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ESTIMATION OF PARAMETERS OF THE GAMMA DISTRIBUTION IN THE PRESENCE OF OUTLIERS GENERATED FROM UNIFORM DISTRIBUTION

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ABSTRACT

The maximum likelihood, moment and mixture of the estimators are derived for samples from the gamma distribution in the presence of outliers generated from uniform distribution. These estimators are compared empirically when all the parameters are unknown; their bias and determinants are investigated with the help of numerical technique. We have shown that these estimators are asymptotically unbiased. At the end, we conclude that mixture estimators are better than the maximum likelihood and moment estimators.

KEYWORDS

Gamma distribution, Outlier, Uniform distribution, Moment Estimator, Maximum likelihood estimator, Mixture estimator, Newton-Raphson.

1. INTRODUCTION

Consider spread from a point source for example, which might a small plot of plants. During favorable weather conditions, the plants release their pollen and it disperses according to a gamma distribution with distance from the source. However, in less favorable conditions, light, rain or mist, not only are the plants less likely to release pollen, but that which is released still falls with a gamma distribution. Dixit, Moore and Barnett (1996) consider the above example in the context of spread disease amongst plants of viral spores such as barley yellow mosaic dwarf virus (BYMDV). By using the methodology as stated in Dixit, Moore and Barnett (1996), it is possible to estimate the average distance (and hence area) of disease spread in a field from a small patch of infested plants in the presence of some spread caused by insects. Also Dixit and Nasiri (2001) estimate parameters of the exponential distribution in the presence of outliers generated from uniform distribution.

According to Dixit, Moore and Barnett (1996), we assume that a set of random variables $\left(X_1, X_2, \ldots, X_n\right)$ represent the distance of an infected sampled plant from a plot of plants inoculated with a virus.

Some of the observations are derived from the airborne dispersal of the spores and are distributed according to the gamma distribution. The other observations out of n random variables (say k) are present. Because, aphids which are known to be carriers of BYMDV have passed the virus into the plants when the aphids feed on the sap. These k (known) aphids are considered to be uniformly distributed. Thus, we assume that the random variables $\left(X_1, X_2, ..., X_n\right)$ are such that k of them are distributed with pdf

$$g(x,\theta) = \frac{1}{\theta}; \quad 0 < x < \theta,\ \theta > 0 \tag{1.1}$$

and remaining $(n-k)$ random variables are distributed with pdf $f(x,\alpha,\theta)$

$$f(x,\alpha,\theta) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right) \tag{1.2}$$

The present paper considers the estimation of θ and α in the model described above. In Section 2, we have obtained the joint distribution of $\left(X_1, X_2, ..., X_n\right)$ in the presence of k outliers. In section 3, 4 and 5, we deal with the method of moment, maximum likelihood and mixture of method of moment and maximum likelihood in estimating θ and α. In Section 6, we compare the Bias and Determinant of both the estimates empirically.

2. JOINT DISTRIBUTION OF $\left(X_1, X_2, ..., X_n\right)$ WITH k OUTLIERS

The joint distribution of $\left(X_1, X_2, ..., X_n\right)$ in the presence of k outliers is given by

$$f(x_1, x_2, ..., x_n, \alpha, \theta) = h_k . \prod_{i=1}^{n-k} x_i^{(n-k)(\alpha-1)} . e^{\frac{-\sum_{i=1}^{n} x_i}{\theta}} . G(x,\theta) \tag{2.1}$$

where

$$h_k = \left[C(n,k).\theta^{k+\alpha(n-k)}.\Gamma(\alpha)\right]^{-1} \tag{2.2}$$

$$C(n,k) = \frac{n!}{(n-k)!k!}, \quad \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt \tag{2.3}$$

$$G(x,\theta) = \sum_{A_1=1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2} ... \sum_{A_k=A_{k-1}+1}^{n} e^{\sum_{i=1}^{k} x_{A_i}/\theta} . \prod_{i=1}^{k} \mathrm{I}\left(\theta - x_{A_i}\right) \tag{2.4}$$

$$\mathrm{I}(u) = \begin{cases} 0 & if \quad u < 0 \\ 1 & if \quad u \geq 0 \end{cases}$$

