELF), the Bayes estimators of θ and λ are $\hat{\theta}=\dfrac{\sqrt{\sum y_i^2}\ \Gamma\left(\sum d_i-1\right)}{\Gamma\left(\sum d_i-\frac{1}{2}\right)}$ and $\hat{\lambda}=\dfrac{\sqrt{\sum y_i^2}\ \Gamma\left(n-\sum d_i-1\right)}{\Gamma\left(n-\sum d_i-\frac{1}{2}\right)}$ respectively. Clearly the Bayes estimators are a constant times their corresponding ML counterpart.

The variances of the Bayes estimators with the Uniform prior are derived as

$$V\left(\hat{\theta}\middle|_{\mathbf{z}}\right)=\frac{\sum y_i^2}{\Gamma^2\left(\sum d_i-\frac{1}{2}\right)}\left\{\Gamma\left(\sum d_i-\frac{1}{2}\right)\Gamma\left(\sum d_i-\frac{3}{2}\right)-\Gamma^2\left(\sum d_i-1\right)\right\}$$

and

$$V\left(\hat{\lambda}\middle|_{\mathbf{z}}\right)=\frac{\sum y_i^2}{\Gamma^2\left(n-\sum d_i-\frac{1}{2}\right)}\left\{\Gamma\left(n-\sum d_i-\frac{1}{2}\right)\Gamma\left(n-\sum d_i-\frac{3}{2}\right)-\Gamma^2\left(n-\sum d_i-1\right)\right\}.$$

Each of these variances is a linear combination of its ML counterpart.

4. THE BAYES ESTIMATORS ASSUMING SRIG PRIOR

Let θ and λ be distributed as SRIG variables with hyper-parameters $\left(a_1,b_1\right)$ and $\left(a_2,b_2\right)$ respectively. Assuming independence, we have a joint prior that is incorporated with the likelihood to yield the following joint posterior.

$$p\left(\theta,\lambda\middle|_{\mathbf{z}}\right)=\frac{\left(\sum y_i^2+b_1\right)^{j\sum d_i+a_1}\left(\sum y_i^2+b_2\right)^{n-\sum d_i+a_2}}{\Gamma\left(\sum d_i+a_1\right)\,\Gamma\left(n-\sum d_i+a_1\right)}$$
$$\times\left(\frac{1}{\theta^2}\right)^{\sum d_i+a_1+\frac{1}{2}}e^{-\frac{\sum y_i^2+b_1}{\theta^2}}\left(\frac{1}{\lambda^2}\right)^{n-\sum d_i+a_2+\frac{1}{2}}e^{-\frac{\sum y_i^2+b_2}{\lambda^2}}.\quad\theta,\lambda>0 \tag{4.1}$$

The marginal posterior distributions of θ and λ are

$$p_1\left(\theta\middle|\mathbf{z}\right)=\frac{\left(\sum y_i^2+b_1\right)^{\sum d_i+a_1}}{\Gamma\left(\sum d_i+a_1\right)}\left(\frac{1}{\theta^2}\right)^{\sum d_i+a_1+\frac{1}{2}}e^{-\frac{\sum y_i^2+b_1}{\theta^2}}.\quad\theta>0 \tag{4.2}$$

and

$$p_2\left(\lambda\middle|\mathbf{z}\right)=\frac{\left(\sum y_i^2+b_2\right)^{n-\sum d_i+a_2}}{\Gamma\left(n-\sum d_i+a_2\right)}\left(\frac{1}{\lambda^2}\right)^{n-\sum d_i+a_2+\frac{1}{2}}e^{-\frac{\sum y_i^2+b_2}{\lambda^2}}.\quad\lambda>0 \tag{4.3}$$

Using the SELF, the Bayes estimators of θ and λ are found to be

$$\hat{\theta}=\frac{\sqrt{\left(\sum y_i^2+b_1\right)}\;\Gamma\left(\sum d_i+a_1-\frac{1}{2}\right)}{\Gamma\left(\sum d_i+a_1\right)}\text{ and }\hat{\lambda}=\frac{\sqrt{\left(\sum y_i^2+b_2\right)}\;\Gamma\left(n-\sum d_i+a_2-\frac{1}{2}\right)}{\Gamma\left(n-\sum d_i+a_2\right)}.$$

Each of these Bayes estimators is a linear combination of its ML as well as its Bayesian (Uniform prior) counterpart.

