Pak. J. Statist.
2009 Vol. 25(1), 5-14

SOME CRISP CENTRAL MOMENTS BASED ON FUZZY RANDOM VARIABLES

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ABSTRACT

Fuzzy set theory deals with concepts of uncertainly and the theory of fuzzy sets is a well known tool for formulation and analysis of imprecise and subjective concepts. Central moments are useful for determination of variance, covariance and correlation coefficient. In this paper we apply crisp central moments based on fuzzy random variables or data based on L_2 metric and Yao-Wu singed distance.

KEYWORDS:

Fuzzy random variable; Fuzzy conditional expectation; L_2 metric; Yao-Wu singed distance.

1. INTRODUCTION

In this paper, we propose some crisp central moments based on fuzzy random variables or observations. In this approach we apply L_2 metric and Yao-Wu singed distance to goal this end.

We organize the matter in the following way: in section 2 we describe some basic concepts of canonical fuzzy numbers, fuzzy random variable and its expectation, and Yao-Wu singed distance. Section 3 is devoted to describe a crisp central moments using fuzzy random variables based on L_2 metric. In section 4 we summarize the research results report in the crisp central moments using fuzzy random variables based on Yao-Wu singed distance. Finally a conclusion is presented in section 5 in order to illustrate our proposed methods.

2. PRELIMINARIES

In this section we introduce fuzzy canonical number, fuzzy random variable, fuzzy expectation and Yao-Wu singed distance.

2.1 Canonical Fuzzy Numbers

Let R be the real line, then a fuzzy subset $\tilde{x}$ of R is defined by its membership function $\mu_{\tilde{x}} : R \to [0,1]$. We denote by $\tilde{x}_\alpha = \{x : \mu_{\tilde{x}}(x) \geq \alpha\}$ the $\alpha-$cut set of $\tilde{x}$ and $\tilde{x}_0$ is the closure of the set $\{x : \mu_{\tilde{x}}(x) > 0\}$, and

1) $\tilde{x}$ is called normal fuzzy set if there exist $x \in R$ such that $\mu_{\tilde{x}}(x) = 1$;
2) $\tilde{x}$ is called convex fuzzy set if $\mu_{\tilde{x}}\left(\lambda x + (1-\lambda)y\right) \geq \min\left\{\mu_{\tilde{x}}(x), \mu_{\tilde{x}}(y)\right\}$ for all $\lambda \in [0,1]$;
3) the fuzzy set $\tilde{x}$ is called a fuzzy number if $\tilde{x}$ is normal convex fuzzy set and its $\alpha-$ cut sets, is bounded $\forall \alpha \neq 0$;
4) $\tilde{x}$ is called a closed fuzzy number if $\tilde{x}$ is fuzzy number and its membership function $\mu_{\tilde{x}}$ is upper semi continues;
5) $\tilde{x}$ is called a bounded fuzzy number if $\tilde{x}$ is a fuzzy number and its membership function $\mu_{\tilde{x}}$ has compact support.

If $\tilde{x}$ is a closed and bounded fuzzy number with $x_{\alpha}^{L} = \inf\left\{x : x \in \tilde{x}_{\alpha}\right\}$ and $x_{\alpha}^{U} = \sup\left\{x : x \in \tilde{x}_{\alpha}\right\}$ and its membership function be strictly increasing on the interval $[x_0^L, x_1^L]$ and strictly decreasing on the interval $\left[x_1^U, x_0^U\right]$ then $\tilde{x}$ is called canonical fuzzy number.

2.2 Fuzzy random variable and its expectation

Let (Ω, F, P) be a measure space, $F(R)$ be a canonical fuzzy real number space and $F(R^n)$ be a $n-$dimensional canonical fuzzy real number space Therefore, a fuzzy random variable (FRV) $\tilde{X}$ can be defined as a Borel measurable function $\tilde{X} : \Omega \rightarrow F(R^n)$.