3. METHOD OF MOMENTS

Let $D = \frac{m_2'}{m_1'^2}$, $D_1 = \frac{m_2'}{m_1'}$, $b = \frac{k}{n}$ and $\bar{b} = \frac{n-k}{n}$, where

$$m_i' = \sum_{j=1}^{n} \frac{x_j^i}{n} \qquad i = 1, 2 \tag{3.1}$$

$$D = \frac{b/3 + \bar{b}\alpha(\alpha+1)}{(b/2 + \bar{b}\alpha)^2} \tag{3.2}$$

$$D.\left[\frac{b}{2} + \bar{b}\alpha\right]^2 = \frac{b}{3} + \bar{b}\alpha(\alpha+1) \tag{3.3}$$

Solving (3.3)

$$A_1\alpha^2 + A_2\alpha + A_3 = 0 \tag{3.4}$$

where,

$$A_1 = D.\bar{b}^2 - \bar{b} \, , \; A_2 = D.b.\bar{b} - \bar{b} \, , \; A_3 = \frac{D.b^2}{4} - \frac{b}{3}$$

If $\Delta = (A_2^2 - 4A_1A_3)$ is non-negative then the roots are real.

Therefore,

$$\hat{\alpha} = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1} \tag{3.5}$$

Next

$$\frac{m_2'}{m_1'} = \frac{\left[\frac{b}{3} + \bar{b}\alpha\left(\alpha+1\right)\right]}{\left[\frac{b}{2} + \bar{b}\alpha\right]}\theta$$

$$D_1.\left[\frac{b}{2} + \bar{b}\alpha\right] = \left[\frac{b}{3} + \bar{b}\alpha\left(\alpha+1\right)\right]\theta \tag{3.6}$$

Hence

$$\hat{\theta} = D_1.\frac{\frac{b}{2} + \bar{b}\hat{\alpha}}{\left[\frac{b}{3} + \bar{b}\hat{\alpha}\left(\hat{\alpha}+1\right)\right]} \tag{3.7}$$

where, $\hat{\alpha}$ is given by (3.5). Now, we shall show that $\hat{\alpha}$ and $\hat{\theta}$ are asymptotically unbiased estimators.

Let $W_1 = \sum_{i=1}^{n} X_i$ and $W_2 = \sum_{i=1}^{n} X_i^2$, then

$$D = \frac{nW_2}{W_1^2}`,\ A_1 = \frac{nW_2}{W_1^2}\bar{b}^2 - \bar{b},\ A_2 = \frac{nW_2}{W_1^2}b\bar{b} - \bar{b} \text{ and } A_3 = \frac{nW_2}{W_1^2}.\frac{b^2}{4} - \frac{b}{3}$$

Now, we can write $\hat{\alpha}$ as a function of W_1 and W_2. Hence

$$\hat{\alpha} = f\left(W_1, W_2\right). \tag{3.8}$$

Let $\mathrm{E}\left(W_1\right) = \mu$ and $\mathrm{E}\left(W_2\right) = \nu$. Expand the function $f\left(W_1, W_2\right)$ around (μ, ν) by Taylor series

$$f\left(W_1, W_2\right) = f(\mu, \nu) + \left(W_1 - \mu\right)\frac{\partial f}{\partial W_1}\Big|_{W_1=\mu, W_2=\nu} + \left(W_2 - \nu\right)\frac{\partial f}{\partial W_2}\Big|_{W_1=\mu, W_2=\nu} + \ldots \tag{3.9}$$

