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PRIORITIZING THE ITEMS THROUGH PAIRED COMPARISON MODELS, A BAYESIAN APPROACH

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ABSTRACT

A probabilistic approach to build a model for the paired comparisons experimentation based on the comparison of two Cauchy random variables is considered. The Cauchy distribution may be used to model a variety of phenomena, such as the impact of light particles being emitted from a point source on a plane surface. By basing the preference criterion on such impact, the device with greater impact may be preferred to the device with the lower impact (or as the theory suggests). Moreover, Adams (2005) also discusses the Cauchy model while studying dominance hierarchies in social animals. The paired comparison model is built and is then studied in Bayesian framework. The inferences about the proposed model parameters are made using the non-informative priors (uniform and Jeffreys), as well as informative and the conjugate priors. The hyperparameters are also elicited. Although any group of individuals may be ranked using the Cauchy model, we illustrate the Bayesian inferential procedure with real data on five top-ranked one-day international (ODI) cricket teams.

KEY WORDS

Paired Comparisons, Cauchy distribution, Worth Parameters, Bayesian Analysis, Uniform, Jeffreys, informative and conjugate priors, Elicitation of Hyperparameters.

1. INTRODUCTION

The method of paired comparisons is a technique in which individuals may be ranked on the basis of wins and losses in their pair-wise encounters. It is primarily used for subjective judgments where quantitative measurement is impossible or impracticable. Its also used in many cases where there may be a substantial effect of sampling error on the measurements. Hence it is widely used by the psychometricians. The most frequent application has been to sensory testing; especially taste testing, to consumer tests, personal rating and choice behavior.

David (1988) provides a detailed review of paired comparison models. Probably the most cited among the applied uses of paired comparisons is the tournament analysis in which the objects are players or teams competing with each other on pairs. Thurstone (1927) assumes the responses to follow normal distribution but Bradley (1953) assumes the Logistic distribution and presents their paired comparison models. Stern (1990) considers an approach to build models for paired comparisons on the comparison of two Gamma random variables with the same shape parameter but different scale parameters.

In Section 2, the logical basis and idea for construction of the Cauchy Model are discussed. Sections 3 and 4 deal with the notations and the prior distributions. The Bayesian analysis is considered in Section 5. Section 6 addresses the appropriateness of the model. All the results are discussed along with the concluding remarks in Section 7. The entire procedure is illustrated using real data on five top-rank ODI cricket teams, namely, Australia, India, New Zealand, Pakistan and South Africa.

2. THE CAUCHY MODEL FOR PAIRED COMPARISONS

Literature suggests that the Cauchy distribution may be used to model the impact of light particles being emitted from a point source, on a plane. By basing the preference criterion on such impact, the device with greater impact may be preferred to the device with the lower impact. Adams (2005) also discusses the Cauchy model.

Considering the standardized Cauchy distribution and using the model building criteria adopted by Bradley and Terry (1952), Bradley (1953) and also discussed by David (1988), the probability $P\left(T_i > T_j \mid \theta_i, \theta_j\right)$ that the treatment (device) T_i is preferred to T_j is given by:

$$\phi_{ij} = F\left(\theta_i - \theta_j\right) = \int_{-\infty}^{\theta_i - \theta_j} \frac{1}{\pi\left(1+y^2\right)} dy = \int_{-(\theta_i - \theta_j)}^{\infty} \frac{1}{\pi\left(1+y^2\right)} dy$$

$$= \frac{\pi + 2\tan^{-1}\left(\theta_i - \theta_j\right)}{2\pi}, \qquad (2.1)$$

where ϕ_{ij} denotes the probability of preferring the treatment T_i over T_j for all $i(< j) = 1, 2, ..., t$ with its complement $\phi_{ji} = 1 - \phi_{ij}$ (ties not being considered at the moment) and $\pi \cong 3.141593$. Here the expression (2.1) serves as a model for the paired comparison experimentation.