Generally, the $\alpha-$ cut set of expected value $\tilde{\mathrm{E}}(\tilde{X})$ of the FRV $\tilde{X}$ is defined by

$$\tilde{\mathrm{E}}_{\alpha}(\tilde{X}) = \left\{\mathrm{E}(X) : X \in \tilde{X}_{\alpha}\right\}.$$

2.3 Yao-Wu singed distance

We define a signed distance between fuzzy numbers which uses later. Several ranking methods have been proposed so far, by Cheng (1998), Modarres and Sadi-Nezhad (2001) and Nojavan and Ghazanfari (2006). In this paper we use another ranking system for canonical fuzzy numbers which is very realistic and is defined by Yao and Wu as the following:

Definition 2.1: For each $a, b \in R$, define the signed distance d^* of a and b by $d^*(a,b) = a - b$. Thus, we have the following way to define the rank of any two numbers on R. For each $a, b \in R$

$$d^*(a,b) > 0 \quad \Leftrightarrow \quad d^*(a,0) > d^*(b,0) \quad \Leftrightarrow \quad a > b$$
$$d^*(a,b) < 0 \quad \Leftrightarrow \quad d^*(a,0) < d^*(b,0) \quad \Leftrightarrow \quad a < b$$
$$d^*(a,b) = 0 \quad \Leftrightarrow \quad d^*(a,0) = d^*(b,0) \quad \Leftrightarrow \quad a = b.$$

Definition 2.2: Let $F(R)$ be all of canonical fuzzy numbers on the real line R. For each $\tilde{a},\tilde{b}\in F(R)$, define the signed distance of $\tilde{a}$ and $\tilde{b}$ as follows:

$$
\begin{aligned}
d(\tilde{a},\tilde{b}) &= \int_0^1 \left(M_\alpha(\tilde{a})-M(\tilde{b})\right)d\alpha \\
&= \int_0^1 d^*\left(M_\alpha(\tilde{a}),M(\tilde{b})\right)d\alpha
\end{aligned}
$$

where $M_\alpha(\tilde{a})$ and $M(\tilde{b})$ are equal to $\frac{a_\alpha^L+a_\alpha^U}{2}$ and $\frac{b_\alpha^L+b_\alpha^U}{2}$ respectively, furthermore $d(\tilde{a},\tilde{b})$ means the distance of $\tilde{a}$ to $\tilde{b}$.

Definition 2.3: (Yao and Wu, 2000) For each $\tilde{a},\tilde{b}\in F(R)$, define the ranking of $\tilde{a}$ and $\tilde{b}$ by:

$$
\begin{aligned}
d(\tilde{a},\tilde{b})>0 \quad &\Leftrightarrow \quad d(\tilde{a},0)>d(\tilde{b},0) \quad \Leftrightarrow \quad \tilde{a}\succ\tilde{b} \\
d(\tilde{a},\tilde{b})<0 \quad &\Leftrightarrow \quad d(\tilde{a},0)<d(\tilde{b},0) \quad \Leftrightarrow \quad \tilde{a}\prec\tilde{b} \\
d(\tilde{a},\tilde{b})=0 \quad &\Leftrightarrow \quad d(\tilde{a},0)=d(\tilde{b},0) \quad \Leftrightarrow \quad \tilde{a}=\tilde{b}.
\end{aligned}
$$

3. SOME CRISP CENTRAL MOMENTS BASED ON FUZZY RANDOM VARIABLES BASED ON L_2 METRIC

In this section, we define variance, covariance and correlation coefficient of fuzzy random variables and study some of their properties. Before, we define support function, which plays an important role in measuring differences between fuzzy quantities.

For each $\alpha-$cuts of $\tilde{a}\in F(R^n)$ the support function $S_{\tilde{a}_\alpha}$ is defined as $S_{\tilde{a}_\alpha}(t)=\sup_{x\in\tilde{a}_\alpha} << x,t >>$, $t\in S^{n-1}$, S^{n-1} the $(n-1)-$dimensional unit sphere in R^n. Using support function we define L_2 metric

$$
\delta_2\left(\tilde{a},\tilde{b}\right)=\left(n\int_0^1\left(\rho_2\left(\tilde{a}_\alpha,\tilde{b}_\alpha\right)\right)^2 d\alpha\right)^{\frac{1}{2}} \qquad \tilde{a},\tilde{b}\in F\left(R^n\right)
$$

where

$$
\rho_2\left(\tilde{a}_\alpha,\tilde{b}_\alpha\right)=\left(\int_{S^{n-1}}| S_{\tilde{a}_\alpha}(t)-S_{\tilde{b}_\alpha}(t)|^2\mu(dt)\right)^{\frac{1}{2}}.
$$

Note that μ is the normalized Lebesgue measure on S^{n-1}.