Hence, from (3.5), (3.8), and (3.9),

$$\mathrm{E}\left(\hat{\alpha}\right) = f(\mu, \nu) = \left[-\frac{n\nu}{\mu^2}b\bar{b} + \bar{b} + \sqrt{\Delta}\right] \times \left[\frac{2n\nu}{\mu^2}\bar{b}^2 - 2\bar{b}\right]^{-1} \tag{3.10}$$

where,

$$\mu = n.\left(\frac{b}{2} + \bar{b}\alpha\right)\theta$$

$$\nu = n\left(\frac{b}{3} + \bar{b}\alpha(\alpha + 1)\right)\theta^2$$

$$\Delta = \left(\frac{n\nu}{\mu^2}b\bar{b} - \bar{b}\right)^2 - 4.\left(\frac{n\nu}{\mu^2}.\bar{b}^2 - \bar{b}\right).\left(\frac{n\nu}{\mu^2}.\frac{b^2}{4} - \frac{b}{3}\right)$$

$$= \bar{b}^2 - \frac{4}{3}b\bar{b} + \frac{n\nu}{\mu^2}\left(b^2\bar{b} - \frac{2}{3}b\bar{b}^2\right)$$

$$= \frac{\left(36\bar{b}^4 + 36b^2\bar{b}^2 - 72b\bar{b}^3\right)\alpha^2 + \left(12b\bar{b}^3 - 12b^2\bar{b}^2\right)\alpha + \bar{b}^2 b^2}{9\left(b + 2\bar{b}\alpha\right)^2} \tag{3.11}$$

From (3.12),

$$\mathrm{E}(\hat{\alpha})=\frac{\left[-\left(\dfrac{b/3+\bar{b}\alpha(\alpha+1)}{(b/2+\bar{b}\alpha)^2}\right).b\bar{b}+\bar{b}+\sqrt{\Delta}\right]}{2\left[\dfrac{b/3+\bar{b}\alpha(\alpha+1)}{(b/2+\bar{b}\alpha)^2}\right]\bar{b}^2-2\bar{b}} \tag{3.12}$$

Hence, from (3.10), (3.11) and (3.12),

$$\begin{aligned}\mathrm{E}(\hat{\alpha})=&\left[\left(12\bar{b}^3-12b\bar{b}^2+4\frac{(b+6\bar{b}\alpha-6b\alpha)\bar{b}}{(b+2\bar{b}\alpha)}\bar{b}^2\right)\alpha^2\right.\\&\left.+4\frac{(b+6\bar{b}\alpha-6b\alpha)\bar{b}}{(b+2\bar{b}\alpha)}.b\bar{b}\alpha+\frac{(b+6\bar{b}\alpha-6b\alpha)\bar{b}}{(b+2\bar{b}\alpha)}.b^2-\bar{b}b^2\right]\\&\times\left[2\bar{b}\left(4b\bar{b}+12\bar{b}^2\alpha-3b^2-12b\bar{b}\alpha\right)\right]^{-1}\end{aligned}$$

Then,

$$\mathrm{E}(\hat{\alpha})=\alpha \tag{3.13}$$

By using (3.9), let $W_1=\sum_{i=1}^{n}X_i$, $W_2=\sum_{i=1}^{n}X_i^2$, $W_3=\hat{\alpha}$, $\mathrm{E}(W_1)=\mu$, $\mathrm{E}(W_2)=\nu$, and $\mathrm{E}(\hat{\alpha})=\alpha$.

$$\begin{aligned}\hat{\theta}=f(W_1,W_2,W_3)=&\,f(\mu,\nu,\alpha)+(W_1-\mu)\frac{\partial f}{\partial W_1}\Big|_{W_1=\mu,W_2=\nu,W_3=\alpha}\\&+(W_2-\nu)\frac{\partial f}{\partial W_2}\Big|_{W_1=\mu,W_2=\nu,W_3=\alpha}+(W_3-\alpha)\frac{\partial f}{\partial W_3}\Big|_{W_1=\mu,W_2=\nu,W_3=\alpha}+\ldots\end{aligned}$$