3. NOTATIONS AND THE LIKELIHOOD FUNCTION OF THE MODEL

Let $n_{ij} = n_{ji}$ be the total comparisons made between the stimuli T_i and T_j, a_{ij} be the number of times the stimulus T_i is preferred over T_j, with $a_{ji} = n_{ij} - a_{ij}$ denoting it reverse. For the present situation, the trials are independent with only two categories of the outcomes for each trial, e.g. winning or losing the match and each outcome with constant probability φ_{ij} of preferring the stimulus T_i over T_j for all $i(\neq j) = 1, 2, ..., t$. The paired comparison experiment is performed a fixed number of times i.e. n_{ij}. Then the random variable a_{ij} follows the $Binomial(n_{ij}, \varphi_{ij})$. Then the likelihood function $L\left(a, \theta\right)$ is given by:

$$L(\mathrm{a};\theta) = \prod_{i(<j)=1}^{t} C_{a_{ij}}^{n_{ij}} \phi_{ij}^{a_{ij}} (1-\phi_{ij})^{n_{ij}-a_{ij}}$$

$$L(\mathrm{a};\theta) = \prod_{i(<j)=1}^{t} \frac{n_{ij}\,!}{(2\pi)^{n_{ij}}\, a_{ij}\,!\left(n_{ij}-a_{ij}\right)!} \times \left[\left\{\pi + 2\tan^{-1}(\theta_i-\theta_j)\right\}^{a_{ij}} \left\{\pi - 2\tan^{-1}(\theta_i-\theta_j)\right\}^{a_{ji}}\right], \qquad (3.1)$$

where θ denotes the vector of the unknown parameters θ_i which refers to the worth parameter of the *i*th of the five ODI cricket teams under consideration with i =1,2,…,5 and the team names given in Section 1.

4. THE PRIOR DISTRIBUTIONS

We use the non-informative –the uniform prior (UP) and the Jeffreys prior (JP)–priors, the informative prior (IP) and the conjugate prior (CP) distributions. We define the uniform prior as $p_U(\theta) \propto 1$, the Jeffreys prior as $\left\{p_J\left(\theta\right)\right\} \propto \sqrt{\det\left\{I\left(\theta\right)\right\}}$, where $I(\theta) = -E\left\{\dfrac{\partial^2 \ln L(a_{ij};\theta)}{\partial\theta^2}\right\}$ is the Fisher Information matrix, the conjugate prior $p_C(\theta) = \prod_{i(<j)=1}^{t} \begin{pmatrix} n_{ij} \\ a_{ij}^{0} \end{pmatrix} \phi_{ij}^{\,a_{ij}^{o}} \phi_{ji}^{\,a_{ji}^{o}}$, and the independent multivariate normal distribution $\theta_i \sim N\left(\mu_i, \sigma_i^2\right), \forall i = 1,2,...,t$ are used as the informative prior $p_I\left(\theta\right)$ distribution with the parametric range $-\infty \le \theta \le \infty$ for all types of the priors.

5. BAYESIAN ANALYSIS OF THE MODEL

The posterior distribution for the vector of model parameters θ given data is:

$$P\left(\theta \mid \mathrm{a}\right) = K^{-1} p\left(\theta\right) L\left(\mathrm{a};\theta\right)$$

$$p\left(\theta \mid \mathrm{a}\right) = K^{-1} p\left(\theta\right) \prod_{i(<j)=1}^{t} \left[\left\{\pi + 2\tan^{-1}\left(\theta_i-\theta_j\right)\right\}^{a_{ij}} \left\{\pi - 2\tan^{-1}\left(\theta_i-\theta_j\right)\right\}^{a_{ji}}\right], \quad -\infty < \theta < \infty \qquad (5.1)$$

where the normalizing constant may be defined as:

$$K = \int_{\theta} p\left(\theta\right) \prod_{i(<j)=1}^{t} \left[\left\{\pi + 2\tan^{-1}\left(\theta_i-\theta_j\right)\right\}^{a_{ij}} \left\{\pi - 2\tan^{-1}\left(\theta_i-\theta_j\right)\right\}^{a_{ji}}\right] d\theta \,.$$

Here $p\left(\theta\right)$ denotes one of the prior types incorporated in the analysis mentioned earlier and the data to be used is displayed in Table 1. The leading column denotes the victors and the first row, the losers.

Table 1:
Data of ODI Cricket Matches

Teams	**Australia**	**India**	**New Zealand**	**Pakistan**	**South Africa**
Australia	-	15	12	10	15
India	4	-	3	9	6
New Zealand	6	7	-	6	6
Pakistan	4	8	11	-	3
South Africa	9	7	10	6	-

5.1 Elicitation of the Hyperparameters

When substantial amount of information is available pertaining to the model or population parameter(s), it becomes desirable to quantify such information in the form of a (prior) probability. The process of quantifying the prior information accurately is known as elicitation. According to Garthwaite et al. (2005), the general algorithm for the elicitation of the hyperparameters may be depicted as Figure 1.