Definition 3.1: The variance of a FRV $\tilde{X}$ is defined as $Var\left(\tilde{X}\right)=\mathrm{E}\left[\delta_2^2\left(\tilde{X},\tilde{\mathrm{E}}\left(\tilde{X}\right)\right)\right]$.

Using $\tilde{\mathrm{E}}_\alpha\left(\tilde{X}\right)=\tilde{\mathrm{E}}\left(\tilde{X}_\alpha\right)$ and $S_{\tilde{\mathrm{E}}_\alpha(\tilde{X})}(t)=\mathrm{E}\left(S_{\tilde{X}_\alpha}(t)\right)$ this can be written as

$$Var\left(\tilde{X}\right) = n\int_0^1 \int_{S^{n-1}} Var\left(S_{\tilde{X}_\alpha}(t)\right)\mu(dt)d\alpha.$$

Nather (2006) defined an scalar multiplication between $\tilde{X}$ and $\tilde{Y}$ given by

$$< \tilde{X}, \tilde{Y} >= n\int_0^1 \int_{S^{n-1}} S_{\tilde{X}_\alpha}(t) S_{\tilde{Y}_\alpha}(t)\mu(dt)d\alpha$$

thus

$$Var\left(\tilde{X}\right) = \mathrm{E} < \tilde{X}, \tilde{X} > - < \tilde{\mathrm{E}}\left(\tilde{X}\right), \tilde{\mathrm{E}}\left(\tilde{X}\right) >$$

and similarly

$$\begin{aligned} Cov\left(\tilde{X}, \tilde{Y}\right) &= n\int_0^1 Cov\left(\int_{S^{n-1}} S_{\tilde{X}_\alpha}(t), S_{\tilde{Y}_\alpha}(t)\right)\mu(dt)d\alpha \\ &= \mathrm{E} < \tilde{X}, \tilde{Y} > - < \tilde{\mathrm{E}}\left(\tilde{X}\right), \tilde{\mathrm{E}}\left(\tilde{Y}\right) >, \end{aligned}$$

then we obtain the correlation coefficient given by

$$\rho\left(\tilde{X}, \tilde{Y}\right) = \frac{Cov\left(\tilde{X}, \tilde{Y}\right)}{\sqrt{Var\left(\tilde{X}\right) Var\left(\tilde{Y}\right)}}.$$

We note that in spite of $\tilde{X}$ and $\tilde{Y}$ are fuzzy quantities, $Var\left(\tilde{X}\right)$ and $Var\left(\tilde{Y}\right)$ are crisp.

Definition 3.2: (Puri and Ralescu (1991)) The conditional expectation of $\tilde{Y}$ with respect to the $\sigma-$algebra is the fuzzy random variable $\tilde{\mathrm{E}}\left(\tilde{Y} \mid G\right)$ with the following properties:

a) $\tilde{\mathrm{E}}\left(\tilde{Y} \mid G\right)$ is G measurable.

b) $\int_T \tilde{\mathrm{E}}\left(\tilde{Y} \mid G\right)dp = \int_T \tilde{Y}dp \qquad \forall\ T \in G.$

Lemma 3.1: If $G \subseteq F$, then

$$\mathrm{E}\left(S_{\tilde{X}_\alpha}(t) \mid G\right) = S_{\tilde{\mathrm{E}}_\alpha(\tilde{Y}|G)}(t) \qquad t \in S^{n-1}.$$