According to the previous procedure

$$\begin{aligned}\mathrm{E}(\hat{\theta})&=\lim_{n\to\infty}\frac{\nu}{\mu}.\frac{b/2+\bar{b}\alpha}{b/3+\bar{b}\alpha(\alpha+1)}\\&=\lim_{n\to\infty}\frac{\left(b/3+\bar{b}\alpha(\alpha+1)\right)\theta^2}{(b/2+\bar{b}\alpha)\theta}.\frac{b/2+\bar{b}\alpha}{b/3+\bar{b}\alpha(\alpha+1)}\end{aligned}$$

$$\mathrm{E}(\hat{\theta})=\theta \tag{3.14}$$

Hence, $\hat{\alpha}$ and $\hat{\theta}$ are asymptotically unbiased.

4. MAXIMUM LIKELIHOOD ESTIMATE

From (2.1), the likelihood of $(x_1,x_2,\ldots,x_n)$ is

$$L(x_1, x_2, \ldots, x_n; \alpha, \theta) = \frac{\prod_{i=1}^{n-k} x_i^{(n-k)(\alpha-1)} e^{-\sum_{i=1}^{n} x_i/\theta}}{C(n,k).\theta^{k+\alpha(n-k)} (\Gamma(\alpha))^{n-k}} .G(x,\theta) \tag{4.1}$$

where,

$$G(x,\theta) = \sum_{A_1=1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2} \cdots \sum_{A_k=A_{k-1}+1}^{n} e^{\sum_{i=1}^{k} x_{A_i}/\theta} . \prod_{i=1}^{k} \mathrm{I}\left(\theta - x_{A_i}\right)$$

If, $l(x;\alpha,\theta) = \ln(L(x;\alpha,\theta))$, let θ is fixed, then,

$$\frac{\partial l}{\partial \alpha} = -(n-k).\left(-\sum_{i=1}^{n-k} \ln(x_i) + \Psi(\alpha) + \ln(\theta) \right) = 0$$

$$\sum_{i=1}^{n-k} \ln(x_i) - \Psi(\alpha) - \ln(\theta) = 0 \tag{4.2}$$

where,

$$\Psi(\alpha) = \frac{\frac{\partial}{\partial \alpha} \Gamma(\alpha)}{\Gamma(\alpha)}$$

So, we must find α from (4.2) for MLE of α.

For MLE of θ, the likelihood equation is

$$l(x,\theta,\hat{\alpha}) = \ln\left(\frac{\prod_{i=1}^{n-k} x_i^{(n-k)(\hat{\alpha}-1)} e^{-\sum_{i=1}^{n} x_i/\theta}}{C(n,k).\theta^{k+\hat{\alpha}(n-k)} \left(\Gamma(\hat{\alpha})\right)^{n-k}} .G(x,\theta) \right)$$

$$= (n-k).(\hat{\alpha}-1).\sum_{i=1}^{n-k} \ln(x_i) - \frac{\sum_{i=1}^{n} x_i}{\theta} + \ln(G(x,\theta)) - \ln(C(n,k))$$

$$-(k+\hat{\alpha}(n-k)).\ln(\theta) - (n-k).\ln(\Gamma(\hat{\alpha})) \tag{4.3}$$

Hence,

$$\frac{\partial}{\partial \theta} l(x,\theta,\hat{\alpha}) = \frac{\sum_{i=1}^{n} x_i}{\theta^2} + \frac{G'(x,\theta)}{G(x,\theta)} - \frac{k+\hat{\alpha}(n-k)}{\theta} = 0 \tag{4.4}$$

where,

$$G'(x,\theta) = \frac{\partial G}{\partial \theta} = -\frac{1}{\theta^2} \sum_{A_1} \sum_{A_2} \cdots \sum_{A_k} \left(\sum_{i=1}^{k} x_{A_i} \right) . e^{\sum_{i=1}^{k} x_{A_i}/\theta}$$