The expressions for the variances of the Bayes estimators with the SRIG prior are derived as

$$V\left(\hat{\theta}\big|_{\mathbf{z}}\right)=\frac{\sqrt{\left(\sum y_i^2+b_1\right)}}{\Gamma^2\left(\sum d_i+a_1\right)}\left\{\Gamma\left(\sum d_i+a_1\right)\Gamma\left(\sum d_i+a_1-1\right)-\Gamma^2\left(\sum d_i+a_1-\frac{1}{2}\right)\right\},$$

and

$$V\left(\hat{\lambda}\big|_{\mathbf{z}}\right)=\frac{\sqrt{\left(\sum y_i^2+b_2\right)}}{\Gamma^2\left(\sum d_i+a_2\right)}\left\{\Gamma\left(\sum d_i+a_2\right)\Gamma\left(\sum d_i+a_2-1\right)-\Gamma^2\left(\sum d_i+a_2-\frac{1}{2}\right)\right\}.$$

Each of these variances is a linear combination of its ML as well as its Bayesian (Uniform prior) counterpart.

5. THE POSTERIOR PREDICTIVE DISTRIBUTION AND THE BAYESIAN PREDICTIVE INTERVALS

The posterior predictive distribution of the future observation, y is defined as

$$p(y\big|_{\mathbf{z}})=\int_0^\infty\int_0^\infty p(\,\theta,\lambda\big|_{\mathbf{z}})\,f(y\big|_{\theta,\lambda})\;d\theta\;d\lambda\;,$$

where $f\left(y\big|_{\theta,\lambda}\right)$ and $p\left(\theta,\lambda\big|_{z}\right)$ are given by (2.2) and (4.1) respectively.

$$p\left(y\big|_{\mathbf{z}}\right)=\frac{2y\;\left(\sum y^2+b_1\right)^{\sum d+a_1}\left(\sum y^2+b_2\right)^{n-\sum d+a_2}}{\left(\sum y^2+b_1+y^2\right)^{\sum d+a_1}\left(\sum y^2+b_2+y^2\right)^{n-\sum d+a_2}}$$

$$\times\left\{\frac{\sum d+a_1}{\sum y^2+b_2+y^2}+\frac{n-\sum d+a_2}{\sum y^2+b_1+y^2}\right\}.\;\; y>0 \tag{5.1}$$

with $a_1=a_2=a$ and $b_1=b_2=b$, it further reduces to

$$p\left(y\big|_{\mathbf{z}}\right)=\frac{2y(n+2a)\left(\sum y^2+b\right)^{n+2a}}{\left(\sum y^2+b+y^2\right)^{n+2a+1}}.\;\; y>0 \tag{5.2}$$

The $(1-\alpha)100\%$ Bayesian predictive interval (L,U) can be obtained by solving the two equations $\int_0^L p\left(y\big|_{\mathbf{z}}\right)dy=\frac{\alpha}{2}=\int_U^\infty p\left(y\big|_{\mathbf{z}}\right)dy$ where $p\left(y\big|_{\mathbf{z}}\right)$ given by (5.2). These equations, on simplification become

$$(n+2a)(n+2a+1)\left(\sum y^2+b\right)^{n+2a}\left\{\frac{1}{\left(\sum y^2+b\right)^{n+2a+2}}-\frac{1}{\left(\sum y^2+b+L^2\right)^{n+2a+2}}\right\}-\frac{\alpha}{2}=0\,,$$

and

$$\frac{(n+2a)(n+2a+1)\left(\sum y^2+b\right)^{n+2a}}{\left(\sum y^2+b+U^2\right)^{n+2a+2}}-\frac{\alpha}{2}=0\ .$$

Evaluating these predictive intervals for various combinations of the hyper-parameters, a trend in hyper-parameters can be determined which leads to enhance the efficiency of the Bayes estimates as discussed in Saleem and Aslam (2008). This trend can also be used to further filter the prior information provided by a number of experts.