Proof: We know that $\mathrm{E}\left(S_{\tilde{Y}_\alpha}(t) \mid G\right)$ and $S_{\tilde{\mathrm{E}}_\alpha(\tilde{Y}|G)}(t)$ are $G-$measurable, thus it is enough to show

$$\int_T \mathrm{E}\left(S_{\tilde{Y}_\alpha}(t) \mid G\right)dp = \int_T S_{\tilde{Y}_\alpha}(t)dp = \int_T S_{\tilde{\mathrm{E}}_\alpha(\tilde{Y}|G)}(t)\,dp \qquad \forall\ T \in G.$$

Let $T \in G$

$$
\begin{aligned}
\int_T S_{\tilde{Y}_\alpha}(t)dp &= \int_\Omega I_T S_{\tilde{Y}_\alpha}(t)dp \\
&= \mathrm{E}\left(I_T S_{\tilde{Y}_\alpha}(t)\right) \\
&= S_{\mathrm{E}(I_T \tilde{Y}_\alpha)}(t) \\
&= S_{\int_T \tilde{Y}_\alpha dp}(t) \\
&= S_{\int_T \mathrm{E}\left(\tilde{Y}_\alpha | G\right) dp}(t) \\
&= \int_T S_{\tilde{\mathrm{E}}_\alpha\left(\tilde{Y}|G\right)}(t)\, dp.
\end{aligned}
$$

Lemma 3.2: (Puri and Ralescu (1991)) For $\alpha \in [0,1]$,

$$\mathrm{E}\left(\tilde{Y}_\alpha \mid G\right) = \mathrm{E}_\alpha\left(\tilde{Y} \mid G\right).$$

Let $\oplus$ or Θ be two binary operations (sum and subtraction) between two canonical fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership functions of $\tilde{a} \oplus \tilde{b}$ and $\tilde{a}\Theta\tilde{b}$ are defined by

$$
\begin{aligned}
\mu_{\tilde{a}\oplus\tilde{b}}(z) &= \sup_{a+b=z} \min\left\{\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)\right\}, \\
\mu_{\tilde{a}\Theta\tilde{b}}(z) &= \sup_{a-b=z} \min\left\{\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)\right\}.
\end{aligned}
$$

Lemma 3.3: Let $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers. Then $\tilde{a} \oplus \tilde{b}$ and $\tilde{a} \Theta \tilde{b}$ are also closed fuzzy numbers. Furthermore, we have

$$
\begin{aligned}
\left(\tilde{a} \oplus \tilde{b}\right)_\alpha &= \left(a_\alpha^L + b_\alpha^L, a_\alpha^U + b_\alpha^U\right), \\
\left(\tilde{a}\Theta\tilde{b}\right)_\alpha &= \left((a_\alpha^L - b_\alpha^U, a_\alpha^U - b_\alpha^L\right).
\end{aligned}
$$

If $G = \sigma\left(\tilde{X}\right)$ is induced by a FRV $\tilde{X}$ we write $\tilde{\mathrm{E}}\left(\tilde{Y} \mid G\right) = \tilde{\mathrm{E}}\left(\tilde{Y} \mid \tilde{X}\right)$ where $\sigma\left(\tilde{X}\right) \subseteq F$ is the smallest $\sigma-$algebra so that $\tilde{X}$ is measurable. In the following theorem, we exhibit some properties of $\tilde{X}$ and $\tilde{Y}$' s central moments.

Theorem 3.1: For given fuzzy random variables $\tilde{X}$ and $\tilde{Y}$

i) $Var\left(a\tilde{X}\right) = a^2 Var\left(\tilde{X}\right)$.

ii) $Var\left(a\tilde{X} \oplus b\tilde{Y}\right) = a^2 Var\left(\tilde{X}\right) + b^2 Var\left(\tilde{Y}\right) + 2abCov\left(\tilde{X}, \tilde{Y}\right)$.

iii) $Var\left(\tilde{X}\right) = Var\left(\tilde{\mathrm{E}}\left[\tilde{X} \mid \tilde{Y}\right]\right) + \mathrm{E}\left(Var\left[\tilde{X} \mid \tilde{Y}\right]\right)$,

where $a, b \in R^+$.