We can solve (4.4) by Newton-Raphson method. Hence, solution of the equation is

$$\theta_{i+1} = \theta_i - \frac{g(\theta_i)}{g'(\theta_i)} \quad ; \quad i = 1, 2, 3, \ldots \tag{4.5}$$

where,

$$g(\theta_i) = (k + \hat{\alpha}(n-k))\theta_i + \frac{\sum\limits_{A_1}\sum\limits_{A_2}\cdots\sum\limits_{A_k}\left(\sum\limits_{i=1}^{k} x_{A_i}\right).e^{\sum_{i=1}^{k} x_{A_i}/\theta}}{\sum\limits_{A_1}\sum\limits_{A_2}\cdots\sum\limits_{A_k} e^{\sum_{i=1}^{k} x_{A_i}/\theta}} - \sum_{i=1}^{n} x_i \tag{4.6}$$

$$g'(\theta_i) = (k + \hat{\alpha}(n-k)) + \frac{1}{\theta_i^2}.\frac{\left\{\sum\limits_{A_1}\sum\limits_{A_2}\cdots\sum\limits_{A_k}\left(\sum\limits_{i=1}^{k} x_{A_i}\right).e^{\sum_{i=1}^{k} x_{A_i}/\theta}\right\}^2}{\left\{\sum\limits_{A_1}\sum\limits_{A_2}\cdots\sum\limits_{A_k} e^{\sum_{i=1}^{k} x_{A_i}/\theta}\right\}^2}$$

$$-\frac{1}{\theta_i^2}.\frac{\left\{\sum\limits_{A_1}\sum\limits_{A_2}\cdots\sum\limits_{A_k}\left(\sum\limits_{i=1}^{k} x_{A_i}\right)^2.e^{\sum_{i=1}^{k} x_{A_i}/\theta}\right\}}{\left\{\sum\limits_{A_1}\sum\limits_{A_2}\cdots\sum\limits_{A_k} e^{\sum_{i=1}^{k} x_{A_i}/\theta}\right\}} \tag{4.7}$$

Here, the initial solution θ_0 should be selected from (3.9).

Note: Estimation of k: If k is unknown, then k can be selected by evaluating the likelihood for different values of k choosing the one that maximizes the likelihood.

5. MIXTURE OF METHOD OF MOMENT AND MAXIMUM LIKELIHOOD

Read (1981) proposed the methods which avoid the difficulty of complicated equations. According to Read (1981), replacement of some, but, not all, of the equations in the system of likelihood may make it more manageable. From (3.9),

$$\hat{\theta} = D_1.\frac{\frac{b}{2} + \bar{b}\hat{\alpha}}{\left[\frac{b}{3} + \bar{b}\hat{\alpha}(\hat{\alpha}+1)\right]} \tag{5.1}$$

and

$$-\sum_{i=1}^{n-k}\ln(x_i)+\Psi(\hat{\alpha})+\ln(\hat{\theta})=0 \qquad (5.2)$$

From (5.1) and (5.2)

$$\Psi(\alpha)-\sum_{i=1}^{n-k}\ln(x_i)+\ln\left(\frac{b}{2}+\bar{b}.\alpha\right)-\ln\left(\frac{b}{3}+\bar{b}\alpha(\alpha+1)\right)+\ln(D_1)=0 \qquad (5.3)$$

With solving (5.3) $\hat{\alpha}$ is obtained. Then we replace $\hat{\alpha}$ in (5.1) to find $\hat{\theta}$.

If $\alpha = 1$ then we imply three estimators are presented in Dixit, U.J. and Nasiri, F.P. (2001).

6. NUMERICAL STUDY

In order to have some idea about Bias and Determinant we perform sampling experiments using a Pentium IV. We have a written program in Maple software to do simulation study.