6. THE BAYESIAN CREDIBLE INTERVALS AND THE HIGHEST POSTERIOR DENSITY (HPD) INTERVALS

Sinha (1998) has presented an interesting discussion of some Bayesian counterparts of confidence intervals. The $(1-\alpha)100\%$ Bayesian credible intervals are obtained using the marginal distribution of the respective parameter of interest. The marginal posterior densities of θ and λ (assuming Uniform prior) lead to the credible intervals

$$\sqrt{\frac{2\sum \mathrm{y}^2}{\chi^2_{\frac{\alpha}{2},2\left(\sum d-\frac{1}{2}\right)}}}\le\theta\le\sqrt{\frac{2\sum \mathrm{y}^2}{\chi^2_{1-\frac{\alpha}{2},2\left(\sum d-\frac{1}{2}\right)}}}\ ,$$

and

$$\sqrt{\frac{2\sum \mathrm{y}^2}{\chi^2_{\frac{\alpha}{2},2\left(n-\sum d-\frac{1}{2}\right)}}}\le\lambda\le\sqrt{\frac{2\sum \mathrm{y}^2}{\chi^2_{1-\frac{\alpha}{2},2\left(n-\sum d-\frac{1}{2}\right)}}}\ .$$

And the marginal posterior densities of θ and λ (assuming SRIG prior) lead to the credible intervals

$$\sqrt{\frac{2\left(\sum \mathrm{y}^2+\mathrm{b}_1\right)}{\chi^2_{\frac{\alpha}{2},2\left(\sum d+a_1-\frac{1}{2}\right)}}}\le\theta\le\sqrt{\frac{2\left(\sum \mathrm{y}^2+\mathrm{b}_1\right)}{\chi^2_{1-\frac{\alpha}{2},2\left(\sum d+a_1-\frac{1}{2}\right)}}}\ ,$$

and

$$\sqrt{\frac{2\left(\sum \mathrm{y}^2+\mathrm{b}_2\right)}{\chi^2_{\frac{\alpha}{2},2\left(n-\sum d+a_2-\frac{1}{2}\right)}}}\le\lambda\le\sqrt{\frac{2\left(\sum \mathrm{y}^2+\mathrm{b}_2\right)}{\chi^2_{1-\frac{\alpha}{2},2\left(n-\sum d+a_2-\frac{1}{2}\right)}}}\ .$$

Here $\chi^2_{\frac{\alpha}{2},m}$ and $\chi^2_{1-\frac{\alpha}{2},m}$ are $(1-\frac{\alpha}{2})100$ th and $(\frac{\alpha}{2})100$ th percentiles respectively.

The HPD interval is defined on the posterior density such that the posterior density at every point inside the HPD interval is greater than the posterior density at every point outside the HPD interval. An interval (θ_1, θ_2) would be a $(1-\alpha)100\%$ HPD interval for θ if it satisfy the following two conditions simultaneously.

$$\int_{\theta_1}^{\theta_2} p_1(\theta \big| \mathbf{z})\, d\theta = 1-\alpha \text{ and } p_1(\theta_1 \big| \mathbf{z}) = p_1(\theta_2 \big| \mathbf{z}) \text{ where } p_1(\theta \big| \mathbf{z}) \text{ is given by } (4.2).$$

On simplification, these two conditions reduce to

$$\Gamma\left(\sum d + a_1 - \frac{1}{2}, \frac{\sum y_i^2 + b_1}{\theta_2^2}\right) - \Gamma\left(\sum d + a_1 - \frac{1}{2}, \frac{\sum y_i^2 + b_1}{\theta_1^2}\right) - (\alpha - 1)\, \Gamma\left(\sum d + a_1 - \frac{1}{2}\right) = 0$$

and

$$\left(\sum d + a_1 + \frac{1}{2}\right) \ln\left(\frac{\theta_2^2}{\theta_1^2}\right) - \left(\frac{1}{\theta_1^2} - \frac{1}{\theta_2^2}\right)\left(\sum y_i^2 + b_1\right) = 0$$