Proof (i):

$$Var\left(a\tilde{X}\right)=n\int_0^1\int_{S^{n-1}}\mathrm{E}\left(S_{a\tilde{X}_\alpha}(t)-S_{\mathrm{E}\left(a\tilde{X}_\alpha\right)}(t)\right)^2\mu\left(dt\right)d\alpha$$

Under the assumption $a>0$, we conclude $S_{\mathrm{E}\left(a\tilde{X}_\alpha\right)}(t)=a\mathrm{E}\left(S_{\tilde{X}_\alpha}(t)\right)$. Then we have

$$\begin{aligned}Var(a\tilde{X})&=n\int_0^1\int_{S^{n-1}}\mathrm{E}\left(aS_{\tilde{X}_\alpha}(t)-a\mathrm{E}\left(S_{\tilde{X}_\alpha}(t)\right)\right)^2\mu(dt)d\alpha\\&=a^2Var\left(\tilde{X}\right).\end{aligned}$$

The proof of **(ii)** is similar to **(i)**.

Proof (iii)

$$Var\left[\tilde{X}\mid\tilde{Y}\right]=n\int_0^1\int_{S^{n-1}}\mathrm{E}\left[S_{\tilde{X}_\alpha}(t)-S_{\mathrm{E}\left(\tilde{X}_\alpha|\tilde{Y}\right)}(t)^2\mid\tilde{Y}\right]\mu(dt)d\alpha.$$

Using Lemma 3.1

$$\begin{aligned}Var\left[[\tilde{X}\mid\tilde{Y}\right]&=n\int_0^1\int_{S^{n-1}}\mathrm{E}\left[S_{\tilde{X}_\alpha}(t)-\mathrm{E}\left(S_{\tilde{X}_\alpha}(t)\mid\tilde{Y}\right)^2\mid\tilde{Y}\right]\mu(dt)d\alpha\\&=n\int_0^1\int_{S^{n-1}}\mathrm{E}\left[S^2_{\tilde{X}_\alpha}(t)\mid\tilde{Y}\right]\mu(dt)d\alpha-n\int_0^1\int_{S^{n-1}}\mathrm{E}^2\left(S_{\tilde{X}_\alpha}(t)\mid\tilde{Y}\right)\mu(dt)d\alpha\end{aligned}$$

therefore,

$$\mathrm{E}\left(Var\left[\tilde{X}\mid\tilde{Y}\right]\right)=n\int_0^1\int_{S^{n-1}}\mathrm{E}\left[S^2_{\tilde{X}_\alpha}(t)\right]\mu(dt)d\alpha-n\int_0^1\int_{S^{n-1}}\mathrm{E}\left\{\mathrm{E}^2\left(S_{\tilde{X}_\alpha}(t)\mid\tilde{Y}\right)\right\}\mu(dt)d\alpha$$

and similarly based on definition of variance for a fuzzy random

$$\begin{aligned}Var\left(\tilde{\mathrm{E}}\left[\tilde{X}\mid\tilde{Y}\right]\right)&=n\int_0^1\int_{S^{n-1}}Var\left(S_{\mathrm{E}\left(\tilde{X}_\alpha|\tilde{Y}\right)}(t)\right)\mu(dt)d\alpha\\&=n\int_0^1\mathrm{E}\left(S^2_{\mathrm{E}\left(\tilde{X}_\alpha|\tilde{Y}\right)}(t)\right)\mu(dt)d\alpha-n\int_0^1\mathrm{E}^2\left(S_{\mathrm{E}\left(\tilde{X}_\alpha|\tilde{Y}\right)}(t)\right)\mu(dt)d\alpha\\&=n\int_0^1\mathrm{E}\left[\mathrm{E}^2\left(S_{\tilde{X}_\alpha}(t)\mid\tilde{Y}\right)\right]\mu(dt)d\alpha-n\int_0^1\mathrm{E}\left[\mathrm{E}\left(S_{\tilde{X}_\alpha}(t)\right)\mid\tilde{Y}\right]^2\mu(dt)d\alpha\\&=n\int_0^1\mathrm{E}\left[\mathrm{E}^2\left(S_{\tilde{X}_\alpha}(t)\mid\tilde{Y}\right)\right]\mu(dt)d\alpha-n\int_0^1\mathrm{E}^2\left(S_{\tilde{X}_\alpha}(t)\right)\mu(dt)d\alpha,\end{aligned}$$