Since in our model we have two parameters, we have calculated variance-covariance matrix of the estimate θ and α. Determinant of the covariance matrix is $V(\hat{\theta}).V(\hat{\alpha})-(Cov(\hat{\alpha},\hat{\theta}))^2$. (See Jayade and Prasad (1990)). The simulation study was carried out for $\theta = 0.2$ and $\alpha = 5$ for $k = 1, 2$ with sample sizes $n = 10(10)80$, and for $k = 3$ with sample size $n = 10, 12, 14, 15, 16, 18, 20, 22, 24, 25, 26, 28, 30$.

Here, we have presented Bias and Determinant in table 1, 2 and 3 for k = 1, 2 and 3. Table 1, 2 and 3 summarizes the results based on one thousand independent replications of each experiment. For all values of k, we can conjecture that in most cases $\mathrm{E}(\hat{\alpha}) < \alpha$ and in some cases $\mathrm{E}(\hat{\alpha}) < \alpha$.

For k = 1, 2 bias for $\hat{\theta}$ is negative in all cases, but for k =3 bias for $\hat{\theta}$ is always positive. In all cases determinant for method of maximum likelihood is larger than the ME and Mixture estimate. But the determinant of mixture estimate is always smaller than the moment method. One more good point about the mixture estimate is that it is easy to calculate.

We also have shown that moment estimates are asymptotically unbiased. It is difficult to show analytically that mixture estimate of α is asymptotically unbiased. But from simulation of study, we conjecture that mixture estimate of α is asymptotically unbiased.

Therefore, we conclude that mixture estimate should be used always.

Table 1

		$k=1$	$\alpha=5$	$\theta=0.2$
n	Method	Bias of $\hat{\alpha}$	Bias of $\hat{\theta}$	Determinant
10	M.E	3.488741850	-0.0733029678	0.001787436627
	M.L.E	0.413643383	0.0195972388	0.1537762039
	Mixture	-2.749637659	0.2290763347	0.06060926403
20	M.E	-0.013840032	0.0003364296	0.000969761209
	M.L.E	-4.557977218	-0.1840235003	0.5570808970
	Mixture	-4.535623602	0.6192896661	0.0000986593490
30	M.E	1.149145500	-0.0298602108	0.001136554591
	M.L.E	-4.277706305	0.0498333508	2.523169886
	Mixture	-4.460928117	0.6051305307	0.000901926384
40	M.E	1.011516556	-0.0276266037	0.002338059393
	M.L.E	-5.416583394	-0.6908539816	13.32242101
	Mixture	-4.332870109	0.6632616043	0.03104640231
50	M.E	0.533064658	-0.0123868353	0.0002366568859
	M.L.E	-5.288902679	-0.4812652874	1.806400581
	Mixture	-4.571491063	0.7051317413	0.001333823940
60	M.E	0.875836996	-0.0233811822	0.000525687222
	M.L.E	-25.55099400	-0.3923208559	1499.989509
	Mixture	-4.646434590	0.7299853236	0.001260777983
70	M.E	0.270849429	-0.0040361071	0.0000738942082
	M.L.E	-12.59520200	-0.5251133859	782.5553946
	Mixture	-4.803326118	0.828141010	0.000009419982002
80	M.E	0.596393417	-0.0147520205	0.000399325930
	M.L.E	-39.80710047	-0.4151478534	13209.29714
	Mixture	-4.834301805	0.836899870	0.00001630611399