We use the minimum chi-square (MCS) and the Confidence Interval (CI) methods to elicit the values of the hyperparameters. Aslam (1995) discusses in detail the CI method of elicitation. The underlying logic of all the elicitations methods is to minimize the difference between the elicited and the fitted probabilities. In MCS method, we try to search for those values of the hyperparameters which minimize the associated chi-square values found by using the posterior estimates found on behalf of the all possible values of the hyperparameters. The procedure is bit lengthy and is based on the data but is free from the objection of subjectivity.

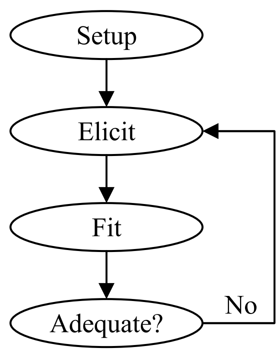

Figure 1: Algorithms for Elicitation Process

However, in the CI method, we find those values of the hyperparameters which minimize the difference between the elicited and the fitted predictive probabilities at some specific value (for discrete random variable case) and in the range (for discrete and continuous random variables) of the random variable. The prior information in the form of wining probabilities p_{ij}, for all $i(< j) = 1, 2, ..., 5,$ of all the teams under consideration in their order Australia, India, New Zealand, Pakistan and South Africa against their competitors are gathered from the cricket experts for $C_2^5 = 10$ pairs of teams as 0.7500, 0.6550, 0.7000, 0.6000, 0.3000, 0.5000, 0.4500, 0.3500, 0.3600 and 0.3000.

Table 2:
Estimates of Hyperparameters

Informative Multivariate Prior				Conjugate Prior			
Hyper-parameters	Estimates via MCS Method	Hyper-parameters	Estimates via MCS Method	Hyper-parameters	Estimates via PPD Approch	Hyper-parameters	Estimates via PPD Approch
μ_1	-2.5	σ_1	1.5	a_{12}^0	16.6945	a_{24}^0	7.1084
μ_2	2.0	σ_2	1.5	a_{13}^0	12.9046	a_{25}^0	3.4553
μ_3	-1.0	σ_3	1.5	a_{14}^0	10.3417	a_{34}^0	3.6656
μ_4	-2.5	σ_4	1.5	a_{15}^0	16.2723	a_{35}^0	3.4496
μ_5	0.5	σ_5	1.5	a_{23}^0	1.5098	a_{45}^0	1.4766

We use these probabilities as the probabilities parameter of the binomial distribution and find the (elicited) probabilities of at most four successes (wins) for the teams against their competitors which are found respectively to be 0.0000, 0.0002, 0.0017, 0.0000, 0.8497, 0.0245, 0.2279, 0.2348, 0.2613 and 0.9012 and estimate those hyperparameters which make the difference between the fitted and the elicited probabilities of the specified range minimum. The estimates of the hyperparameters of the informative normal are found by MCS method by using a computer program in 'C' language and those of the conjugate priors are found by developing a computer program in *SAS* language using its *PROC SYSNLIN* command and the results are displayed in Table 2.

5.2 Posterior Means

Using the data given in Table (1), and assuming the squared error loss function and we find the posterior mean estimate for the worth parameter of the *i*th team θ_i as:

$$E\left(\theta_i \mid \mathrm{a}\right) = \int_{-\infty}^{\infty} p\left(\theta_i \mid \mathrm{a}\right) d\theta_i \,, \tag{5.2}$$

where $p_U(\theta_i \mid \mathrm{a})$ is the posterior marginal distribution of the model parameter θ_i under uniform prior and may be given as:

$$p_U\left(\theta_i \mid \mathrm{a}\right) = \int_{-\infty}^{\infty} p\left(\theta \mid \mathrm{a}\right) d\theta' \,,\ -\infty \le \theta_i \le \infty \,. \tag{5.3}$$

Here θ' is such that $\theta' \cap \theta_i = \varnothing$ and $\theta' \cup \theta_i = \Omega$.