Hence

$$Var\left(\tilde{X}\right)=Var\left(\tilde{\mathrm{E}}\left[\tilde{X}\mid\tilde{Y}\right]\right)+\mathrm{E}\left(Var\left[\tilde{X}\mid\tilde{Y}\right]\right).$$

In order to define the variance of difference between two fuzzy random variables we apply Hukuhara (1967) difference. Let two fuzzy sets $\tilde{a}$ and $\tilde{b}$ be given. If there exist a fuzzy set $\tilde{c}$ with $\tilde{a} = \tilde{b} \oplus \tilde{c}$ then $\tilde{c}$ is called Hukuhara difference between $\tilde{a}$ and $\tilde{b}$, denoted by

$$\tilde{c} := \tilde{a} \Theta_H \tilde{b}.$$

Based on support function, for three fuzzy sets $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ with $\tilde{c} = \tilde{a} \oplus \tilde{b}$, it can be easily seen that

$$S_{\tilde{c}_\alpha}(t) = S_{\tilde{a}_\alpha}(t) + S_{\tilde{b}_\alpha}(t),$$

but it does not admit $S_{\tilde{c}_\alpha}(t) = S_{\tilde{a}_\alpha}(t) - S_{\tilde{b}_\alpha}(t)$ for $\tilde{c} = \tilde{a} \Theta \tilde{b}$. Therefore proceed with Hukuhara difference dealing with the difference of $S_{\tilde{a}_\alpha}(t) - S_{\tilde{b}_\alpha}(t)$.

Lemma 3.4: Given three fuzzy sets $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ with $\tilde{c} := \tilde{a} \Theta_H \tilde{b}$, we have

$$S_{\tilde{c}_\alpha}(t) = S_{\tilde{a}_\alpha}(t) - S_{\tilde{b}_\alpha}(t).$$

It can be easily shown that

$$Var\left(a\tilde{X} \Theta_H b\tilde{Y}\right) = a^2 Var\left(\tilde{X}\right) + b^2 Var\left(\tilde{Y}\right) - 2ab Cov\left(\tilde{X}, \tilde{Y}\right).$$

Theorem 3.2: For fuzzy random variables $\tilde{X}$ and $\tilde{Y}$

$$\left|\rho\left(\tilde{X}, \tilde{Y}\right)\right| \leq 1.$$

Proof: We have

$$Var\left(\frac{\tilde{X}}{\sqrt{Var\left(\tilde{X}\right)}} \oplus \frac{\tilde{Y}}{\sqrt{Var\left(\tilde{Y}\right)}}\right) = \frac{Var\left(\tilde{X}\right)}{Var\left(\tilde{X}\right)} + \frac{Var\left(\tilde{Y}\right)}{Var\left(\tilde{Y}\right)} + 2\frac{Cov\left(\tilde{X}, \tilde{Y}\right)}{\sqrt{Var\left(\tilde{X}\right)}\sqrt{Var\left(\tilde{Y}\right)}}$$

$$= 1 + 1 + 2\rho\left(\tilde{X}, \tilde{Y}\right) \geq 0$$

thus $\rho\left(\tilde{X}, \tilde{Y}\right) \geq -1$. Similarly

$$Var\left(\frac{\tilde{X}}{\sqrt{Var\left(\tilde{X}\right)}} \Theta_H \frac{\tilde{Y}}{\sqrt{Var\left(\tilde{Y}\right)}}\right) = \frac{Var\left(\tilde{X}\right)}{Var\left(\tilde{X}\right)} + \frac{Var\left(\tilde{Y}\right)}{Var\left(\tilde{Y}\right)} - 2\frac{Cov\left(\tilde{X}, \tilde{Y}\right)}{\sqrt{Var\left(\tilde{X}\right)}\sqrt{Var\left(\tilde{Y}\right)}}$$

$$= 1 + 1 - 2\rho\left(\tilde{X}, \tilde{Y}\right) \geq 0$$

Thus $\rho\left(\tilde{X}, \tilde{Y}\right) \leq 1$, hence $|\rho\left(\tilde{X}, \tilde{Y}\right)| \leq 1$.