Table 2

		$k=2$	$\alpha=5$	$\theta=0.2$
n	Method	Bias of $\hat{\alpha}$	Bias of $\hat{\theta}$	Determinant
10	M.E	13.35182328	-0.0022714571	18.5026775
	M.L.E	-1.569805998	0.0629471500	0.1219826260
	Mixture	0.764417596	0.3224406422	4.925517589
20	M.E	-0.792425730	0.0386366882	0.001536310152
	M.L.E	-4.428627095	0.3308673388	0.8043224612
	Mixture	-4.760313186	2.657060492	4.516386401
30	M.E	0.205415602	-0.0080052998	0.001179166170
	M.L.E	-5.230706928	-0.4685688926	1.08011157
	Mixture	-4.665644017	0.7158060076	0.0001507602704
40	M.E	-0.153004002	0.0038219908	0.0001720753876
	M.L.E	-5.470429578	-0.7081654235	0.4063478857
	Mixture	-4.783724053	0.815312178	0.000005113992568
50	M.E	-0.051575552	-0.0040325840	0.0005007159607
	M.L.E	-8.618237180	-0.5011490165	0.01897486659
	Mixture	-4.861137575	0.858773252	0.000002223722712
60	M.E	0.106216393	-0.0061264027	0.001101899594
	M.L.E	-9.878458290	-0.4932626364	0.4181462927
	Mixture	-4.869828786	0.869598233	0.000002198128274
70	M.E	-0.361788086	0.0120308856	0.0005922667790
	M.L.E	-11.97556324	-0.2819743724	-0.1285862768
	Mixture	-4.891478472	0.915779374	7.230026550 e-9
80	M.E	0.270943410	-0.0139914474	0.001696061493
	M.L.E	-11.25087107	2.622543573	1263.569207
	Mixture	-4.914838562	0.894371417	0.000001063092796

Table 3

$k=3$ $\alpha=5$ $\theta=0.2$

n	Method	Bias of $\hat{\alpha}$	Bias of $\hat{\theta}$	Determinant
10	M.E	3.379724556	-0.0646621755	0.03284198464
	M.L.E	-1.0011654951	-0.0496348076	0.06792547153
	Mixture	-3.483640265	0.3511699641	0.02069831114
12	M.E	6.66461434	-0.0994196150	0.01383959560
	M.L	2.597993427	-0.1186284889	0.001473688376
	Mixture	-2.568723545	0.1769715760	0.006213659944
14	M.E	1.786005170	0.0094176184	0.2398595925
	M.L	-1.254273870	0.0417130323	0.1606929935
	Mixture	-2.896381455	0.2861564657	0.07013970251
15	M.E	0.467018427	0.0134723135	0.03302228334
	M.L	-1.755881948	0.1646107280	0.5515751860
	Mixture	-1.224658770	0.2115245763	0.7058711514
16	M.E	0.772101043	0.0069027999	0.08080617253
	M.L	-4.056102109	-0.1326248532	0.4807609538
	Mixture	-4.048964771	0.4636453060	0.005201544608
18	M.E	-1.168964133	0.0713465936	0.00008251878050
	M.L	-3.365142214	-0.5523543643	2.846076379
	Mixture	-3.571826776	0.4654426427	0.08615811151
20	M.E	-0.642516687	0.0552915900	0.005028178381
	M.L.E	-1.748825713	0.2380938453	0.7623168867
	Mixture	-2.238349130	0.3058901730	0.3914909957
22	M.E	-0.806557347	0.0557440726	0.0008018714224
	M.L	-3.566792029	0.3364167110	0.0004566259226
	Mixture	-4.290696078	0.6005366733	0.00001459588212
24	M.E	-0.789716850	0.0505681921	0.0001538203975
	M.L	-3.707531554	0.4274502070	0.002576005775
	Mixture	-4.334212793	0.6210979970	0.0001613079987
25	M.E	-0.696201610	0.0357388848	0.0003767224630
	M.L	-4.256616061	0.820562165	0.002786054764
	Mixture	-4.592696564	0.7390169877	0.00002609126744
26	M.E	-0.782144553	0.0366590847	0.0002583174244
	M.L	-4.744040586	0.0177199732	1.233814643
	Mixture	-4.627389361	0.7590893643	0.00007886447984
28	M.E	-0.480934603	0.0199105801	0.001051959254
	M.L	-5.186997608	-0.8231092427	0.8787766085
	Mixture	-4.674536508	0.7685210127	0.00007155473861
30	M.E	-0.129194213	0.0054998097	0.0002843612458
	M.L.E	-5.133702382	-0.8910042847	0.9592839448
	Mixture	-4.650544824	0.7490507790	0.00007690142346

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