Due to the complicated nature of the expressions (5.2) and (5.3), we may use different methods, like Markov Chain Monte Carlo (MCMC), Gibbs sampling, Quadrature method etc. But, we use the Quadrature method of numerical integration, which refers to any method for numerically approximating the value of a definite integral $\int_a^b p(\theta) d\theta$. The procedure is to calculate it at a number of points in the range *a* to *b* and find the result as a weighted average as $\int_a^b p(\theta)d\theta = \sum_{i=1}^{n} \varepsilon_i p(\theta)$, where ε_i denotes the increment used to b through a. Here it is important the note that the accuracy and the size of increment are inversely proportional to each other. The two dimensional case integration may be found by the relation $\int_{\theta_i}\int_{\theta_j} p\left(\theta_i,\theta_j\right)d\theta_i d\theta_j \cong \sum_{i=o}^{n_i}\sum_{j=0}^{n_j} \varepsilon_i\varepsilon_j p\left(\theta_i,\theta_j\right)$, where the notations are pre-defined. The higher dimensions may similarly be accounted for.

Table 3:
Posterior Means Estimates of the Worth Parameters

Teams	**Parameters**	**UP**	**JP**	**IP**	**CP**
Australia	θ_1	0.66710	0.57748	0.60533	0.80272
India	θ_2	-0.37157	-0.31907	-0.29772	-0.61489
New Zealand	θ_3	-0.26416	-0.22768	-0.26313	-0.36955
Pakistan	θ_4	-0.18406	-0.15955	-0.20979	-0.14633
South Africa	θ_5	0.15269	0.12882	0.16530	0.32804

To solve the expressions (5.2) and (5.3), we deploy the Quadrature method of numerical integration and a computer program is developed in '*C*' language along with the condition $\sum_{i=1}^{t}\theta_i = 0$, and the results thus obtained are displayed in Table 3.

From the results, we observe that the Kangaroos stand first, South Africans the second, Pakistanis being third, Kiwis being the fourth and finally come Indians with the lowest rank. It is important to note that same ranking order prevails for all the priors used.

5.3 The Posterior Marginal Distributions of Parameters

The joint posterior distribution (5.1) has complicated expression and the posterior marginal distributions can not be obtained in the closed form. However, we may have an idea about the nature of variation of the worth parameters by plotting their ordinates against their values. The resulting information are displayed in the Figure 2.

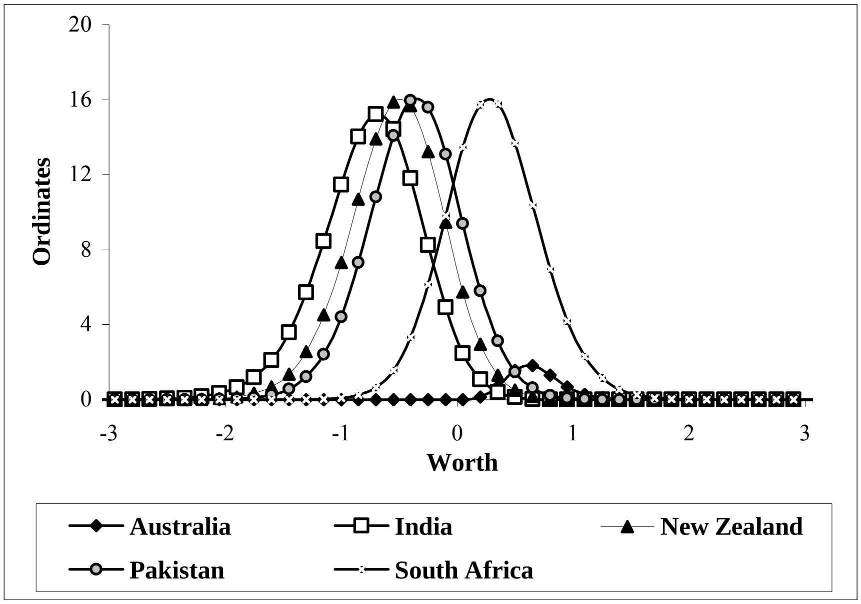

Figure 2: The Posterior Marginal Distributions of the Worth Parameters

From the figure we see that the posterior distributions of the worth parameters are almost symmetric around the posterior modes with varying dispersions.

5.4 The Preference Probabilities

The probabilities indicating the expected chances of preferences of any one treatment over the other in any one comparison are termed as the preference probabilities and are calculated using the posterior estimates and the model **Error! Reference source not found.**. Such probabilities are calculated and are displayed in Table 4.