Example 3.1: Let (X, Y) are jointly distributed as bivariate normal denoted by $(X, Y) \sim N_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho\sigma_x^2\sigma_y^2)$. If $\left(\tilde{X}, \tilde{Y}\right)$ be triangular fuzzy random variables given by $(X-a, X, X+b)$ and $(Y-a, Y, Y+b)$ respectively, then

$$Var\left(\tilde{X}\right) = \frac{2}{3}\sigma_x^2 \qquad Var\left(\tilde{Y}\right) = \frac{2}{3}\sigma_y^2 \qquad Cov\left(\tilde{X}, \tilde{Y}\right) = \frac{2}{3}\rho\sigma_x^2\sigma_y^2,$$

moreover $\rho\left(\tilde{X}, \tilde{Y}\right) = \rho$.

If we take $\tilde{Y}$ to be as bell shape, with the membership function given by

$$\mu_{\tilde{Y}}(t) = \exp\left\{-\left(\frac{t-y}{y}\right)^2\right\} \qquad t \in R$$

then we obtain

$$Var\left(\tilde{X}\right) = \frac{2}{3}\sigma_x^2 \qquad Var\left(\tilde{Y}\right) = \frac{4}{3}\sigma_y^2 \qquad Cov\left(\tilde{X}, \tilde{Y}\right) = \frac{2}{3}\rho\sigma_x^2\sigma_y^2,$$

hence $\rho\left(\tilde{X}, \tilde{Y}\right) = \frac{1}{\sqrt{2}}\rho$.

4. SOME CRISP CENTRAL MOMENTS BASED ON FUZZY RANDOM SAMPLE BASED ON YAO-WU SINGED DISTANCE

In this section we describe on way to get correlation and its mathematical properties using fuzzy data.

Let there be two canonical fuzzy numbers $\left((\tilde{x}_i, \tilde{y}_i);\ i = 1, 2, ..., n\right)$ defined on the same universe R.

Definition 4.1: For $(\tilde{x}, \tilde{y})$, define the correlation of $\tilde{x}$ and $\tilde{y}$ as follows:

$$\rho_{(\tilde{x},\tilde{y})} = \frac{Cov(\tilde{x}, \tilde{y})}{\sigma(\tilde{x})\sigma(\tilde{y})},$$

where $Cov(\tilde{x}, \tilde{y})$, $\sigma^2(\tilde{x})$ and $\sigma^2(\tilde{y})$ are equal to $\frac{1}{n}\sum_{i=1}^{n} d\left(\tilde{x}_i, \tilde{\bar{x}}\right) d\left(\tilde{y}_i, \tilde{\bar{y}}\right)$, $\frac{1}{n}\sum_{i=1}^{n} d^2\left(\tilde{x}_i, \tilde{\bar{x}}\right)$ and $\frac{1}{n}\sum_{i=1}^{n} d^2\left(\tilde{y}_i, \tilde{\bar{y}}\right)$ respectively, furthermore $\tilde{\bar{x}} = \frac{1}{n}\oplus_{i=1}^{n}\tilde{x}_i$ and d is Yao-Wu signed distance.

Lemma 4.1: $\rho_{(\tilde{x},\tilde{x})} = 1$.

This means that the correlation between two identical fuzzy data is always 1.

Lemma 4.2: $\rho_{(\tilde{x},\tilde{y})} = \rho_{(\tilde{y},\tilde{x})}$.

This ensures that the measure is symmetric.

Lemma 4.3: $|\rho_{(\tilde{x},\tilde{y})}| \leq 1$ **.**

Proof: We have

$$\frac{1}{n}\sum_{i=1}^{n} d^2\left(\frac{\tilde{x}_i}{\sigma(\tilde{x})}\Theta\frac{\tilde{y}_{i,}}{\sigma(\tilde{y})}, \frac{\tilde{\bar{x}}}{\sigma(\tilde{x})}\Theta\frac{\tilde{\bar{y}}}{\sigma(\tilde{y})}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(d\left(\frac{\tilde{x}_i}{\sigma(\tilde{x})}, \frac{\tilde{\bar{x}}}{\sigma(\tilde{x})}\right) - d\left(\frac{\tilde{y}_{i,}}{\sigma(\tilde{y})}, \frac{\tilde{\bar{y}}}{\sigma(\tilde{y})}\right)\right)^2$$
$$= \frac{\sigma^2(\tilde{x})}{\sigma^2(\tilde{x})} + \frac{\sigma^2(\tilde{y})}{\sigma^2(\tilde{y})} - 2\frac{Cov(\tilde{x},\tilde{y})}{\sigma(\tilde{x})\sigma(\tilde{y})}$$
$$\geq 0$$