Solving these two equations simultaneously gives the HPD interval (θ_1, θ_2) for θ. Here $\Gamma\left(\sum d + a_1 - \frac{1}{2},\ \frac{\sum y_i^2 + b_1}{\theta_1^2}\right) = \int_{\frac{\sum y_i^2 + b_1}{\theta_1^2}}^{\infty} \theta^{\sum d + a_1 - \frac{3}{2}} e^{-\theta} d\theta$ is the incomplete Gamma function. While $\Gamma\left(\sum d + a_1 - 2\right) = \int_0^{\infty} \theta^{\sum d + a_1 - 3} e^{-\theta} d\theta$ is the Gamma function. Similarly a $(1-\alpha)100\%$ HPD interval (λ_1, λ_2) for λ is obtained by solving the following two equations simultaneously.

$$\int_{\lambda_1}^{\lambda_2} p_2\left(\lambda \big|_{\mathbf{z}}\right) d\lambda = 1-\alpha \text{ and } p_2\left(\lambda_1 \big|_{\mathrm{z}}\right) = p_2\left(\lambda_2 \big|_{\mathrm{z}}\right) \text{ where } p_2\left(\lambda \big|_{\mathrm{z}}\right) \text{ is given by } (4.3).$$

$$\left(n - \sum d + a_2 - \frac{1}{2}\right) \ln\left(\frac{\lambda_2^2}{\lambda_1^2}\right) - \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right)\left(\sum y_i^2 + b_2\right) = 0$$

and

$$\Gamma\left(n - \sum d + a_2 - \frac{1}{2}, \frac{\left(\sum y_i^2 + b_2\right)}{\theta_2^2}\right) - \Gamma\left(n - \sum d + a_2 - \frac{1}{2}, \frac{\left(\sum y_i^2 + b_2\right)}{\theta_1^2}\right)$$
$$-(\alpha - 1)\, \Gamma\left(n - \sum d + a_2 - \frac{1}{2}\right) = 0$$

7. A SIMULATION STUDY

We simulate random samples of various sizes ranging from small to large i.e., $n = 50, 100, 150, 200, 300, 400$ with various combinations of the parameter values which cover the parameter space as well as $(\theta, \lambda) = (1, 0.7), (5, 3.5), (10, 7), (15, 3.5)$. As one data set does not help clarify performance of the methods, each data set is based on repeated simulations a large number of times using computer program in Minitab. The numerical calculations are conducted in Mathematica.

Table 7.1
The ML estimates and the Bayes estimates assuming the Uniform and the SRIG priors of parameters $\theta = 1,\ \lambda = 0.7$

Sample Size	Methods					
	ML estimates		Bayes estimates (Uniform)		Bayes estimates (SRIG)	
n	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
50	0.99878 (0.01515)	0.69966 (0.00365)	1.03895 (0.01819)	0.71306 (0.00398)	1.05322 (0.013980)	0.702791 (0.003345)
100	0.995018 (0.007489)	0.699135 (0.001825)	1.01437 (0.008185)	0.705752 (0.001906)	1.02576 (0.007221)	0.700821 (0.001746)
150	0.998632 (0.005071)	0.697313 (0.001206)	1.01157 (0.005381)	0.701675 (0.001241)	1.01979 (0.004948)	0.698522 (0.001171)
200	0.999018 (0.003790)	0.69979 (0.000912)	1.00864 (0.003961)	0.703072 (0.000932)	1.01512 (0.003722)	0.700636 (0.000892)
300	1.00059 (0.002539)	0.70005 (0.000608)	1.00699 (0.002614)	0.702232 (0.000617)	1.01149 (0.002509)	0.700614 (0.000599)
400	0.999525 (0.001896)	0.700295 (0.000457)	1.0043 (0.001939)	0.701932 (0.000462)	1.00781 (0.001880)	0.700718 (0.000452)