Table 4:
The Pair-wise Preference Probabilities ϕ_{ij}

Team-pairs	UP	JP	IP	CP
(AU, IN)	0.75604	0.73265	0.73380	0.80445
(AU, NZ)	0.73867	0.71578	0.72763	0.77519
(AU, PA)	0.72446	0.70217	0.71769	0.74168
(AU, SA)	0.65123	0.63424	0.63195	0.64107
(IN, NZ)	0.46594	0.47099	0.48899	0.42342
(IN, PA)	0.44100	0.44965	0.47208	0.36052
(IN, SA)	0.34630	0.36596	0.36197	0.25935
(NZ, PA)	0.47456	0.47835	0.48304	0.43009
(NZ, SA)	0.37429	0.39099	0.37116	0.30612
(PA, SA)	0.39661	0.41063	0.38577	0.35901

Obviously these probabilities are compatible with the posterior estimates showing worth of the competing teams.

5.5 Bayesian Testing of Hypotheses

We consider the following hypotheses for comparing two parameters:

$H H_{ij}: \theta_i > \theta_j$ and $H_{ji}: \theta_i \le \theta_j$, $\forall i (< j) = 1, 2, 3, ..., t$.

We derive the density $p\left\{(\varphi_{ij}, \xi_i, \theta'') \mid \mathrm{a}\right\}$ by re-parameterization in the joint posterior probability distribution given in expression (5.1) as $\varphi_{ij} = (\theta_i - \theta_j)$ and $\xi_i = \theta_i, \forall i (< j) = 1, 2, ..., t$. For the specific team pair of Australia and India, the posterior probability p_{12} of the hypothesis H_{12} is found to be:

$$p_{12} = P\left(H_{12} \mid \mathrm{a}\right) = P\left(\theta_1 > \theta_2 \mid \mathrm{a}\right) = P\left(\varphi_{12} > 0 \mid \mathrm{a}\right) = \int_{\varphi_{12}=0}^{\infty} p\left(\varphi_{12} \mid \mathrm{a}\right) d\varphi_{12}, \tag{5.4}$$

where $p\left(\varphi_{12} \mid \mathrm{a}\right)$ is the posterior marginal distribution of φ_{12} and is given by:

$$p\left(\varphi_{12} \mid \mathrm{a}\right) = \int_{\theta''=-\infty}^{\infty} \int_{\xi_1=-\infty}^{\infty} p\left\{\left(\varphi_{12}, \xi_1, \theta''\right) \mid \mathrm{a}\right\} d\xi_1 d\theta'', -\infty \le \varphi_{ij} \le \infty, \tag{5.5}$$

where θ'' is a vector of parameters such that $\theta'' \cap \theta_1 \cap \theta_2 = \varnothing \wedge \theta'' \cup \theta_1 \cup \theta_2 = \Omega$ and $\sum_{i=1}^{5} \theta_i = 0$. The expressions (5.4) and (5.5) have complicated forms and are algebraically intractable. So we use the Quadrature method of numerical integration (discussed in Section 5.2) and develop a computer program in *SAS* language and the obtained results are displayed in Table 5.

Table 5:
The Pair-wise Probabilities of Hypotheses p_{ij}

Team-pairs	UP	JP	IP	CP
(AU , IN)	0.99982	0.99873	0.99973	1.00000
(AU, NZ)	0.99927	0.99466	0.99606	1.00000
(AU, PA)	0.99748	0.98564	0.98645	0.99991
(AU, SA)	0.96855	0.89224	0.91641	0.96636
(IN, NZ)	0.39997	0.21150	0.2155	0.09387
(IN, PA)	0.30482	0.16927	0.19094	0.00514
(IN, SA)	0.06159	0.02620	0.04004	0.00001
(NZ, PA)	0.42478	0.25927	0.23304	0.05539
(NZ, SA)	0.10029	0.04419	0.04782	0.00008
(PA, SA)	0.16779	0.08561	0.55038	0.05926

The following rule is applied to draw conclusion about the hypotheses regarding the teams being compared. Let

$$s = \min\left(p_{ij}, q_{ij}\right). \tag{5.6}$$

If p_{ij} is small, then H_{ji} is accepted with high probability. If q_{ij} is small then H_{ji} is accepted with high probability. This implies that if $'s'$ is small, we can reject one hypothesis otherwise if s > 0.1 (say) then the evidence is inconclusive.

From these results we can obtain the probabilities of the hypotheses in accordance with the posterior means given in Tables 2 and 3. For instance, while testing the hypotheses H_{12}, H_{13}, H_{14} and H_{15}, there exist high probabilities in favor of the Kangaroos against all the competing teams for all types of the priors used, which lead us to the acceptance of all the hypotheses and declare the Kangaroos to be the top-ranked team. The probabilities of the hypotheses for rest of the pairs may similarly be interpreted.