thus we verify $\rho_{(\tilde{x},\tilde{y})} \leq 1$ and similarly

$$\frac{1}{n}\sum_{i=1}^{n} d^2\left(\frac{\tilde{x}_i}{\sigma(\tilde{x})}\oplus\frac{\tilde{y}_{i,}}{\sigma(\tilde{y})}, \frac{\tilde{\bar{x}}}{\sigma(\tilde{x})}\oplus\frac{\tilde{\bar{y}}}{\sigma(\tilde{y})}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(d\left(\frac{\tilde{x}_i}{\sigma(\tilde{x})}, \frac{\tilde{\bar{x}}}{\sigma(\tilde{x})}\right) + d\left(\frac{\tilde{y}_{i,}}{\sigma(\tilde{y})}, \frac{\tilde{\bar{y}}}{\sigma(\tilde{y})}\right)\right)^2$$
$$= \frac{\sigma^2(\tilde{x})}{\sigma^2(\tilde{x})} + \frac{\sigma^2(\tilde{y})}{\sigma^2(\tilde{y})} + 2\frac{Cov(\tilde{x},\tilde{y})}{\sigma(\tilde{x})\sigma(\tilde{y})}$$
$$\geq 0$$

also we get $\rho_{(\tilde{x},\tilde{y})} \geq 1$.

Lemma 4.4: $Cov(\tilde{x},\tilde{y}) \leq \sqrt{\sigma^2(\tilde{x})\sigma^2(\tilde{y})}$.

It is a special condition of Holder's inequality.

Lemma 4.5: If a and b be two real numbers and, $\tilde{y}_i = a\tilde{x}_i + b$ then $\rho_{(\tilde{x},\tilde{y})} = 1$ if $a > 0$ and $\rho_{(\tilde{x},\tilde{y})} = -1$ if $a < 0$.

This ensure that the correlation is equal -1 or 1, if and only if there have been a linear relation between $\tilde{x}_i$'s and $\tilde{y}_i$'s.

Example 4.1: Suppose that we have taken a two fuzzy random sample of size $n = 10$ from a population and we observed the following triangular fuzzy data:

Table 1
Two fuzzy random sample of size $n = 10$ from a population

N	Observation
1	((19,20,22), (12,13,14))
2	((20,22,23), (11,12,12))
3	((16,19,22), (14,14,15))
4	((25,27,28), (12,13,17))
5	((27,30,30), (12,14,15))
6	((23,23,26), (13,15,17))
7	((31,32,33), (12,16,17))
8	((33,35,36), (15,17,19))
9	((22,23,23), (14,15,17))
10	((29,30,32), (13,14,15))

Then, $Cov\left(\tilde{x}, \tilde{y}\right) = 3.985$, $\sigma(\tilde{x}) = 5.074$, and the correlation between two fuzzy samples is 0.578.

Example 4.2: Consider Table 1. According to the L_2 metri**c** $Cov\left(\tilde{X}, \tilde{Y}\right) = 26.122$, $\sqrt{Var\left(\tilde{X}\right)} = 13.143$, $\sqrt{Var\left(\tilde{Y}\right)} = 3.706$ and the correlation between two fuzzy samples is 0.536.

5. CONCLUSIONS

In this paper we introduce correlation based on L_2 metric and Yao-Wu singed distance. As for this paper, it sound the correlation based on Yao-Wu singed distance is better than L_2 metric because the true value based on non-fuzzy data is 0.622 and the correlation based on Yao-Wu singed distance (0.578) is near to 0.622.

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