Table 7.2
The ML estimates and the Bayes estimates assuming the Uniform and the SRIG priors of parameters $\theta = 5,\ \lambda = 3.5$

Sample Size	Methods					
	ML estimates		Bayes estimates (Uniform)		Bayes estimates (SRIG)	
n	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
50	5.03975 (0.392251)	3.48714 (0.089910)	5.24601 (0.472337)	3.5534 (0.098068)	5.01338 (0.321153)	3.50386 (0.082533)
100	4.97459 (0.187088)	3.49659 (0.045666)	5.07129 (0.204471)	3.52969 (0.047692)	4.96688 (0.169231)	3.50497 (0.043682)
150	5.01464 (0.12854)	3.48796 (0.030086)	5.07994 (0.136441)	3.50974 (0.030960)	5.00646 (0.119849)	3.49393 (0.029214)
200	4.96972 (0.092905)	3.50599 (0.023012)	5.0171 (0.097057)	3.52252 (0.023515)	4.96595 (0.088280)	3.51004 (0.022497)
300	4.99724 (0.063224)	3.5004 (0.015221)	5.02917 (0.065105)	3.51132 (0.015440)	4.99368 (0.061047)	3.50322 (0.014994)
400	4.9983 (0.047516)	3.49692 (0.011384)	5.02226 (0.048572)	3.50508 (0.011506)	4.9956 (0.046276)	3.49909 (0.011256)

Table 7.3
The ML estimates and the Bayes estimates assuming the Uniform and the SRIG priors of parameters $\theta = 10,\ \lambda = 7$

Sample Size	Methods					
	ML estimates		Bayes estimates (Uniform)		Bayes estimates (SRIG)	
n	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
50	9.97981 (1.5106)	6.99854 (0.365334)	10.3805 (1.8127)	7.13271 (0.398793)	10.0449 (1.27011)	7.02969 (0.334862)
100	10.0188 (0.765648)	6.99555 (0.181992)	10.2153 (0.837466)	7.06149 (0.190031)	10.0509 (0.698592)	7.01211 (0.174112)
150	9.9767 (0.502822)	7.00048 (0.121893)	10.1051 (0.533347)	7.04441 (0.125455)	10.0018 (0.473082)	7.01158 (0.118307)
200	9.97016 (0.376856)	6.99265 (0.091187)	10.066 (0.393836)	7.02548 (0.093175)	9.98971 (0.359911)	7.00127 (0.089167)
300	10.0149 (0.254399)	7.00539 (0.060907)	10.079 (0.261984)	7.02722 (0.061785)	10.0265 (0.246547)	7.01094 (0.059998)
400	9.99707 (0.189951)	6.99765 (0.455993)	10.0449 (0.194171)	7.014 (0.046092)	10.0064 (0.185544)	7.00194 (0.045088)

Table 7.4
The ML estimates and the Bayes estimates assuming the Uniform and the SRIG priors of parameters $\theta = 15,\ \lambda = 10.5$

Sample Size	Methods					
	ML estimates		Bayes estimates (Uniform)		Bayes estimates (SRIG)	
n	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
50	15.1065 (3.49493)	10.5176 (0.821204)	15.7193 (4.20168)	10.7182 (0.896034)	15.0136 (2.86033)	10.4986 (0.743685)
100	14.9572 (1.69781)	10.4833 (0.40972)	15.249 (1.8562)	10.5824 (0.427863)	14.9215 (1.53263)	10.4751 (0.389473)
150	153.0081 (1.14066)	10.5117 (0.274504)	15.2017 (1.21008)	10.5776 (0.282516)	14.98 (1.06365)	10.5052 (0.265264)
200	15.0142 (0.856779)	10.5105 (0.205761)	15.1588 (0.895482)	1.5598 (0.210240)	14.9924 (0.812592)	10.5056 (0.200525)
300	14.9722 (0.56741)	10.4892 (0.136686)	15.0678 (0.584293)	10.5219 (0.138660)	14.9589 (0.547689)	10.4862 (0.134356)
400	15.0025 (0.427518)	10.5062 (0.10282)	15.0743 (0.437011)	10.5308 (0.103930)	14.9917 (0.416225)	10.5038 (0.101496)