5.6 The Predictive Distribution

The predictive distribution represents our current predictions of the variable a_{ij} taking into account both the uncertainty about the parameters θ and the residual uncertainty about the variable a_{ij} when the parameters θ are unknown, (Lee, 1989). The predictive probability $p_{(ij)}$ for the future random variable a'_{ij} is given by:

$$p_{(ij)} = p\left(a'_{ij} \mid a_{ij}\right) = \int_{\theta} p\left(\theta \mid a_{ij}\right) p\left(a'_{ij} \mid \theta\right) d\theta. \tag{7.1}$$

These indicate the future probabilities of preferring the team T_i over T_j and are given in Table 6.

Table 6:
The Pair-wise Predictive Probabilities $p_{(ij)}$

Team-Pairs	UP	JP	IP	CP
(AU , IN)	0.74651	0.72444	0.72470	0.80012
(AU, NZ)	0.72853	0.70740	0.71805	0.77026
(AU, PA)	0.71374	0.69346	0.70779	0.73578
(AU, SA)	0.64204	0.62717	0.62419	0.63623
(AU, NZ)	0.46889	0.47333	0.48980	0.42728
(AU, PA)	0.44585	0.45334	0.47428	0.36630
(AU, SA)	0.35699	0.37412	0.37128	0.26576
(NZ, PA)	0.47642	0.47966	0.48421	0.43279
(NZ, SA)	0.38292	0.39719	0.37958	0.31191
(PA, SA)	0.40465	0.41632	0.39403	0.36467

Obviously these probabilities for all the priors are compatible with the posterior preference probabilities.

7. APPROPRIATENESS OF THE MODEL

A model is said to give appropriate or good fit to the observed data if the expected frequencies obtained by using the model are considerably close to the observed frequencies of the data. To test the hypothesis of goodness of fit of a model, we use χ^2. The smaller the value of χ^2, the better the fit. Let the ordered pairs $\left(a_{ij}, \hat{a}_{ij}\right)$ and $\left(a_{ji}, \hat{a}_{ji}\right)$, where $\hat{a}_{ij} = n_{ij}.\varphi_{ij}$, respectively denote the observed and the corresponding expected frequencies for the preferences of the treatments T_i and T_j and vice versa. Then the χ^2-statistic attains the form:

$$\chi^2 = \sum_{i<j}^{t}\left\{\frac{\left(a_{ij} - \hat{a}_{ij}\right)^2}{\hat{a}_{ij}} + \frac{\left(a_{ji} - \hat{a}_{ji}\right)^2}{\hat{a}_{ji}}\right\} \tag{7.1}$$

with $(t-1)(t-2)/2$ degrees of freedom [Aslam, (1995) and Stern (1990)]. We consider the model equation (5.1) for the study and the null hypothesis H_0 and the alternative hypothesis H_1 are formed as:

H_0 : The model is true for some values of $\theta = \theta_0$.

H_1 : The model is not true for any values of the parameters.

The expected frequencies are calculated by using the relation $\hat{a}_{ij} = n_{ij}\dfrac{\pi + 2\tan^{-1}\left(\theta_i - \theta_j\right)}{2\pi}$. The p-values associated with the values of the χ^2 statistic for all types of the priors are found and are displayed in Table 7.

Table 7:
***p*-values at 6 Degrees of Freedom**

UP	JP	IP	CP
0.64344	0.67960	0.66123	0.26766

From the p-values, it is quite evident that under no circumstances the Cauchy model may be regarded as providing a poor fit to the data.

8. CONCLUDING REMARKS WITH DISCUSSION

An alternative model given in equation (2.1) is proposed for the paired comparisons experimentation based on the Cauchy distribution. Having a view of the facts and figures of the analysis given in the form of the posterior estimates, we see that the five ODI cricket teams under study may be ranked as Australia being the top most, South Africa the second one, Pakistan being the third one, New Zealand with the fourth position and finally India being the fifth and the last one. It is also worth mentioning that the estimates obtained in the forms of the posterior means under all the priors considerably agree. The preference and the predictive probabilities also favor the same ranking. The test for goodness of fit of the model conducted through the χ^2-statistic also indicates the appropriateness of model with insignificant p-values. The results of hypothesis-testing also justify the same ranking order.

To see how different forms of prior differ in the inferences to which they lead, it is worth mentioning that all the priors lead us to same sort of results regarding the ranking order, which necessarily means that the prior information obtained in the forms of the experts opinion are also harmonious with data information. Since ties are not studied here, they may be considered as future study of the model.

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