It is immediate from Tables 7.1-7.4 (the round brackets contain variances) that the ML estimates and the Bayes estimates assuming the Uniform prior are not only almost equal but are equally efficient as well for all sample sizes and parameter values. While the Bayes estimates assuming the SRIG prior turn out to be more precise as compared to their ML and the Bayes (Uniform) counterparts. This establishes the supremacy of the

Informative prior (the SRIG) over the uninformative prior (the Uniform). This precision may further be increased with the use of more effective prior information and consequently by adopting the more suitable hyper-parameters. This increase in efficiency is because of the fact that using a prior information is equivalent to adding a number of observations to the given sample size as discussed in Bolstad (2004). The proposed Bayes estimators assuming the SRIG prior behaved well for the small samples as well for different combinations of the unknown parameters as depicted in Tables 7.1-7.4.

Table 7.5
95% Bayesian Credible Intervals and HPD Intervals Assuming Uniform and SRIG Prior $\theta = 15,\ \lambda = 10$

Parameters	Credible Intervals (Uniform prior)	HPD Intervals (Uniform prior)	Credible Intervals (SRIG prior)	HPD Intervals (SRIG prior)
$\theta = 15$	(13.9387,16.3297)	(13.90,16.287)	(13.1679,15.2834)	(13.136,15.248)
$\lambda = 10$	(9.5229,10.582)	(9.511,10.57)	(9.21189,10.1995)	(9.2,10.19)

Although, all the confidence intervals contain the values of the respective unknown parameters, yet Table 7.5 depicts that both the credible intervals and the HPD intervals assuming the Informative prior (SRIG) are pretty shorter than those obtained by assuming the state of ignorance i.e.; the Uniform prior. This again highlights the advantage of incorporating informative prior information (based on a proper probability density function) into the analysis rather than an uninformative one that has either no proper density or involves no hyper-parameters. The HPD intervals are not only narrower than their corresponding credible intervals but are also slightly left aligned to capture the denser parts of the respective marginal distributions. This is supported by the fact that the marginal distributions of the unknown parameters are positively skewed. A more extensive analysis can be managed by considering a number of different parameter points and a variety of sample sizes as done in the above point estimation case.

8. CONCLUSION

The ML and the Bayes estimates assuming the Uniform prior are almost identical and so are their variances for small and large sample sizes. However, the Bayes estimates assuming the SRIG prior have the least variances. This shows that the use of an informative prior may be more paying than an uninformative prior. For all the parameter values, small or large, a reduction in variances of both the estimates is observed with the increase in sample size. For all the sample sizes and parameter values, variance of lifetime parameter is observed to be larger than that of censoring time parameter. For a fixed sample size, small or large, estimates of both the parameters have smaller variances when parameter values are small and have larger variances when parameter values are large. In other words, variances of estimates of larger parameters have larger variances as compared to that of estimates of smaller parameter values whatever be the sample size.

The credible intervals assuming the SRIG prior are much narrower than the credible intervals assuming the Uniform prior. Also, the HPD intervals assuming the SRIG prior are more precise than the HPD intervals assuming the state of ignorance. It is the use of prior information that makes a difference in terms of gain in precision. As the marginal posterior densities are positively skewed, so the HPD intervals are slightly left aligned as compared to the corresponding credible intervals. Also, the lengths of the HPD intervals are shorter than the lengths of the corresponding credible intervals. The predictive intervals may be used to discover a range of the hyper-parameters that ensure more precise estimates. This can also be used to further filter the pieces of prior information provided by a number of experts.

ACKNOWLEDGEMENT

The authors are grateful to the referees and the Editor for their valuable comments for the considerable improvement of the manuscript.

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